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Toward the Application of a Critical-Chain-Project-Management-based Framework on Max-plus Linear Systems

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Abstract. We focus on discrete event systems with a structure of parallel processing, synchronization, and no-concurrency. We use max-plus algebra, which is an effective approach for controller design for this type of system, for modeling and formulation. Since a typical feature of this type of system is that the initial schedule is frequently changed due to unpredictable disturbances, we use a simple model and numerical examples to examine the possibility of applying the concepts of the feeding buffer and the project buffer of critical chain project management (CCPM) on max-plus linear discrete event systems in order to control the occurrence of an undesirable state change. The application of a CCPM-based framework on a max-plus linear discrete event system was proven to be effective.

Keywords: Max-plus Linear Systems, Critical Chain Project Management, Discrete Event Systems, State-space Representation, Scheduling

1. INTRODUCTION

We focus on discrete event systems with a structure of parallel processing, synchronization, and non-concurrency. Typical examples of this type of system include a class of manufacturing systems, transportation systems, and project management. The behavior of this type of system can be described with max-plus algebra (Cohen *et al.*, 1989, Heidergott *et al.*, 2006), which is a subclass of dioid algebra (Baccelli *et al.*, 1992). A typical feature of this type of system is that the initial schedule is frequently changed due to unpredictable disturbances.

In this context, we examine a method of controlling the occurrence of an undesirable state change in these systems. This state change indicates a significant change in tasks from an initial schedule. In general, assigning buffers in the systems and monitoring and controlling the tasks in a wider range are effective for controlling such changes.

Focusing on max-plus linear discrete event systems, several studies have considered the uncertainty of the execution time of the task (Heidergott, 2006). However, if the relevant parameter contains stochastic variations, there is a strong nonlinearity. Thus, it is difficult to deal with large-scale problems.

On the other hand, the method based on Critical Chain Project Management (CCPM) (e.g., Lawrence, P. L., 2005 and reference therein) has an advantage in that

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it can handle easily large-scale problems. The CCPM method has been found to be an effective tool to protect projects from delays. The CCPM method is an outgrowth of the theory of constraints (TOC), developed by Goldratt (1990), for scheduling and management of manufacturing. In the CCPM method, an empirical value is used to obtain an estimate of the process duration for each process. Moreover, the CCPM method provides a method for determining locations at which time buffers should be inserted in order to prevent unplanned delays in completing the project. This is because the method does not consider the change in execution points of individual tasks, and a buffer is incorporated into the cluster of tasks.

Therefore, using a simple model and a numerical example, we examine the possibility of applying the concept of the feeding and project buffers in CCPM to max-plus linear discrete event systems which are based on Goto and Masuda (2008).

The remainder of the present paper is organized as follows. In Section 2, we present an overview of the max-plus algebra and the max-plus linear system. In Section 3, we briefly review the concept of critical chain project management. In Section 4, a simple model and a numerical example are presented. In Section 5, we summarize the results and present concluding remarks.

2. MAX-PLUS LINEAR SYSTEM

In this section, we briefly review the max-plus algebra and the max-plus linear discrete event systems, both of which are fundamental to the present study.

2.1 Max-plus Algebra

Max-plus algebra is an algebraic system that is suitable for describing a certain class of discrete event systems. In a field $\mathbf{D} = \mathbf{R} \cup \{-\infty\}$, where $\mathbf{R}$ is a real field, the operators for addition and multiplication are defined as:

$$x \oplus y = \max(x,\ y), \quad x \otimes y = x + y. \tag{1}$$

The symbol $\otimes$ corresponds to multiplication in conventional algebra, and we often omit this notification when no confusion is likely to occur. For example, we simply write xy as the simplified expression of $x \otimes y$. The commutative, associative, and distributive properties hold for this operation. By definition, the unit elements for these operators are given by $\varepsilon(\equiv -\infty)$ and $e(\equiv 0)$, respectively. The following relationships are satisfied for an arbitrary $x \in \mathbf{D}$.

$$x \oplus \varepsilon = \varepsilon \oplus x = x\ , \quad x \otimes e = e \otimes x = x\ . \tag{2}$$

Furthermore, the following two operators are defined for subsequent discussions:

$$x \wedge y = mm(x,\ y), \ x \setminus y = -x + y. \tag{3}$$

We denote the unit element of $\wedge$ by $\bar{\varepsilon}(\equiv +\infty)$. An operator for the powers of real numbers is introduced as:

$$x^{\otimes \alpha} = \alpha \times x, \quad \text{for} \quad \alpha \in \mathbf{R} \tag{4}$$

If $m \le n$, then the operators for multiple numbers are:

$$\bigoplus_{k=m}^{n} x_k = x_m \oplus x_{m+1} \oplus \cdots \oplus x_n = \max(x_m,\ x_{m+1},\ \cdots,\ x_n), \tag{5}$$

$$\bigwedge_{k=m}^{n} x_k = x_m \wedge x_{m+1} \wedge \cdots \wedge x_n = \min(x_m,\ x_{m+1},\ \cdots,\ x_n). \tag{6}$$

For matrices $\mathbf{X} \in \mathbf{D}^{m \times n}$, $[\mathbf{X}]_{ij}$ express the $(i,\ j)$-th element of $\mathbf{X}$, and $\mathbf{X}^T$ is the transpose matrix of $\mathbf{X}$. For $\mathbf{X},\ \mathbf{Y} \in \mathbf{D}^{m \times n}$,

$$[\mathbf{X} \oplus \mathbf{Y}]_{ij} = [\mathbf{X}]_{ij} \oplus [\mathbf{Y}]_{ij} = \max([\mathbf{X}]_{ij},\ [\mathbf{Y}]_{ij}), \tag{7}$$

$$[\mathbf{X} \wedge \mathbf{Y}]_{ij} = [\mathbf{X}]_{ij} \wedge [\mathbf{Y}]_{ij} = \min([\mathbf{X}]_{ij},\ [\mathbf{Y}]_{ij}). \tag{8}$$

If $\mathbf{X} \in \mathbf{D}^{m \times l}$, $\mathbf{Y} \in \mathbf{D}^{l \times p}$,

$$[\mathbf{X} \otimes \mathbf{Y}]_{ij} = \bigoplus_{k=1}^{l}([\mathbf{X}]_{ik} \otimes [\mathbf{Y}]_{kj}) = \max_{k=1,\ \cdots,\ l}([\mathbf{X}]_{ik} + [\mathbf{Y}]_{kj}), \tag{9}$$

$$[\mathbf{X} \odot \mathbf{Y}]_{ij} = \bigwedge_{k=1}^{l}([\mathbf{X}]_{ik} \setminus [\mathbf{Y}]_{kj}) = \max_{k=1,\ \cdots,\ l}(-[\mathbf{X}]_{ik} + [\mathbf{Y}]_{kj}), \tag{10}$$

where the priorities of operators $\otimes$, $\setminus$, and $\odot$ are higher than those of operators $\oplus$ and $\wedge$. Unit elements for matrices are denoted as ε_{mn}, which is a matrix in which all of elements are ε in $\varepsilon_{mn} \in D^{m \times n}$, and $\mathbf{e}_m$ is a matrix in which the diagonal elements are e and the off-diagonal elements are ε in $\mathbf{e}_m \in D^{m \times n}$. In $x,\ y \in D^m$, if $[x]_i \le [y]_i$ holds for all $i (1 \le i \le m)$, then we simply write $x \le y$.

The operators introduced above have other interesting and attractive properties that are not used in the present paper. See (Heidergott *et al.*, 2006) or (Baccelli *et al.*, 1992) for details.

2.2 Max-plus Linear System

The max-plus linear discrete event system is defined as a system for which the behavior can be described in linear form in max-plus algebra. The max-plus linear discrete event system is similar to the state-space equations in modern control theory:

$$\mathbf{x}(k) = \mathbf{A}(k)\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k), \tag{11}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k), \tag{12}$$

where k is the event counter that represents the number of event occurrences from the initial state. In addition, $\mathbf{x}(k) \in \boldsymbol{D}^n$, $\mathbf{u}(k) \in \boldsymbol{D}^p$, and $\mathbf{y}(k) \in \boldsymbol{D}^q$ are the states, and the input and output variables, respectively, and n, p, and q are the corresponding dimensions. $\mathbf{A} \in \boldsymbol{D}^{m \times n}$, $\mathbf{B} \in \boldsymbol{D}^{m \times p}$ and $\mathbf{C} \in \boldsymbol{D}^{q \times n}$ are the system input and output matrices, respectively.

Let us consider the simple production system shown in Figure 1 as an example. In this system, Process 1 receives the raw material from the input lane, and Processes 2 and 3 manufacture the parts simultaneously. Process 4 processes the parts received from Processes 2 and 3, and then sends the resulting part to the output lane. The processing times in Processes 1 though 4 are denoted as d_1 through d_4, respectively. Each process is assumed to have the following specific constraints:

- While the machines are in operation, they cannot initiate processing for subsequent parts.
- Processes 2 through 4, which have precedence constraints, cannot begin processing until they have received the manufactured parts from the preceding processes.
- Process 1, which has an external input, cannot begin processing until it receives the corresponding materials.
- If the machine is empty, processing starts as soon as all required materials from the preceding processes and external inputs become readily available.

For the k-th batch, assume that $u(k)$, $\mathbf{x}(k)$, and $y(k)$ represent the feeding, processing start, and processing finish times, respectively. Then, we have the following relations.

$$x_1(k) = \max\{u(k),\ x_1(k-1) + d_1\}, \tag{13}$$

$$x_2(k) = \max\{x_2(k-1) + d_2,\ x_1(k) + d_1\}, \tag{14}$$

$$x_3(k) = \max\{x_3(k-1) + d_3,\ x_1(k) + d_1\}, \tag{15}$$

$$x_4(k) = \max\{x_4(k-1) + d_4,\ x_2(k) + d_2,\ x_3(k) + d_3\}, \tag{16}$$

$$y(k) = x_4(k) + d_4 \tag{17}$$

Equations (13) through (17) can be expressed in the form of Eqs. (11) and (12), where

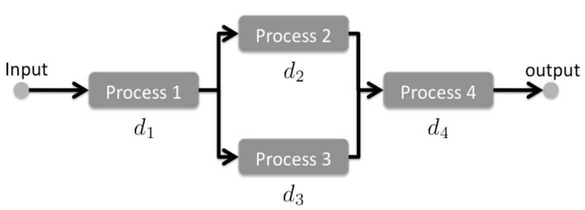

Figure 1. Manufacturing sequence of a simple production system.

$$\mathbf{A} = \begin{pmatrix} d_1 & \varepsilon & \varepsilon & \varepsilon \\ d_1^2 & d_2 & \varepsilon & \varepsilon \\ d_1^2 & \varepsilon & d_3 & \varepsilon \\ a_{41} & d_2^2 & d_3^2 & d_4 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} e \\ d_1 \\ d_1 \\ b_4 \end{pmatrix},\ \mathbf{C} = \begin{pmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ d_4 \end{pmatrix}^T, \tag{18}$$

$$a_{41} = d_1^2(d_2 \oplus d_3), \quad b_4 = d_1(d_2 \oplus d_3). \tag{19}$$

Equations (14) through (16) must be transformed into equations that do not contain the term $x_i(k)$. This implies that the equations are expressed in the form of Eq. (11). There have been few discussions about the domain, in which the equations are described as a general form of the system matrix and input/output matrices. Therefore, in the next subsection, we review a general formulation of the max-plus linear equations for systems with precedence constraints or synchronizations.

2.3 Max-plus Linear Representation

We briefly review the process for deriving the max-plus linear representation for a certain class of discrete event systems that were developed by Goto and Masuda (2008). We assume that the relevant constraints are imposed on the focused system in the following manner:

- The number of processes is n, the number of external inputs is p, and the number of external outputs is q.
- All processes are used only once for a single batch.
- The subsequent batch cannot start processing while the current batch is being processed.
- Processes that have precedence constraints cannot start processing until they have received all required parts from the preceding processes.
- For processes that have external inputs, processing cannot start until all required materials have arrived.
- Processing starts as soon as all of the conditions stipulated above are satisfied.

For the k-th job in process $i (1 \le i \le n)$, let $[\mathbf{x}(k)]_i$ and (k) (≥ 0) be the starting time and the processing time, respectively, for each process, and let the initial condition be $\mathbf{x}(0) = \boldsymbol{\varepsilon}_{n1}$. For external input $j (1 \le j \le p)$, $[\mathbf{u}(k)]_j$ represents the material feeding time. For external output $r (1 \le r \le q)$, $[\mathbf{y}(k)]_r$ denotes the output time. Matrices $\mathbf{A}_k$, $\mathbf{F}_k$, $\mathbf{B}^0$ and $\mathbf{C}_k$ are introduced to represent the structures of systems as follows:

$$[\mathbf{A}_k]_{ij} = \begin{cases} d_i(k) & : \text{if } i = j. \\ \varepsilon & : \text{otherwise}. \end{cases} \tag{20}$$

$$[\mathbf{F}_k]_{ij} = \begin{cases} d_j(k) : \text{if process } i \text{ has a preceding process } j. \\ \varepsilon : \text{if process } i \text{ does not have a preceding process } j. \end{cases} \quad (21)$$

$$[\mathbf{B}^0]_{ij} = \begin{cases} e : \text{if process } i \text{ has an external input } j. \\ \varepsilon : \text{if process } i \text{ does not have an expernal input } j. \end{cases} \quad (22)$$

$$[\mathbf{C}_k]_{ij} = \begin{cases} d_j(k) : \text{if process } j \text{ has an external output } i. \\ \varepsilon : \text{if process } j \text{ does not have an external output } i. \end{cases} \quad (23)$$

where $\mathbf{F}_k$ is referred to as the adjacency matrix.

For the k-th job in process i $(1 \le i \le n)$, $[\mathbf{A}_{k-1}\mathbf{x}(k-1)]_i$ is the finishing time, $[\mathbf{F}_k\mathbf{x}(k)]_i$ is the latest time among finishing times in the preceding processes, and $[\mathbf{B}^0\mathbf{u}(k)]_i$ is equal to the latest feeding time from external inputs. Furthermore, $[\mathbf{C}_k\mathbf{x}(k)]_i$ is the latest time among the finishing times in the processes attached to the corresponding output. The earliest starting time is defined as the minimum value on which the corresponding process can begin processing immediately

Using the above discussions, the earliest starting times of any of the processes are given by Goto and Masuda (2008), as follows:

$$\mathbf{x}_E(k) = \mathbf{F}_k^*[\mathbf{A}_{k-1}\mathbf{x}(k-1) \oplus \mathbf{B}^0\mathbf{u}(k)], \quad (24)$$

where

$$\mathbf{F}_k^* = \mathbf{e}_n \oplus \mathbf{F}_k \oplus \mathbf{F}_k^2 \oplus \cdots \oplus \mathbf{F}_k^{l-1}, \quad (25)$$

$$\mathbf{F}_k^l = \varepsilon_{nn} \quad (26)$$

An instance l $(1 \le l \le n)$ depends on the precedence-relations of the systems. The corresponding output times are given by:

$$\mathbf{y}_E(k) = \mathbf{C}_k\mathbf{x}_E(k). \quad (27)$$

Furthermore, the latest starting time is defined as the maximum value for which the same output time by the earliest time is accomplished. The latest starting times of any of the processes are given by Goto and Masuda (2008), as follows:

$$\mathbf{x}_L(k) = \left(\mathbf{A}_k\mathbf{F}_k^*\right)^T \odot \mathbf{x}(k+1) \wedge \left(\mathbf{C}_k\mathbf{F}_k^*\right)^T \odot \mathbf{y}(k). \quad (28)$$

The latest feeding times are also given by:

$$\mathbf{u}_L(k) = (\mathbf{C}_k\mathbf{F}_k^*\mathbf{B}^0)^T \odot \mathbf{y}(k). \quad (29)$$

A critical path is defined as the processes for which the total floats are zero. Moreover, the total float is defined as the total sum of the float times of the corresponding processes. The total float can also be described as the difference between two primary times, one of which is the minimum value among the latest starting times of the succeeding processes, by which the output time is unchanged, and the other of which is the completion time in the corresponding process caused by the earliest starting time. The total floats $\mathbf{w}(k)$ regarding all processes are obtained as:

$$\mathbf{w}(k) = \mathbf{x}_L(k) - \mathbf{x}_E(k). \quad (30)$$

The critical path is determined by a collection of numbers α that satisfy:

$$\text{Critical paths : } \{\alpha \mid [\mathbf{w}(k)]_\alpha = 0\}. \quad (31)$$

3. CRITICAL CHAIN PROJECT MANAGEMENT

We briefly review the concept of the critical chain project management (CCPM) method. Projects often exceed their initial planned schedule, usually as the result of unforeseen uncertainties in the external factors. In order to resolve this dilemma, the CCPM method has been considered in various studies (e.g., Lawrence, P. L., 2005 and reference therein). Critical chain project management addresses several shortcomings of the Program Evaluation and Review Technique (PERT), which is the most widely used tool for project management. The PERT is based on identifying a critical path, which is the longest path of linked processes in the entire project. Focusing only on the longest path of processes may result in several problems, such as multitasking. Critical chain project management instead asserts that, in addition to process dependencies, a good project management should address resource constraints only if they are absolutely required. Critical chain project management provides a method for determining locations at which time buffers should be inserted in order to prevent unplanned delays in the completion of the project. Using the PERT, each process in the project consists of a set of four times, namely, the earliest start, the earliest output, the latest start, and the latest output times. Since these times are communicated to everyone involved in the project, they can be closely monitored. The difference between the earliest and latest start times is equivalent to the slack. Processes on the critical path do not have any slack time, and significant attention should be paid on it. In order to estimate the process duration, optimistic estimates, which include significant safety margins, are included in order to ensure completion in the specified time frame. This is often referred to as the 90% estimate.

Instead, CCPM uses an empirical value to obtain an estimate of the Aggressive But Possible (ABP) time for each process as well as for the process duration. In the present paper, we use $\mathrm{ABP} = \mathrm{HP}^{\otimes 1/3}$, where Highly Possible (HP) is the time to completion with a probability of 90%.

The next step is to determine a buffer to encapsulate the uncertainty of task durations. This buffer is referred to as the project buffer, and it absorbs variations in the critical path. A project buffer is embedded between the final process on the critical path and the external output. The position of the project buffer is determined using the output matrix $[\mathbf{C}_k]_{ij}$. We consider the element $[\mathbf{C}_k]_{i\alpha}$, in which α is the collection of processes on the critical path. If the element $[\mathbf{C}_k]_{i\alpha}$ has a finite value, by the definition of matrix $[\mathbf{C}_k]_{ij}$, process α has an external output i. Then, we embed a project buffer after process α. The size of the project buffer is estimated as follows:

$$\mathrm{PB} = \mathbf{y}_E^{\otimes 1/3}. \tag{32}$$

This method is based on the cut and paste method or 50% of the chain (Tukel, O. I. *et al.*, 2006, Lawrence, P. L., 2005).

We assume that the original processing time of each process d_i is equal to the HP time. Then, we reduce the processing time of each process by the ABP time, and let the buffer size equal one half the duration of the longest path feeding into the buffer:

$$\text{Buffer size} = (\mathrm{HP} - \mathrm{ABP})^{\otimes 1/2} = (\mathrm{HP})^{1/3} \tag{33}$$

Then, the power 1/3 appears in Eq. (32). The two major advantages of this method are that it is simple to apply, and that it usually provides a sufficiently large buffer.

In order to protect the critical path from the delay of the non-critical path, a feeding buffer is embedded. A feeding buffer is inserted before the process that joins the critical path but is not on the critical path. In order to determine the position of the feeding buffer, we define a new vector, as follows:

$$[\mathbf{v}]_i = \begin{cases} \varepsilon : i \in \alpha, \\ e : i \in \beta, \end{cases} \tag{34}$$

where β is the collection of processes on the non-critical path. Moreover, we introduce the following new vector:

$$\mathbf{k} \equiv \mathbf{F}_k^* \otimes \mathbf{v}. \tag{35}$$

If $[\mathbf{k}]_j \neq \varepsilon$ and $j \in \alpha$, then we embed a feeding buffer between processes β and j. The size of the feeding buffer is estimated as:

$$\mathrm{FB} = \sum_{\beta} \frac{1}{2} \times \left(\mathrm{HP}_\beta - \mathrm{APB}_\beta\right) = \sum_{\beta} \mathrm{HP}_\beta^{\otimes 1/3}. \tag{36}$$

Equation (36) indicates that the size of the feeding buffer is defined as 1/3 the sum of the processing times (HP times) of the processes on the non-critical path between the juncture and the joint.

Note that the project and feeding buffers are time buffers and not inventory buffers. That is, variation is protected by capacity, rather than inventory.

Finally, the critical path is monitored by closely following the rate at which the project buffer is consumed. Thus, instead of reporting the due dates for each process, and thereby promoting the student syndrome, CCPM suggests simply informing the project team on a regular basis as to whether the consumption rate of the project buffer is under control.

4. NUMERICAL EXAMPLE

Using a simple model and a numerical example, we examine the possibility of applying the concepts of the feeding and the project buffers in CCPM on Max-Plus linear discrete event systems.

4.1 Simple Max-plus Linear System

We apply the method introduced in Section 2.3 to calculate the earliest/latest starting time and finding the critical path. Figure 2 shows a simple production system with one-input, one-output, and six processes. Considering these structures, the matrices defined in Eqs. (20) through (23) are set as follows:

$$\mathbf{A}_{k-1} = \mathrm{diag}(3,\ 9,\ 3,\ 15,\ 6,\ 6),\ \mathbf{F} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 9 & 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 9 & 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 15 & 6 & \varepsilon \end{pmatrix}, \tag{37}$$

$$\mathbf{B}^0 = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}^T,\ \mathbf{C} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 6 \end{pmatrix}. \tag{38}$$

By calculating $\mathbf{F}_k^3$, we obtain $\mathbf{F}_k^4 = \boldsymbol{\varepsilon}_{44}$ and

$$\mathbf{F}^* = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\ 12 & 9 & 3 & e & \varepsilon & \varepsilon \\ 12 & 9 & 3 & \varepsilon & e & \varepsilon \\ 27 & 24 & 18 & 15 & 6 & e \end{pmatrix}, \tag{39}$$

Assuming the initial condition as $\mathbf{x}(k) = \varepsilon_{61}$ and the input times from the external input as $\mathbf{u} = 0$, the earliest starting time $\mathbf{x}_E$ and the output time $\mathbf{y}_E$ are calculated using Eq. (24) and Eq. (27), as follows:

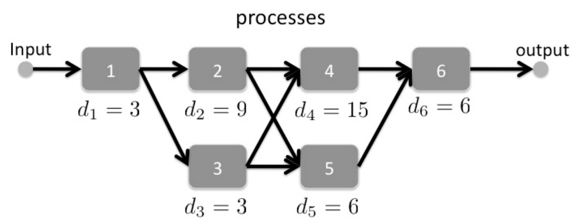

Figure 2. A simple production system.

$$\begin{aligned}\mathbf{x}_E &= \mathbf{F}^*[\mathbf{A}_{k-1}\ \mathbf{x}(k-1) \oplus \mathbf{B}^0\mathbf{u}] \\ &= \mathbf{F}^*\mathbf{B}^0\mathbf{u} = \begin{pmatrix} e & 3 & 3 & 12 & 12 & 27 \end{pmatrix}^T,\end{aligned} \tag{40}$$

$$\mathbf{y}_E = \mathbf{C}\mathbf{x}_E = 33. \tag{41}$$

From Eqs. (28) through (30), the latest starting time $\mathbf{x}_L$, feeding time $\mathbf{u}_L$, and total float $\mathbf{w}$ are obtained as:

$$\mathbf{x}_L = (\mathbf{C}\mathbf{F}^*)^T \odot \mathbf{y} = \begin{pmatrix} e & 3 & 9 & 12 & 21 & 27 \end{pmatrix}^T, \tag{42}$$

$$\mathbf{u}_L = 0, \tag{43}$$

$$\mathbf{w} = \mathbf{x}_L - \mathbf{x}_E = \begin{pmatrix} e & e & 6 & e & 9 & e \end{pmatrix}^T. \tag{44}$$

Therefore, the critical path can be identified as $\alpha = \{1, 2, 4, 6\}$

4.2 Application of a CCPM-based Framework on a Simple Max-plus Linear System

We apply the concepts of the feeding and project buffers of CCPM to a simple max-plus linear system shown in the previous subsection. We assume that each processing time is equivalent to the ABP time. Then, we obtain the respective processing times as (3, 9, 3, 15, 6, 6)$\times$1/3. From the discussion in Section 4.1, the critical path is {1, 2, 4, 6}. Furthermore, we embed a project buffer after process 6, because the value of the output matrix $[\mathbf{C}]_{16}$ in Eq. (38) is finite, and process 6 is on the critical path. We define a new vector $\mathbf{v} \equiv (\varepsilon, \varepsilon, e, \varepsilon, e, \varepsilon)^T$. Then, we can obtain $\mathbf{k} = \mathbf{F}^* \otimes \mathbf{v} = (\varepsilon, \varepsilon, e, 3, 3, 18)^T$. Since $[\mathbf{k}]_4 \neq \varepsilon$ and process 4 is on the critical path, we embed a feeding buffer after process 3. Moreover, since $[\mathbf{k}]_6 \neq \varepsilon$ and process 6 is on the critical path, we embed a feeding buffer after process 5. The size of the project and feeding buffers are estimated by Eq. (32) and Eq. (36), respectively. We then obtain PB=11 and FB_3=1 FB_5=2. Figure 3 shows a production system in which the concepts of the feeding and the project buffers of CCPM are applied to the system shown in Figure 2. Considering these structures, the matrices defined in Eqs. (20) through (23) are set as follows:

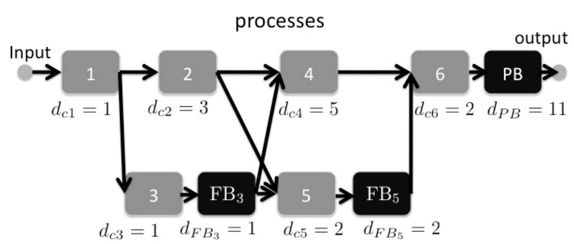

Figure 3. Production system shown in Figure 2 after application of CCPM.

$$\mathbf{A}_{ck-1} = \mathrm{diag}(1, 3, 1, 5, 2, 2, 1, 2, 11), \tag{45}$$

$$\mathbf{F}_c = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \mathbf{B}_c^0 = \begin{pmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}, \mathbf{C}_c^0 = \begin{pmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ 11 \end{pmatrix}^T, \tag{46}$$

where the subscript c indicates that CCPM is applied. By calculating $\mathbf{F}_{ck}^2 \cdots \mathbf{F}_{ck}^6$, we obtain $\mathbf{F}_{ck}^7 = \varepsilon_{99}$ and

$$\mathbf{F}_c^* = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 4 & 3 & 2 & e & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ 4 & 3 & 2 & \varepsilon & e & \varepsilon & 1 & \varepsilon & \varepsilon \\ 9 & 8 & 7 & 5 & 4 & e & 6 & 2 & \varepsilon \\ 2 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ 6 & 5 & 4 & \varepsilon & 2 & \varepsilon & 3 & e & \varepsilon \\ 11 & 10 & 9 & 7 & 6 & 2 & 8 & 4 & e \end{pmatrix}. \tag{47}$$

Assuming that the initial condition is $\mathbf{x}_c(k) = \varepsilon_{61}$ and that the input times from the external input is $\mathbf{u}_c = 0$, the earliest starting time $\mathbf{x}_{cE}$ and the output time $\mathbf{y}_{cE}$ can be calculated using Eq. (24) and Eq. (27), as follows:

$$\begin{aligned}\mathbf{x}_{cE} &= \mathbf{F}_c^*[\mathbf{A}_{ck-1}\mathbf{x}_c(k-1) \oplus \mathbf{B}_c^0\mathbf{u}_c] \\ &= \mathbf{F}_c^*\mathbf{B}_c^0\mathbf{u}_c = \begin{pmatrix} e & 1 & 1 & 4 & 4 & 9 & 2 & 6 & 11 \end{pmatrix}^T,\end{aligned} \tag{48}$$

$$\mathbf{y}_{cE} = \mathbf{C}_c\mathbf{x}_{cE} = 22. \tag{49}$$

From Eq. (28) and Eq. (30), the latest starting time $\mathbf{x}_{cL}$ and the total float $\mathbf{w}_c$ are obtained as:

$$\mathbf{x}_{cL} = (\mathbf{C}\mathbf{F}^*)^T \odot \mathbf{y} = \begin{pmatrix} e & 1 & 2 & 4 & 5 & 9 & 3 & 7 & 11 \end{pmatrix}^T, \tag{50}$$

$$\begin{aligned}\mathbf{w}_c &= \mathbf{x}_{cL} - \mathbf{x}_{cE} \\ &= \begin{pmatrix} e & e & 1 & e & 1 & e & 1 & 1 & e \end{pmatrix}^T.\end{aligned} \tag{51}$$

Therefore, the critical path can be identified as $\alpha = \{1, 2, 4, 6, \mathrm{PB}\}$. It can be confirmed that by applying CCPM, the earliest output time is reduced to 2/3 the original time. This indicates that the application of the concepts of CCPM on max-plus linear discrete event systems is effective.

5. CONCLUDING REMARKS

We examined a method of controlling the occurrence of an undesirable state change of the systems. In general, a method of assigning buffers and monitoring and controlling the tasks in a wider range are effective for controlling such changes. Focusing on max-plus linear discrete event systems, if the relevant parameter contains stochastic variations, there is a strong nonlinearity in the state of the systems. Thus, it is difficult to handle large-scale problems. In contrast, the CCPM-based method has an advantage in that such problems can be handled easily because it does not consider the change in the execution times of individual tasks. Instead, a buffer is assigned to the cluster of tasks.

In the present paper, we examined the possibility of applying the concept of the feeding and project buffers of CCPM on max-plus linear discrete event systems, using a simple model and numerical examples. We confirmed that, by applying CCPM, the earliest output time is reduced to 2/3 the original time. This indicates that application of the concepts of CCPM on max-plus linear discrete event systems was effective.

For the more practical operation, it is necessary to take account of competition for resources and online management of buffer penetration that are discussed by the framework of CCPM. These extensions and additional details of a general formulation for applying the CCPM-based framework on max-plus linear discrete event systems should be discussed in the future.

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