J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 16, Number 2 (May 2009), Pages 243–254

SPATIALLY HOMOGENEOUS GLOBAL PRICE DYNAMICS ON A CHAIN OF LOCAL MARKETS

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ABSTRACT. The main purpose of this paper is to use the methods of Lattice Dynamical System to establish a global model, which extends the Walrasian evolutionary cobweb model in an independent single local market to the global market evolution over an infinite chain of many local markets interacting each other through a diffusion of prices between them.

For brevity of the model, we assume linear decreasing demands and quadratic supplies with naive predictors, and investigate the spatially homogeneous global price dynamics and show that the dynamics is topologically conjugate to that of well-known logistic map and hence undergoes a period-doubling bifurcation route to chaos as a parameter varies through a critical value.

1. INTRODUCTION

In this paper, we try to establish a global model for the global market which consists of many local markets located continually along a long chain by applying the theory and methods of Lattice Dynamical Systems (LDS) and investigate the dynamics of the spatially homogeneous solutions of the resulting global model. We assume that the local markets are located along an infinitely long 1D chain and there is an interaction between them through diffusion of prices of neighboring local markets. The actual examples of this kind of model can be easily found, for example, in the local fish markets distributed along a long coastline, say, that of the Korean peninsula.

Although the number of local fish markets along the coastline is finite, it is more convenient to assume that it is infinite in the case of non-circular coastline, since we want the rightmost or leftmost site to still have an interaction between neighboring sites and also it is a little awkward to impose a boundary condition to the rightmost

Received by the editors January 11, 2009. Accepted May 13, 2009.
2000 *Mathematics Subject Classification.* 22E46, 53C35, 57S20.
Key words and phrases. Cobweb model, lattice dynamical system, Hénon map.

or leftmost site. The local fish markets located in the inland area, of course, may be assumed to be disregarded in this model.

However, in the case of circular coastline, for example, the Australian coastline, it will be reasonable to assume that the number of local markets is finite and they satisfy a periodic boundary condition. Hence, in the case of infinitely long chain, we need only to assume that the prices are positive and finite at $\pm\infty$ for all times.

As a matter of fact, the price of each local market is affected by the prices of the neighboring local markets. In our everyday life, we frequently notice that if the prices of some good in the neighboring sites are higher (or lower) than the price at our local site, then the price of the good at our site also gradually gets higher (or lower) until those prices make balance among the sites.

On the other hand, the endogenous price dynamics what is called Walrasian dynamics also exists internally at each local markets. It depends on the consumer's demand and the producer's supply which is based on some mechanism of expectations for the tomorrow's price of a good and some optimization process to maximize profit or minimize cost for the production of the good. On the problems of various methods of expectations, there have been existing a lot of researches, say, backward-looking expectations ([17]), adaptive learning methods ([17]), fading memory learning ([2]), and so on. About the local market Walrasian dynamics of cobweb type, there also have been a great deal of researches, say, Brock and Hommes ([6], [7]), Goeree and Hommes ([16]), Böhm and Chiarella ([5]), and Chiarella et al. ([11], [12], [13]) and so on.

As is noticed, most of the above researches are confined in a single independent local market, say, agricultural market or financial market, and are concentrated on developing the appropriate methods for the prediction of the price or on describing the time evolution of the local market dynamics which is resulted from the interaction between endogenous variables, say, price and fractions of agents with heterogeneous beliefs ([6]), old and new generations in stock markets ([4]), fundamentalists and chartists ([18]) and so on.

In fact, there have been some researches which dealt with several local markets and considered the resulting global dynamics. For instance, Beckmann and Puu ([3]) have considered the interaction between a finite number of local markets which are located on a plane and studied the continuous commodity flow fields among them by using the physical concepts such as divergence, gradient etc in vector analysis. Hence, this model is different from our lattice dynamical system. Choudhary and

Orszag([14]) have considered a circular lattice model with the effects of the nearest neighbors. In the area of physics, we can find easily some chain or lattice models which are discrete versions of continuous time models, e.g., a discrete version of reaction-diffusion equations ([1]). There is another similar model called Ising model ([20]). This Ising model was invented to describe the phase transition in magnetism or in gas-liquid and each site can have two values 1 or 0, and so can be viewed as a cellular automation which is a special case of LDS with discrete state values, and hence, needless to say, cannot be regarded as economic models with continuous state values. Thus, it seems to us that our paper considering the global market dynamics on a chain of interacting local markets might be one of a few attempts in the field of economic dynamics.

Now, we briefly mention about the contents of this paper. In Section 2, we briefly describe the Cobweb model for the local market dynamics. In Section 3, we introduce the LDS model for the global market dynamics, and under the assumption of homogeneous beliefs of producers, we establish our global market model by connecting the LDS model and Cobweb model. In Section 4, we investigate the global market dynamics by examining the spatially homogeneous solutions. The proofs of the theorems will be given in the Appendix. In Section 5, we make a concluding remark.

2. The Local Market Dynamics – the Cobweb Model

The Cobweb model for the local market dynamics has been well introduced and studied by many researchers (e.g., [6], [7], [8], [14]). The Cobweb model describes the dynamics of equilibrium prices in a single independent local market for a non–storable good, that takes one time period to produce, so that producers must form price expectations one period ahead using the past history of prices.

Let $p_n^e = H(\mathbf{P}_{n-1})$, where p_n^e is the expected price by the producers at time n and $\mathbf{P}_{n-1} = (p_{n-1}, p_{n-2}, \cdots, p_{n-L})$ is a vector of past prices of lag-length L and $H(\cdot) : \mathbf{R}^L \to \mathbf{R}$ is a real-valued function, so called *predictor*. Let p_n be the actual market price at time n by the consumers, and let $D(p_n)$ be the consumer demand and $S(p_n^e)$ be the producer supply for the goods. The supply $S(p_n^e)$ is derived from producer's maximizing expected profit with a cost function $c(q)$, i.e.,

$$S(p_n^e) = arg \max_{q_n} \{p_n^e q_n - c(q_n)\}. \tag{2.1}$$

The demand function $D(\cdot)$ depends on the current market price p_n and is assumed to be strictly decreasing in the price p_n to ensure that its inverse D^{-1} is well-defined. The supply function $S(\cdot)$ depends on the expected price p_n^e and will be assumed to be quadratic in our paper. The intersection point p^* of the demand and supply curve such that $D(p^*) = S(p^*)$ is called the steady state equilibrium price.

If the beliefs of producers are homogeneous, i.e., all producers use the same predictor H, then the market equilibrium price dynamics in the cobweb model is described by

$$D(p_n) = S(H(\mathbf{P}_{n-1})), \quad \text{or,} \quad p_n = D^{-1}(S(H(\mathbf{P}_{n-1}))). \tag{2.2}$$

Thus, the actual equilibrium price dynamics in a local market depends on the demand D, the supply S, and the predictor H used by the producers.

3. THE GLOBAL MARKET DYNAMICS – THE LDS MODEL

Over the last decade, a new class of infinite dimensional dynamical systems, so called Lattice Dynamical Systems(LDS) have been introduced and studied by many researchers (e.g., [1], [9], [10]). These LDS's have been proved to be one of the most efficient tools to analyze space-time behaviors of the extended systems.

To begin with, we define the phase space (or state space) of the LDS. Suppose that at each site j of a d-dimensional lattice $\mathbf{Z}^d$, we have a finite dimensional local dynamical system which is defined by some map $f_j : M_j \longrightarrow M_j$, where M_j is a local phase space at the site j. For simplicity and applicability to our model, we will confine our attention to an infinite chain ($d = 1$) and the identical local map, i.e., $f_j = f, M_j = \mathbf{R}^1\, \forall j \in \mathbf{Z}$, where $\mathbf{R}^1$ is a 1-dimensional real Euclidean space with ordinary inner product $(\cdot,\cdot)$ and the norm $|\cdot| = \sqrt{(\cdot,\cdot)}$. Then we have an infinite dimensional dynamical system on a space

$$M = \prod_{j\in\mathbf{Z}} M_j = \{p = \{p_j\} | p_j \in \mathbf{R}, j \in \mathbf{Z}\} \tag{3.1}$$

where M is obviously a linear space with respect to componentwise addition and scalar multiplication. A point (or, a state) $p = \{p_j\} \in M$ can be thought of as a bi-infinite sequence of real numbers. To make the linear space M be a Hilbert space, we equip M with the inner product defined by

$$\langle p, q\rangle_\rho = \sum_{j\in\mathbf{Z}} \frac{(p_j, q_j)}{\rho^{|j|}} \quad \forall p, q \in M, \tag{3.2}$$

where $\rho > 1$ is some fixed number depending on the particular problem. Then the norm $\|\cdot\|_\rho$ is induced by

$$\|\cdot\|_\rho = \sqrt{\langle \cdot, \cdot \rangle_\rho}$$

and now we can define the phase space of our LDS by

$$B_\rho = \{p \in M | \, \|p\|_\rho < \infty\}. \tag{3.3}$$

Then it can be easily shown that B_ρ is a Hilbert space (e.g., [1]). Next, we define the evolution operator on B_ρ in the following.

Definition 3.1. Define the evolution operator $\Phi : B_\rho \to B_\rho$ by

$$(\Phi p)_j = F(\{p_j\}^s), \forall j \in \mathbf{Z}, \tag{3.4}$$

where $\{p_j\}^s = \{p_i | \, |i-j| \leq s, \; s \geq 1 \text{ integer}\}$ for each $j \in \mathbf{Z}$, i.e., $\{p_j\}^s$ is the set of values p_i at the site i which are within the distance of radius s from the site j, and $F : \mathbf{R}^{2s+1} \to \mathbf{R}$ is a differentiable map of class C^2 such that

$$\left| \frac{\partial F}{\partial p_i} \right| \leq K, \quad \left| \frac{\partial^2 F}{\partial p_i \partial p_k} \right| \leq K, \tag{3.5}$$

for any collection $\{p_j\}^s$ and some constant $K > 0$.

Then it is easy to verify that under the condition (3.5), $\Phi(B_\rho) \subset B_\rho$ and Φ is Lipschitz continuous with the constant $L = K(2s+1)^{\frac{3}{2}} \rho^{\frac{s}{2}}$ (e.g., [1]).

Definition 3.2. Given a state $p(n) = \{p_j(n)\}_{j=-\infty}^{\infty} \in B_\rho$ at the moment n, we can obtain via (3.4) the next state $p(n+1)$, that is,

$$\begin{aligned} p(n+1) &= \Phi(p(n)), \quad \text{or}, \\ p_j(n+1) &= (\Phi(p(n)))_j = F(\{p_j(n)\}^s). \end{aligned} \tag{3.6}$$

The dynamical system $(\Phi^n, B_\rho)_{n \in \mathbf{Z}^+}$ is called a *Lattice Dynamical System* (LDS).

Formula (3.6) implies that given a state $p(n) \in B_\rho$, we can calculate its next state $p(n+1)$, so we can obtain the forward orbit of the evolution operator Φ, i.e.,

$$p(0), p(1) = \Phi(p(0)), p(2) = \Phi(p(1)) = \Phi^2(p(0)), \cdots.$$

Before ending this section, let us consider several kinds of basic motions (or solutions) in the LDS (3.6).

Definition 3.3. (i) A state (or solution) $p(n) = \{p_j(n)\}$ for the LDS (3.6) is *spatially homogeneous* if $p_j(n) = \psi(n) \, \forall j \in \mathbf{Z}$, i.e., a spatially homogeneous solution $\{\psi(n)\}$ does not depend on the space coordinates j and so has the same value at each site j.

(ii) A solution $p(n) = \{p_j(n)\}$ is *static* (or *stationary, steady state,standing wave*) if $p_j(n) = \phi_j \; \forall n \in \mathbf{Z}^+$, i.e., a static solution $\{\phi_j\}$ does not depend on time n, and is standing there along the space coordinates j at all times n.
(iii) A solution $p(n) = \{p_j(n)\}$ is a *traveling wave with wave velocity* m/l if $p_j(n) = \xi(lj + mn)$, where $l > 0, m \in \mathbf{Z}$ and $(l, m) = 1$(i.e.,relatively prime). Here, the ratio m/l is called the *wave velocity* of the traveling wave.

For instance, suppose that the local system $f : M_j \to M_j$ has a fixed point p^*. Then the state $p = \{p_j\}, p_j = p^* \forall j \in \mathbf{Z}$ is a spatially homogeneous static solution, i.e., a fixed point of the evolution map Φ and also can be thought of as a travelling wave with arbitrary velocity.

Now, as our LDS model for the global market dynamics, we will take the following form:

$$
\begin{aligned}
p_j(n+1) &= (\Phi(p(n)))_j \\
&= (1-\alpha)p_j(n) + \alpha f(p_j(n)) + \varepsilon(p_{j-1}(n) - 2p_j(n) + p_{j+1}(n)),
\end{aligned}
\tag{3.7}
$$

where a solution $p_j(n)$, $j \in \mathbf{Z}$, $n \in \mathbf{Z}^+$ represents the price of a good at the site (or local market)j at the time n, and $f : \mathbf{R} \to \mathbf{R}$ is a Walrasian local market price dynamics at each site j, and $\alpha \in [0, 1]$ is a parameter denoting the weighted average between $p_j(n)$ and $f(p_j(n))$, and the parameter ε is a diffusion coefficient measuring the intensity of interaction between the neighboring local markets. Thus, in this global market model, the price $p_j(n+1)$ at site j and at time $n+1$ is determined by several factors, i.e., the previous price $p_j(n)$, the local market dynamics f, the weight $\alpha \in [0, 1]$ of the average between them, and the diffusion coefficient $\varepsilon > 0$. Notice that the parameter α plays a role of controlling each local market in such a way that if $\alpha = 1$ then $p_j(n+1)$ is determined completely by the local market dynamics f together with diffusion term and if $\alpha = 0$ then the local market dynamics is suppressed completely and $p_j(n+1)$ depends only on the present price and diffusion term.

Remark 3.4. For a solution $p_j(n)$ of our model (3.7) to have a meaning in economic sense, we impose a boundary condition at infinity that $p_j(n)$ must be bounded,i.e., $|p_j(n)| \le C \; \forall j \in \mathbf{Z}$, $n \in \mathbf{Z}^+$ for some $C > 0$. Also, we require that a solution $p_j(n)$ must have nonnegative value for all $j \in \mathbf{Z}$, $n \in \mathbf{Z}^+$. If a solution of (3.7) does not satisfy these conditions, then it would not be an admissible solution for our model.

Remark 3.5. Besides the solutions given in the Definition 3.3, there can also be many other solutions, e.g., spatially and/or temporally periodic solutions, spatially and/or temporally chaotic bounded solutions, and so on. In this paper, we restrict our attention only to those periodic solutions or bounded chaotic solutions which are the basic solutions mentioned in the Definition 3.3, e.g., spatially periodic static solutions, temporally periodic spatially homogeneous solutions, spatially and temporally periodic traveling wave solutions, etc.

4. Global Price Dynamics

We assume that the predictor H is naive, the demand D is linear decreasing, and the supply S is quadratic, that is, they are given by

$$\begin{aligned} p_n^e &= H(\mathbf{P}_{n-1}) = p_{n-1}, \quad D(p_n) = 1 - p_n, \\ S(p_n^e) &= S(p_{n-1}) = 4p_{n-1}(1 - p_{n-1}), \end{aligned} \tag{4.1}$$

respectively. Note that the price p_n in (4.1) is a scaled price such that $0 \leq p_n \leq 1$, and the quadratic supply function $S(x) = 4x(1-x)$ is a so called *logistic map*, which is known to exhibit chaotic dynamics on the whole interval $[0, 1]$ (e.g., [15]). This kind of non-monotonic supply curve can be justified in an actual market, e.g., by an *income effect* in an agricultural market (e.g., [17], pp 339). This income effect, of course, may be applied to our fish market as well. In other words, as prices of fish are getting higher, the income of fishermen is getting higher, and so after arriving at a peak point, the production of fish might be getting less due to their taking more leisure time.

Now, with these choices of H, D, and S, the local market equilibrium price dynamics, $D(p_n) = S(p_n^e)$, is given by

$$\begin{aligned} 1 - p_n &= 4p_{n-1}(1 - p_{n-1}), \quad \text{or,} \\ p_n &= 1 - 4p_{n-1}(1 - p_{n-1}). \end{aligned} \tag{4.2}$$

Hence, our local market dynamics f for the global market model (3.7) is given by

$$f(x) = 1 - 4x(1 - x) = (1 - 2x)^2, \tag{4.3}$$

The map f has two fixed points, one at $p^* = \frac{1}{4}$ where $f'(p^*) = -2 < -1$, and the other at $q^* = 1$ where $f'(q^*) = 4 > 1$, and so both fixed points are repellors. Note that at $p^* = \frac{1}{4}$, $D(\frac{1}{4}) = S(\frac{1}{4}) = \frac{3}{4}$, and prices near p^* diverges in an oscillatory way from p^*, while at $q^* = 1$, $D(1) = S(1) = 0$, and prices near and less than q^* decreases in a monotone way from q^* and so prices fluctuate between these two repellors p^* and q^* in a chaotic way as is shown in Figure 1.

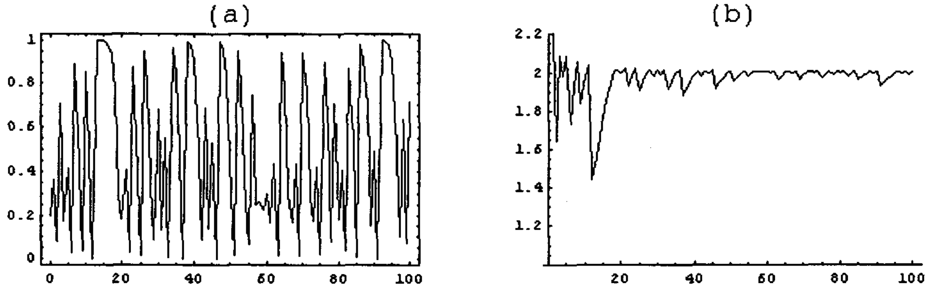

Figure 1. *For an arbitrarily given initial point 0.2, (a) Time Series plot of f. Apparently, the dynamics of f is chaotic on the whole interval* $[0,1]$. *(b) The Lyapunov number of f. This graph indicates that the orbit starting at 0.2 is chaotic, as its Lyapunov number is clearly greater than 1.*

Now, let us consider the global market dynamics (3.7), where the local market dynamics f is given by (4.3), i.e.,

(4.4) $p_j(n+1) = (1-\alpha)p_j(n) + \alpha\{1-2p_j(n)\}^2 + \varepsilon\{p_{j-1}(n) - 2p_j(n) + p_{j+1}(n)\}.$

Hereafter, we will slightly loosen our restriction $0 \le p_j(n) \le 1$ so that $p_j(n) \ge 0$ because interesting dynamics can occur for $p_j(n) > 1$ in the global market dynamics. To avoid making our paper too lengthy, we will restrict our attention only to the spatially homogeneous solutions of (4.4) in this paper. Other solutions such as static solutions or traveling wave solutions will be considered later in another paper.

To obtain the spatially homogeneous solutions, we set $p_j(n) = \psi(n)$ in (4.4), then we have

(4.5) $$\psi(n+1) = (1-\alpha)\psi(n) + \alpha\{1-2\psi(n)\}^2, \quad \psi(n) \ge 0.$$

Equation (4.5) is a 1st order nonlinear difference equation and writing $\psi(n) = x_n$, its dynamics can be described by a 1D quadratic map $F_\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ given by

(4.6) $$F_\alpha(x) = (1-\alpha)x + \alpha(1-2x)^2, \quad x \ge 0, \ \alpha \in [0,1].$$

If $0 < \alpha \le 1$, the map $F_\alpha(x)$ has two fixed points $x_1^* = \frac{1}{4}$ and $x_2^* = 1$, which are the same as the fixed points of the local market dynamics f. If $\alpha = 0$, $F_\alpha(x)$ becomes the identity map $F_\alpha(x) = x$, and so any point $x_0 \ge 0$ is a fixed point of $F_\alpha(x)$. If $\alpha = 1$, then $F_\alpha(x)$ reduces to $F_\alpha(x) = (1-2x)^2 = f(x)$, i.e., is coincident with the local market dynamics f, and so obviously chaotic on the interval $0 \le x \le 1$. Hence, we assume that $0 < \alpha < 1$ from now on. Since $F'_\alpha(\frac{1}{4}) = 1 - 3\alpha$ and $F'_\alpha(1) = 1 + 3\alpha$, it follows that the fixed point $x_1^* = \frac{1}{4}$ is asymptotically stable for $0 < \alpha < \frac{2}{3}$, unstable for $\frac{2}{3} < \alpha < 1$, and has a non-hyperbolic eigenvalue -1 for $\alpha = \frac{2}{3}$; while the fixed point $x_2^* = 1$ is unstable for all $0 < \alpha < 1$. From this, we can expect that as the values of the parameter α vary from 0 to 1, the map $F_\alpha(x)$ undergoes a cascade of period-doubling bifurcation starting at $\alpha = \frac{2}{3}$, and becomes chaotic right after this cascade and ends up with chaotic motion spread out over the whole interval $0 \leq x \leq 1$ at $\alpha = 1$. Now we show in the below that this is indeed the case.

Lemma 4.1. *The map $F_\alpha(x)$ given by* (4.6) *on* $[0, 1]$ *is topologically conjugate to the quadratic map defined by $Q_\mu(x) = \mu x(1 - x)$ on* $[0, 1]$ *via the homeomorphism $h(x) = \frac{4\alpha}{\mu}(1 - x)$, with the relationship $\mu = 3\alpha + 1$ between the parameters.*

Note that $F_\alpha(x)$ with $\alpha = 1$ is conjugate to the map $Q_\mu(x)$ with $\mu = 4$, via $h(x) = 1 - x$, which is coincident with the supply function $S(x)$. The bifurcation diagrams for F_α and Q_μ is shown in the Figure 2 below.

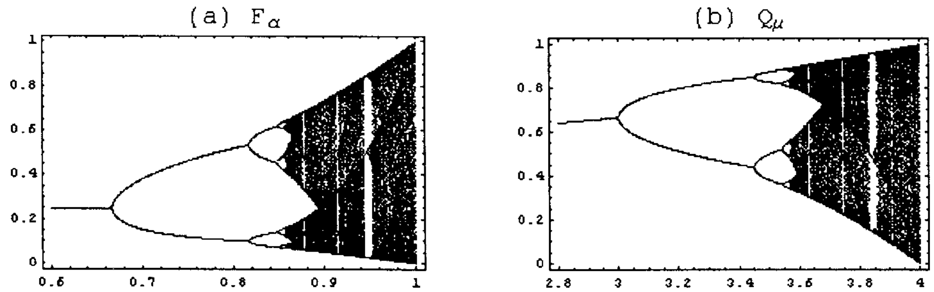

Figure 2. (a) The bifurcation diagram for F_α. (b) the same diagram for Q_μ. The conjugacy between them is obvious. Note that the parameter values $\alpha = \frac{2}{3}$ in (a) and $\mu = 3$ in (b) at which period-doubling bifurcation starts correspond to each other.

Hence, according to the Lemma 4.1, the dynamics of $F_\alpha(x)$ as α varies is the same as the dynamics of $Q_\mu(x)$ as μ changes and so can be obtained from the well-known results about the dynamics of $Q_\mu(x)$ through the relationship $\mu = 3\alpha + 1$. Now we restate the dynamics of F_α in terms of the spatially homogeneous solutions in the below:

Theorem 4.2. (i) *If $0 < \alpha \leq 1$, the static spatially homogeneous solution $p(n) = \{\psi(n)\}_{j \in \mathbf{Z}} = \{\frac{1}{4}\}_{j \in \mathbf{Z}}$ of the evolution operator $\Phi_\alpha : B_\rho \to B_\rho$ undergoes a period-doubling bifurcation route to chaos. That is, for $0 < \alpha < \frac{2}{3}$, Φ_α has an attracting*

fixed point $\{\psi(n)\} = \{\frac{1}{4}\}$ *and then the cascade of period-doubling bifurcation begins at* $\alpha = \frac{2}{3}$ *and ends at* $\alpha_\infty = 0.8566$. *For* $\alpha_\infty < \alpha \leq 1$, *the orbit of a spatially homogeneous solution under* Φ_α *exhibits chaotic behavior in the infinite strip* $I = [0,1]^{\mathbf{Z}}$.
(ii) *If* $\alpha = 1$, *this chaotic motion spreads out over the whole strip* I. *Orbits starting outside this strip* I *become unbounded as* $n \to +\infty$.
(iii) *If* $\alpha = 0$, *then* Φ_α *has no dynamics, i.e., any spatially homogeneous solution is a fixed point of* Φ_α.

Therefore, according to Theorem 4.2, the bounded spatially homogeneous solutions are as follows:

(i) the static spatially homogeneous solutions, i.e., $\{\psi(n)\} = \{\frac{1}{4}\}$ and $\{\psi(n)\} = \{1\}$ for $0 < \alpha \leq 1$, and $\{\psi(n)\} = \{\psi(0)\}$ for $\alpha = 0$.
(ii) the infinitely many temporally-periodic spatially homogeneous solutions with any period created through the period-doubling bifurcation.
(iii) the bounded temporally-chaotic spatially homogeneous solutions created right after the period-doubling bifurcation.

5. Concluding Remarks

In Section 4, we have noticed that bounded spatially homogeneous solutions do exist and are directly affected by the local market dynamics because of the non-presence of the diffusion. Furthermore, even if the local market dynamics is unstable, the spatially homogeneous solutions of the global market can be controlled via market control parameter α, so that it can converge to the static spatially homogeneous solutions corresponding to the fixed points of the local market dynamics.

Appendix

Proof of Lemma 4.1. By using the relationship $\mu = 3\alpha + 1$, we can immediately check that the commutativity relation $h \circ F_\alpha(x) = Q_\mu \circ h(x)$ holds. □

Proof of Theorem 4.2. It has already been a well known fact that the quadratic family $Q_\mu(x)$ undergoes a period-doubling bifurcation route to chaos (e.g., [15], [21]). Stating more specifically, let $\mu_1 = 3 < \mu_2 = 1 + \sqrt{6} < \cdots < \mu_n < \cdots$ be the parameter values of μ for which Q_μ has an orbit of period-2, period-4, $\cdots$, period-2^n, $\cdots$, respectively. The limiting parameter value is known to be $\mu_\infty = 3.5699456$. As

μ varies from 1 to 4, the fixed point $p_\mu = \frac{\mu-1}{\mu}$ of Q_μ is attracting for $1 < \mu < \mu_1 = 3$ and has eigenvalue -1 at $\mu = \mu_1$. For $\mu > \mu_1$, the fixed point becomes repelling and a new attracting period-2 orbit is created. At $\mu = \mu_2$, the period-2 orbit has eigenvalue -1 and for $\mu > \mu_2$, the period-2 orbit becomes repelling and a new attracting period-4 orbit is created. This process repeats itself; at $\mu = \mu_n$, the period-2^n orbit begins to be created, and it is attracting for $\mu_n < \mu < \mu_{n+1}$ and becomes repelling for $\mu > \mu_{n+1}$.

Now the dynamics of Q_μ is converted to that of F_α in a one to one way via the relation $\mu = 3\alpha + 1$. That is, the sequence of parameter values $\mu_1 = 3, \mu_2 = 1 + \sqrt{6}, \cdots, \mu_\infty = 3.5699$ correspond to the sequence of parameter values $\alpha_1 = \frac{2}{3}, \alpha_2 = \frac{\sqrt{6}}{3}, \cdots, \alpha_\infty = 0.8566$ respectively and $\mu = 1, \mu = 4$ correspond to $\alpha = 0, \alpha = 1$ respectively. From the dynamics of F_α, the dynamics of spatially homogeneous solutions follow immediately. □

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