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Estimation of Smoothing Constant of Minimum Variance and Its Application to Shipping Data with Trend Removal Method

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Abstract. Focusing on the idea that the equation of exponential smoothing method (ESM) is equivalent to (1, 1) order ARMA model equation, new method of estimation of smoothing constant in exponential smoothing method is proposed before by us which satisfies minimum variance of forecasting error. Theoretical solution was derived in a simple way. Mere application of ESM does not make good forecasting accuracy for the time series which has non-linear trend and/or trend by month. A new method to cope with this issue is required. In this paper, combining the trend removal method with this method, we aim to improve forecasting accuracy. An approach to this method is executed in the following method. Trend removal by a linear function is applied to the original shipping data of consumer goods. The combination of linear and non-linear function is also introduced in trend removal. For the comparison, monthly trend is removed after that. Theoretical solution of smoothing constant of ESM is calculated for both of the monthly trend removing data and the non monthly trend removing data. Then forecasting is executed on these data. The new method shows that it is useful especially for the time series that has stable characteristics and has rather strong seasonal trend and also the case that has non-linear trend. The effectiveness of this method should be examined in various cases.

Keywords: Minimum Variance, Exponential Smoothing Method, Forecasting, Trend

1. INTRODUCTION

Many methods for time series analysis have been presented such as Autoregressive model (AR Model), Autoregressive Moving Average Model (ARMA Model) and Exponential Smoothing Method (ESM) (Jenkins, 1994, Brown, 1963, Tokumaru *et al*., 1982, Kobayashi 1993). Among these, ESM is said to be a practical simple method. For this method, various improving method such as adding compensating item for time lag, coping with the time series with trend (Winters, 1984), utilizing Kalman Filter (Maeda, 1984), Bayes Forecasting (West and Harrison, 1989), adaptive ESM (Ekern, 1982), exponentially weighted Moving Averages with irregular updating periods (Johnston, 1993), making averages of forecasts using plural method (Makridakis and Winkler, 1983) are presented. For example, Maeda (1984) calculated smoothing constant in relationship with S/N ratio under the assumption that the observation noise was added to the system. But he had to calculate under supposed noise because he couldn't grasp observation noise. It can be said that it doesn't pursue optimum solution from the very data themselves which should be derived by those estimation. Ishii *et al*. (1991) pointed out that the optimal smoothing constant was the solution of infinite order equation, but he didn't show analytical solu-

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tion. Double Exponential Smoothing Method is well known for the case to apply with trend. But if there is a non-linear trend, this method does not necessarily work well. Another approach for forecasting can be seen in such as Takeyasu *et al*. (2003), Takeyasu and Amemiya (2005a), Takeyasu and Higuchi (2005b). Based on these facts, we proposed a new method of estimation of smoothing constant in ESM before (Takeyasu and Nagao 2008). Focusing that the equation of ESM is equivalent to (1, 1) order ARMA model equation, a new method of estimation of smoothing constant in ESM was derived.

Mere application of ESM does not make good forecasting accuracy for the time series which has non-linear trend and/or trend by month. How to cope with this problem is a big issue. We have to devise a new method. Therefore, utilizing above stated method, revised forecasting method is proposed in this paper to improve forecasting accuracy. An approach to this method is executed in the following method. Trend removing by a linear function is applied to the original shipping data of consumer goods. The combination of linear and non-linear function is also introduced in trend removal. For the comparison, monthly trend is removed after that. Theoretical solution of smoothing constant of ESM is calculated for both of the monthly trend removing data and the non monthly trend removing data. Then forecasting is executed on these data. The new method shows that it is useful especially for the time series that has stable characteristics and has rather strong seasonal trend and also the case that has non-linear trend.

The rest of the paper is organized as follows. In section 2, ESM is stated by ARMA model and estimation method of smoothing constant is derived using ARMA model identification. The combination of linear and non-linear function is introduced for trend removing in section 3. The Monthly Ratio is referred in section 4. Forecasting is executed in section 5, and estimation accuracy is examined.

2. DESCRIPTION OF ESM USING ARMA MODEL

In ESM, forecasting at time $t+1$ is stated in the following equation.

$$\hat{x}_{t+1} = \hat{x}_t + \alpha\left(x_t - \hat{x}_t\right) \quad (1)$$

$$= \alpha x_t + (1-\alpha)\hat{x}_t \quad (2)$$

Here,

$\hat{x}_{t+1}$: forecasting at $t+1$

x_t: realized value at t

α: smoothing constant $(0<\alpha<1)$

(2) is re-stated as:

$$\hat{x}_{t+1} = \sum_{l=0}^{\infty} \alpha(1-\alpha)^l x_{t-l} \quad (3)$$

By the way, we consider the following (1, 1) order ARMA model.

$$x_t - x_{t-1} = e_t - \beta e_{t-1} \quad (4)$$

Generally, (p, q) order ARMA model is stated as:

$$x_t + \sum_{i=1}^{p} a_i x_{t-i} = e_t + \sum_{j=1}^{q} b_j e_{t-j} \quad (5)$$

Here,

$\{x_t\}$: Sample process of Stationary Ergodic Gaussian Process $x(t)$ $t = 1, 2, \cdots, N, \cdots$

$\{e_t\}$: Gaussian White Noise with 0 mean σ_e^2 variance

MA process in (5) is supposed to satisfy convertibility condition.

Utilizing the relation that:

$$E\left[e_t \mid e_{t-1}, e_{t-2}, \cdots\right] = 0$$

we get the following equation from (4).

$$\hat{x}_t = x_{t-1} - \beta e_{t-1} \quad (6)$$

Operating this scheme on $t+1$, we finally get:

$$\hat{x}_{t+1} = \hat{x}_t + (1-\beta)e_t$$
$$= \hat{x}_t + (1-\beta)(x_t - \hat{x}_t) \quad (7)$$

If we set $1-\beta = \alpha$, the above equation is the same with (1), i.e., equation of ESM is equivalent to (1, 1) order ARMA model.

Comparing (4) with (5) and using (1) and (7), we get:

$$\begin{cases} a_1 = -1 \\ b_1 = -\beta = \alpha - 1 \end{cases} \quad (8)$$

From above, we can get estimation of smoothing constant after we identify the parameter of MA part of ARMA model. But, generally MA part of ARMA model become non-linear equations which are described below.

Let (5) be:

$$\tilde{x}_t = x_t + \sum_{i=1}^{p} a_i x_{t-i} \quad (9)$$

$$\tilde{x}_t = e_t + \sum_{j=1}^{q} b_j e_{t-j} \quad (10)$$

We express the autocorrelation function of $\tilde{x}_t$ as

$\tilde{r}_k$ and from (9), (10), we get the following non-linear equations which are well known (Tokumaru, 1982).

$$\left.\begin{array}{l}\begin{cases}\tilde{r}_k = \sigma_e^2 \sum_{j=0}^{q-k} b_j b_{k+j} & (k \le q) \\ 0 & (k \ge q+1)\end{cases} \\ \tilde{r}_0 = \sigma_e^2 \sum_{j=0}^{q} b_j^2 \end{array}\right\} \quad (11)$$

For these equations, recursive algorithm has been developed. In this paper, parameter to be estimated is only b_1, so it can be solved in the following way.

From (4) (5) (8) (11), we get:

$$\begin{aligned} & q = 1 \\ & a_1 = -1 \\ & b_1 = -\beta = \alpha - 1 \\ & \tilde{r}_0 = \left(1 + b_1^2\right)\sigma_e^2 \\ & \tilde{r}_1 = b_1 \sigma_e^2 \end{aligned} \quad (12)$$

If we set:

$$\rho_k = \frac{\tilde{r}_k}{\tilde{r}_0} \quad (13)$$

the following equation is derived.

$$\rho_1 = \frac{b_1}{1 + b_1^2} \quad (14)$$

We can get b_1 as follows.

$$b_1 = \frac{1 \pm \sqrt{1 - 4\rho_1^2}}{2\rho_1} \quad (15)$$

In order to have real roots, ρ_1 must satisfy

$$|\rho_1| \le \frac{1}{2} \quad (16)$$

As

$$\alpha = b_1 + 1$$

b_1 is within the range of

$$-1 < b_1 < 0$$

Finally we get

$$\begin{aligned} b_1 &= \frac{1 - \sqrt{1 - 4\rho_1^2}}{2\rho_1} \\ \alpha &= \frac{1 + 2\rho_1 - \sqrt{1 - 4\rho_1^2}}{2\rho_1} \end{aligned} \quad (17)$$

which satisfy above condition. Thus we can obtain a theoretical solution by a simple way.

3. TREND REMOVAL METHOD

As ESM is a one of a linear model, forecasting accuracy for the time series with non-linear trend is not necessarily good. How to remove trend for the time series with non-linear trend is a big issue in improving forecasting accuracy. In this paper, we devise to remove this non-linear trend by utilizing non-linear function.

As trend removal method, we describe linear and non-linear function, and the combination of these.

[1] Linear function
We set:

$$y = c_1 x + d_1 \quad (18)$$

as a linear function, where x is a variable, for example, time and y is a variable, for example, shipping amount, c_1 and d_1 are parameters which are estimated by using least square method.

[2] Non-linear function
We set:

$$y = c_2 x^2 + d_2 x + e_2 \quad (19)$$

$$y = c_3 x^3 + d_3 x^2 + e_3 x + f_3 \quad (20)$$

as a 2nd and a 3rd order non-linear function. (c_2, d_2, e_2) and (c_3, d_3, e_3, f_3) are also parameters for a 2nd and a 3rd order non-linear functions which are estimated by using least square method.

[3] The combination of a linear and a non-linear function
We set:

$$y = \gamma_1\left(c_1 x + d_1\right) + \gamma_2\left(c_2 x^2 + d_2 x + e_2\right) \quad (21)$$

as the combination of a linear and a 2nd order non-linear function, and

$$y = \delta_1\left(c_1 x + d_1\right) + \delta_2\left(c_3 x^3 + d_3 x^2 + e_3 x + f_3\right) \quad (22)$$

as the combination of a linear and a 3rd order non-linear function. Here, $\gamma_2 = 1 - \gamma_1$ and $\delta_2 = 1 - \delta_1$. We use these (18), (21) and (22) for trend removal. Comparative discussion concerning (18), (21) and (22) is described in section 5.

4. MONTHLY RATIO

For example, if there is the monthly data of L years as stated bellow:

$$\{x_{ij}\} \quad (i = 1, \cdots, L) \quad (j = 1, \cdots, 12)$$

where $x_{ij} \in R$ in which j means month and i means year and x_{ij} is a shipping data of i-th year, j-th month. Then, monthly ratio $\tilde{x}_j$ $(j = 1, \cdots, 12)$ is calculated as follows.

$$\tilde{x}_j = \frac{\frac{1}{L}\sum_{i=1}^{L} x_{ij}}{\frac{1}{L}\cdot\frac{1}{12}\sum_{i=1}^{L}\sum_{j=1}^{12} x_{ij}} \quad (23)$$

5. FORECASTING THE SHIPPING DATA OF MANUFACTURER

5.1 Analysis Procedure

The shipping data of food manufacture for 3 cases (Food A, Food B, Food C) from January 1999 to December 2001 are analysed. First of all, graphical charts of these time series data are exhibited in Figure 5-1, 5-2, 5-3.

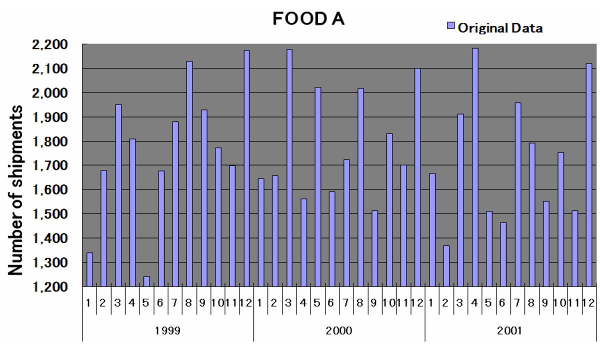

Figure 5-1. Shipping data of Food A.

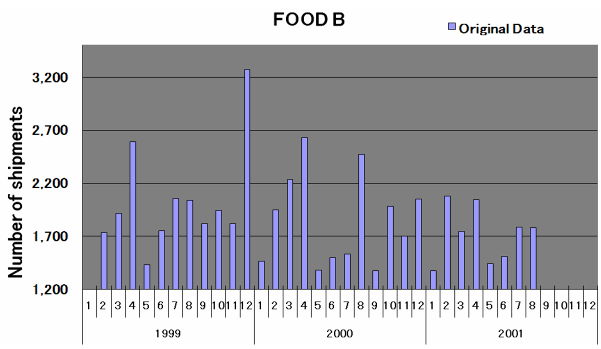

Figure 5-2. Shipping data of Food B.

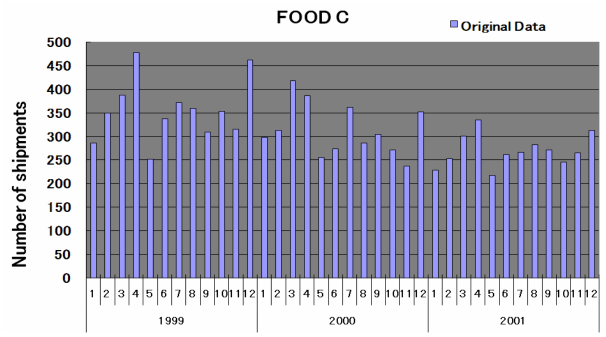

Figure 5-3. Shipping data of Food C.

Analysis procedure is as follows. There are 36 monthly data for each case. We use 24 data (1 to 24) and remove trend by the method stated in 3. Then we calculate monthly ratio by the method stated in 4. After removing monthly trend, the method stated in 2 is applied and Exponential Smoothing Constant with minimum variance of forecasting error is estimated. Then 1 step forecast is executed. Thus, data is shifted to 2nd to 25th and the forecast for 26th data is executed consecutively, which finally reaches forecast of 36th data. Final forecasting data is obtained by multiplying monthly ratio and trend. To examine the accuracy of forecasting, variance of forecasting error is calculated for the data of 25th to 36th data, where we have actual data and therefore we can calculate the forecasting error.

Forecasting error is expressed as:

$$\varepsilon_i = \hat{x}_i - x_i \quad (24)$$

By using (25),

$$\bar{\varepsilon} = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_i \quad (25)$$

variance of forecasting error is calculated by (26).

$$\sigma_\varepsilon^2 = \frac{1}{N-1}\sum_{i=1}^{N}(\varepsilon_i - \bar{\varepsilon})^2 \quad (26)$$

5.2 Trend Removing

Trend is removed by dividing the original data by (18), (21), (22). Parameter estimation result is exhibited in Table 5-1 for the case of calculation result of 1st to

Table 5-1. Parameter Estimation Result.

	1st		2nd					3rd					
	parameter		weight		parameter			weight		parameter			
	c_1	d_1	γ_1	γ_2	c_2	d_2	e_2	δ_1	δ_2	c_3	d_3	e_3	f_3
Food A	7.6	1688.8	0.5	0.5	-1.2	36.7	1562.9	0.5	0.5	0.3	-10.9	135.8	1335.6
Food B	3.5	1861.0	0.5	0.5	-3.0	78.9	1534.4	0.5	0.5	0.4	-17.8	229.5	1189.0
Food C	-2.8	369.8	0.5	0.5	-0.3	4.3	339.0	0.5	0.5	0.03	-1.3	14.7	315.0

24th data. Where the weight of $\gamma_1, \gamma_2, \delta_1, \delta_2$ are set 0.5 in (21), (22).
Graphical chart of trend is exhibited in Figure 5-4, 5-5, 5-6 for Food A, Food B, Food C respectively.

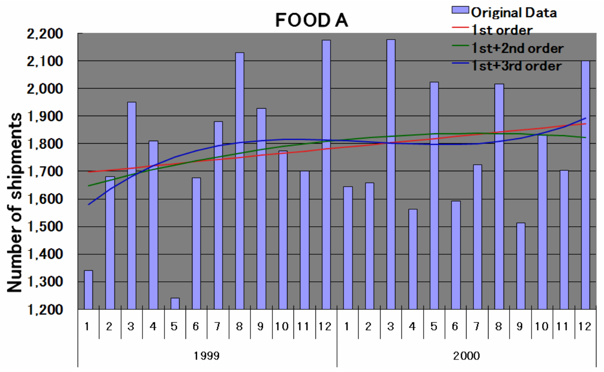

Figure 5-4. Trend of Food A.

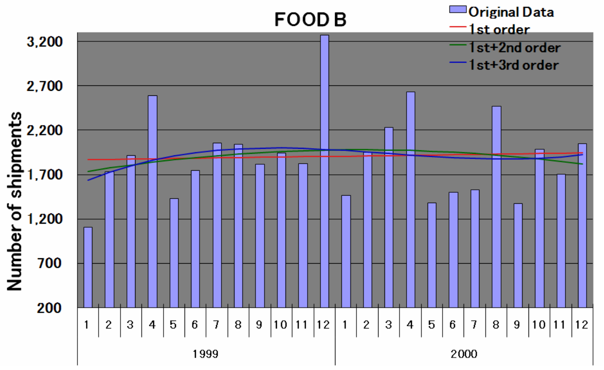

Figure 5-5. Trend of Food B.

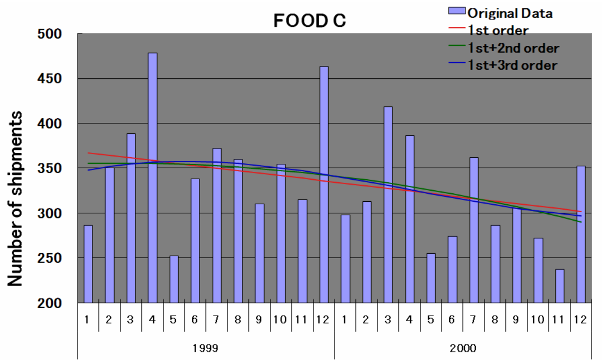

Figure 5-6. Trend of Food C.

5.3 Removing trend of monthly ratio

After removing the trend, monthly ratio is calculated by the method stated in 4. Calculation result for 1st to 24th data is exhibited in Table 5-2, 5-3, 5-4 for the 1st order case, the 1st+2nd order case, the 1st+3rd order case respectively.

5.4 Estimation of Smoothing Constant with Minimum Variance of Forecasting Error

After removing monthly trend, Smoothing Constant with minimum variance of forecasting error is estimated utilizing (17). There are cases that we cannot obtain a theoretical solution because they do not satisfy the condition of (16). In those cases, Smoothing Constant with minimum variance of forecasting error is derived by shifting variable from 0.01 to 0.99 with 0.01 increments. Calculation result for 1st to 24th data is exhibited in

Table 5-2. Monthly ratio (1st order).

Month	1	2	3	4	5	6	7	8	9	10	11	12
Food A	0.85	0.95	1.17	0.96	0.92	0.92	1.01	1.16	0.96	0.99	0.94	1.17
Food B	0.68	0.97	1.10	1.38	0.74	0.85	0.94	1.18	0.84	1.02	0.92	1.39
Food C	0.84	0.95	1.18	1.26	0.75	0.91	1.10	0.98	0.94	0.96	0.85	1.27

Table 5-3. Monthly ratio (1st + 2nd order).

Month	1	2	3	4	5	6	7	8	9	10	11	12
Food A	0.86	0.96	1.17	0.96	0.91	0.92	1.01	1.15	0.95	0.99	0.94	1.18
Food B	0.69	0.98	1.10	1.37	0.74	0.85	0.93	1.17	0.83	1.03	0.92	1.39
Food C	0.84	0.96	1.17	1.26	0.75	0.90	1.10	0.97	0.94	0.96	0.86	1.28

Table 5-4. Monthly ratio (1st + 3rd order).

Month	1	2	3	4	5	6	7	8	9	10	11	12
Food A	0.88	0.97	1.18	0.96	0.92	0.92	1.00	1.15	0.95	0.99	0.93	1.16
Food B	0.71	1.00	1.11	1.38	0.74	0.85	0.93	1.17	0.82	1.01	0.91	1.36
Food C	0.85	0.97	1.18	1.26	0.75	0.91	1.10	0.97	0.94	0.96	0.85	1.27

Table 5-5, 5-6, 5-7 for the 1st order case, the 1st+2nd order case, the 1st+3rd order case respectively.

Table 5-5. Estimated Smoothing Constant with Minimum Variance (1st order).

	ρ_1	α
Food A	-0.201381	0.789713
Food B	-0.699840 (Does not satisfy (16))	0.930000
Food C	-0.110217	0.888411

Table 5-6. Estimated Smoothing Constant with Minimum Variance (1st + 2nd order).

	ρ_1	α
Food A	-0.214793	0.774262
Food B	-0.271424	0.704947
Food C	-0.229036	0.757494

Table 5-7. Estimated Smoothing Constant with Minimum Variance (1st + 3rd order).

	ρ_1	α
Food A	-0.254405	0.726574
Food B	-0.289939	0.680455
Food C	-0.220829	0.767203

5.5 Forecasting and Variance of Forecasting Error

Utilizing smoothing constant estimated in the previous section, forecasting is executed for the data of 25th to 36th data. Final forecasting data is obtained by multiplying monthly ratio and trend. Variance of forecasting error is calculated by (26). Forecasting results are exhibited in Figure 5-7, 5-8, 5-9 for Food A, Food B, Food C respectively.

Variance of forecasting error in the case that monthly ratio is used are exhibited in Table 5-8. Variance of forecasting error in the case that monthly ratio is not used is exhibited in Table 5-9.

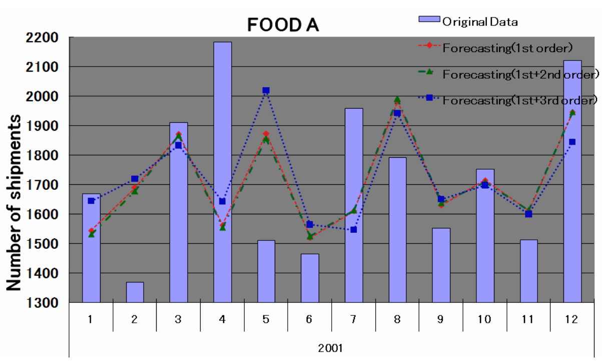

Figure 5-7. Forecasting Result of Food A.

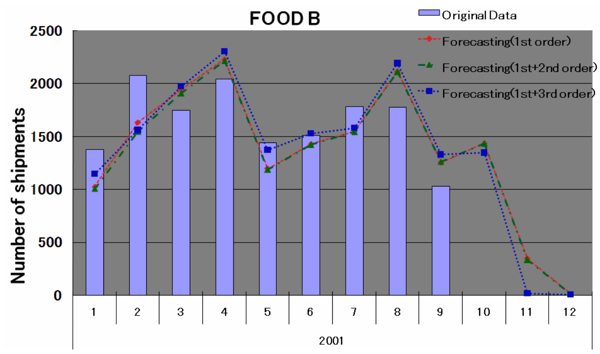

Figure 5-8. Forecasting Result of Food B.

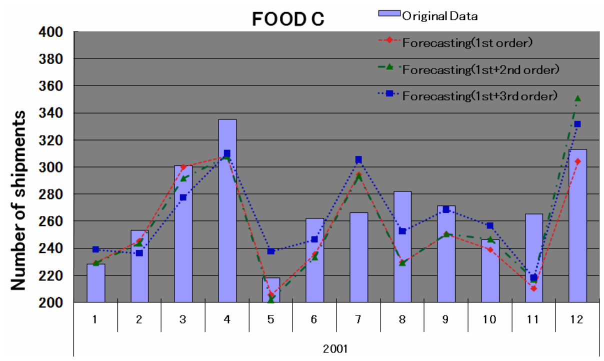

Figure 5-9. Forecasting Result of Food C.

Table 5-8. Variance of Forecasting Error (Monthly Ratio is used).

	1st order	1st + 2nd order	1st + 3rd order
Food A	76,333.551	87,591.677	103,857.817
Food B	245,522.121	221,628.057	203,140.765
Food C	526.053	851.037	1,022.110

Table 5-9. Variance of Forecasting Error (Monthly Ratio is not used).

	1st order	1st + 2nd order	1st + 3rd order
Food A	153,213.802	153,418.972	170,306.541
Food B	254,043.900	252,638.491	251,830.969
Food C	2,983.489	2,753.866	3,235.376

5.6 Remarks

These time series have non-linear trend and trend by month. Applying only an ESM does not make good forecasting accuracy. Forecasting accuracy is apparently improved by introducing monthly ratio. The combination of linear and non-linear function in trend removing is also examined. In Food A and C, the 1st order trend removal has better forecasting accuracy than other methods. On the other hand, the 1st+3rd order trend removal has better forecasting accuracy in Food B. In the case of Food B, it had a non-linear trend and it was proved that our newly proposed non-linear trend removal was effective for these data. Much more cases should be examined hereafter. The 1st order trend removal has better forecasting accuracy as is in Food A and C. Thus, the 1st order trend removal method would be effective for the time series data which have stable characteristics. Comparing Double Exponential Method with this newly proposed method would be our future work.

6. CONCLUSION

Focusing on the idea that the equation of exponential smoothing method(ESM) was equivalent to (1, 1) order ARMA model equation, new method of estimation of smoothing constant in exponential smoothing method was proposed before by us which satisfied minimum variance of forecasting error. Generally, smoothing constant was selected arbitrary. But in this paper, we utilized above stated theoretical solution. Firstly, we made estimation of ARMA model parameter and then estimated smoothing constants. Thus theoretical solution was derived in a simple way and it might be utilized in various fields.

Mere application of ESM does not make good forecasting accuracy for the time series which has non-linear trend and/or trend by month. A new method to cope with this issue is required. Therefore, utilizing above stated method, revised forecasting method is proposed in this paper to improve forecasting accuracy. An approach to this method was executed in the following method. Trend removal by a linear function was applied to the original shipping data of consumer goods. The combination of linear and non-linear function was also introduced in trend removing. For the comparison, monthly trend was removed after that. Theoretical solution of smoothing constant of ESM was calculated for both of the monthly trend removing data and the non monthly trend removing data. Then forecasting was executed on these data. The new method showed that it was useful especially for the time series that had stable characteristics and had rather strong seasonal trend and also the case that had non-linear trend. The effectiveness of this method should be examined in various cases.

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