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ON CONTINUOUS LINEAR JORDAN DERIVATIONS OF BANACH ALGEBRAS

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ABSTRACT. Let A be a Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $[D(x), x]D(x)^2[D(x), x] \in \mathrm{rad}(A)$ for all $x \in A$. Then we have $D(A) \subseteq \mathrm{rad}(A)$.

1. INTRODUCTION

Throughout, R represents an associative ring and A will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. Let $\mathrm{rad}(R)$ denote the *(Jacobson) radical* of a ring R. And a ring R is said to be *(Jacobson) semisimple* if its Jacobson radical $\mathrm{rad}(R)$ is zero.

A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. (see F. F. Bonsall and J. Duncan [1]).

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

B. E. Johnson and A. M. Sinclair [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. A result of I. M. Singer and J. Wermer [11] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra.

M. P. Thomas [13] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

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J. Vukman [15] has proved the following: let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he has proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

B.-D. Kim [6] has showed that the following results hold: let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \to R$ such that

$$[D(x), x]D(x)[D(x), x] = 0$$

for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$. And, B. D. Kim [7] has showed that the following results hold: let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)[D(x), x]D(x) \in \text{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \text{rad}(A)$.

The aims of this paper are to prove the following results in the ring theory in order to apply it to the Banach algebra theory:

Let R be a 7!-torsion free semiprime ring.

Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$[D(x), x]D(x)^2[D(x), x] = 0$$

for all $x \in R$. In this case, we obtain $[D(x), x]^6 = 0$ for all $x \in R$.

Using the above result, we have the following: let A be a Banach algebra and let $D : A \longrightarrow A$ be a continuous linear Jordan derivation. Suppose that

$$[D(x), x]D(x)^2[D(x), x] \in \text{rad}(A)$$

holds for all $x \in A$. In this case, $D(A) \subseteq \text{rad}(A)$.

For further results, see B.-D. Kim [8], [9], K.-H. Park [12], J. Vukman [14] and J. Vukman [16] for the results of the left Jordan derivations on semisimple Banach algebras.

2. Preliminaries

The following lemma is due to L.O. Chung and J. Luh [4].

Lemma 2.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \cdots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \le k \le n$.*

The following theorem is due to M. Brešar [3].

Theorem 2.2. *Let R be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

We write $Q(A)$ for the set of all quasinilpotent elements in A. M. Brešar [2] has proved the following theorem.

Theorem 2.3. *Let D be a bounded derivation of a Banach algebra A. Suppose that $[D(x), x] \in Q(A)$ for every $x \in A$. Then D maps A into $rad(A)$.*

After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer.

We need Theorem 2.2 and 2.3 to obtain the main theorems for Banach algebra theory.

3. Main Results

We need the following statement to obtain the main theorems for Banach algebra theory. Thus it is very important to obtain the following theorem to prove Theorem 3.2-3.3.

Theorem 3.1. *Let R be a 7!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[D(x), x]D(x)^2[D(x), x] = 0$$

for all $x \in R$. In this case we have

$$[D(x), x]^6 = 0$$

for all $x \in R$.

Proof. It suffices to prove the case that R is noncommutative. We define a bi-additive function $B : R \times R \longrightarrow R$ and functions $f, g : R \to R$ by

$$B(x, y) = [D(x), y] + [D(y), x],\ f(x) = [D(x), x],\ g(x) = [f(x), x],$$
$$h(x) = [g(x), x]$$

for all $x, y \in R$ respectively.

And we get the following relations as follows:

$$B(x, y) = B(y, x),$$
$$B(x, yz) = D(y)[z, x] + [y, x]D(z) + B(x, y)z + yB(x, z),$$
$$B(x, yx) = [y, x]D(x) + B(x, y)x + 2yf(x),$$

$$B(x,xy)=D(x)[y,x]+2f(x)y+xB(x,y),$$
$$B(x,x^2)=B(x,x)x+xB(x,x)=2(f(x)x+xf(x)),\ x,y,z\in R.$$

By Theorem 2.2, we can see that D is a derivation on R. Suppose

$$f(x)D(x)^2f(x)=0,\ x\in R. \tag{1}$$

Replacing $x+ty$ for x in (1), we have

$$\begin{aligned}&f(x+ty)D(x+ty)^2f(x+ty)\\&\equiv f(x)D(x)^2f(x)+t\{B(x,y)D(x)^2f(x)+f(x)D(y)D(x)f(x)\\&\quad+f(x)D(x)D(y)f(x)+f(x)D(x)^2B(x,y)\}+t^2H_1(x,y)\\&\quad+t^3H_2(x,y)+t^4H_3(x,y)+t^5H_4(x,y)+t^6f(y)D(y)^2f(y)\\&=0,\ x,y\in R,\ t\in S_5\end{aligned} \tag{2}$$

where $H_i, 1\le i\le 4$, denotes the term satisfying the identity (2).
From (1) and (2), we obtain

$$\begin{aligned}&t\{B(x,y)D(x)^2f(x)+f(x)D(y)D(x)f(x)\\&\quad+f(x)D(x)D(y)f(x)+f(x)D(x)^2B(x,y)\}+t^2H_1(x,y)\\&\quad+t^3H_2(x,y)+t^4H_3(x,y)+t^5H_4(x,y)\\&=0,\ x,y\in R,\ t\in S_5.\end{aligned}$$

Since R is 5!-torsion free by assumption, by Lemma 2.1 the above relation yields

$$\begin{aligned}&B(x,y)D(x)^2f(x)+f(x)D(y)D(x)f(x)+f(x)D(x)D(y)f(x)\\&+f(x)D(x)^2B(x,y)=0,\ x,y\in R.\end{aligned} \tag{3}$$

Replacing xy for y in (3), we have

$$\begin{aligned}&xB(x,y)D(x)^2f(x)+D(x)[y,x]D(x)^2f(x)+2f(x)yD(x)^2f(x)\\&\quad+f(x)D(x)yD(x)f(x)+f(x)xD(y)D(x)f(x)\\&\quad+f(x)D(x)xD(y)f(x)+f(x)D(x)^2yf(x)\\&\quad+f(x)D(x)^2xB(x,y)+f(x)D(x)^3[y,x]+2f(x)D(x)^2f(x)y\\&=0,\ x,y\in R.\end{aligned} \tag{4}$$

From (1) and (4),

$$\begin{aligned}&xB(x,y)D(x)^2f(x)+D(x)[y,x]D(x)^2f(x)+2f(x)yD(x)^2f(x)\\&+f(x)D(x)yD(x)f(x)+f(x)xD(y)D(x)f(x)\end{aligned}$$

$$\begin{aligned}&+f(x)D(x)xD(y)f(x)+f(x)D(x)^2yf(x)\\ &+f(x)D(x)^2xB(x,y)+f(x)D(x)^3[y,x]=0,\ x,y\in R. \qquad (5)\end{aligned}$$

Left multiplication of x in (3) leads to

$$\begin{aligned}&xB(x,y)D(x)^2f(x)+xf(x)D(y)D(x)f(x)+xf(x)D(x)D(y)f(x)\\ &+xf(x)D(x)^2B(x,y)=0,\ x,y\in R. \qquad (6)\end{aligned}$$

Combining (5) with (6),

$$\begin{aligned}&D(x)[y,x]D(x)^2f(x)+2f(x)yD(x)^2f(x)+f(x)D(x)yD(x)f(x)\\ &+g(x)D(y)D(x)f(x)+g(x)D(x)D(y)f(x)+f(x)^2D(y)f(x)\\ &+f(x)D(x)^2yf(x)+g(x)D(x)^2B(x,y)+f(x)^2D(x)B(x,y)\\ &+f(x)D(x)f(x)B(x,y)+f(x)D(x)^3[y,x]=0,\ x,y\in R. \qquad (7)\end{aligned}$$

Replacing yx for y in (3), we have

$$\begin{aligned}&B(x,y)xD(x)^2f(x)+[y,x]D(x)^3f(x)+2yf(x)D(x)^2f(x)\\ &+f(x)D(y)xD(x)f(x)+f(x)yD(x)^2f(x)\\ &+f(x)D(x)yD(x)f(x)+f(x)D(x)D(y)xf(x)\\ &+f(x)D(x)^2B(x,y)x+f(x)D(x)^2[y,x]D(x)+2f(x)D(x)^2yf(x)=0,\\ &x,y\in R. \qquad (8)\end{aligned}$$

From (1) and (8),

$$\begin{aligned}&B(x,y)xD(x)^2f(x)+[y,x]D(x)^3f(x)\\ &+f(x)D(y)xD(x)f(x)+f(x)yD(x)^2f(x)\\ &+f(x)D(x)yD(x)f(x)+f(x)D(x)D(y)xf(x)\\ &+f(x)D(x)^2B(x,y)x+f(x)D(x)^2[y,x]D(x)+2f(x)D(x)^2yf(x)=0,\\ &x,y\in R. \qquad (9)\end{aligned}$$

Right multiplication of x in (3) leads to

$$\begin{aligned}&B(x,y)D(x)^2f(x)x+f(x)D(y)D(x)f(x)x+f(x)D(x)D(y)f(x)x\\ &+f(x)D(x)^2B(x,y)x=0,\ x,y\in R. \qquad (10)\end{aligned}$$

Comparing (9) and (10),

$$-B(x,y)f(x)D(x)f(x)-B(x,y)D(x)f(x)^2-B(x,y)D(x)^2g(x)$$

$$+[y,x]D(x)^3f(x) - f(x)D(y)D(x)g(x) - f(x)D(y)f(x)^2$$
$$+f(x)yD(x)^2f(x) + f(x)D(x)yD(x)f(x) - f(x)D(x)D(y)g(x)$$
$$(11) \qquad +f(x)D(x)^2[y,x]D(x) + 2f(x)D(x)^2yf(x) = 0, \ x,y \in R.$$

Let $y = x$ in (7). Then we get

$$2f(x)D(x)f(x)^2 + 4f(x)^2D(x)f(x) + 7g(x)D(x)^2f(x)$$
$$(12) \qquad -f(x)D(x)^2g(x) = 0, \ x \in R.$$

And let $y = x$ in (11). Then it is obvious that

$$-3f(x)D(x)f(x)^2 - f(x)^2D(x)f(x) - 6f(x)D(x)^2g(x)$$
$$(13) \qquad +2g(x)D(x)^2f(x) = 0, \ x \in R.$$

Combining (12) with (13),

$$-15f(x)D(x)f(x)^2 - 25f(x)^2D(x)f(x) - 40g(x)D(x)^2f(x)$$
$$(14) \qquad = 0, \ x \in R.$$

Since R is 5-torsion free, we get from (14)

$$(15) \qquad 3f(x)D(x)f(x)^2 + 5f(x)^2D(x)f(x) + 8g(x)D(x)^2f(x) = 0, \ x \in R.$$

Let $y = x^2$ in (3). Then we obtain

$$5g(x)D(x)^2f(x) + 2xf(x)D(x)^2f(x) + 2f(x)^2D(x)f(x)$$
$$(16) \qquad -3f(x)D(x)^2g(x) + f(x)D(x)^2f(x)x = 0, \ x,y \in R.$$

From (1) and (16), it follows that

$$(17) \qquad 8g(x)D(x)^2f(x) + 5f(x)^2D(x)f(x) + 3f(x)D(x)f(x)^2 = 0, \ x \in R.$$

On the other hand, we have from (1)

$$(18) \qquad 2g(x)D(x)^2f(x) + 2f(x)D(x)f(x)^2 + 2f(x)^2D(x)f(x) = 0, \ x \in R.$$

Since R is a 2-torsion free, from (18) we get

$$(19) \qquad g(x)D(x)^2f(x) + f(x)D(x)f(x)^2 + f(x)^2D(x)f(x) = 0, \ x \in R.$$

Combining (17) with (19),

$$(20) \qquad 3f(x)^2D(x)f(x) + 5f(x)D(x)f(x)^2 = 0, \ x \in R.$$

From (17) and (20),

$$(21) \qquad 24g(x)D(x)^2f(x) - 16f(x)D(x)f(x)^2 = 0, \ x \in R.$$

Since R is a 2-torsion free, we get from (21)

$$3g(x)D(x)^2f(x) - 2f(x)D(x)f(x)^2 = 0, \ x \in R. \tag{22}$$

The relation (22) can be rewritten as

$$-3f(x)xD(x)^2f(x) - 2f(x)D(x)f(x)^2 = 0, \ x \in R. \tag{23}$$

And also, using (1), the relation (23) gives

$$3f(x)D(x)^2g(x) + f(x)D(x)f(x)^2 + 3f(x)^2D(x)f(x) = 0, \ x \in R. \tag{24}$$

Comparing (12) and (24),

$$7f(x)D(x)f(x)^2 + 15f(x)^2D(x)f(x) + 21g(x)D(x)^2f(x) = 0, \ x \in R. \tag{25}$$

From (22) and (25),

$$21f(x)D(x)f(x)^2 + 15f(x)^2D(x)f(x) = 0, \ x \in R. \tag{26}$$

Since R is a 3!-torsion free, we get from (26)

$$7f(x)D(x)f(x)^2 + 5f(x)^2D(x)f(x) = 0, \ x \in R. \tag{27}$$

Combining (20) with (27), we have

$$4f(x)D(x)f(x)^2 = 0, \ x \in R \tag{28}$$

and

$$4f(x)^2D(x)f(x) = 0, \ x \in R. \tag{29}$$

Since R is a 2-torsion free, from (28) and (29) we obtain

$$f(x)D(x)f(x)^2 = 0, \ x \in R \tag{30}$$

and

$$f(x)^2D(x)f(x) = 0, \ x \in R, \tag{31}$$

respectively. Thus from (19), (30) and (31), we have

$$g(x)D(x)^2f(x) = 0, \ x \in R. \tag{32}$$

And it follows from (24), (30), (31) and (32) that

$$3f(x)D(x)^2g(x) = 0, \ x \in R. \tag{33}$$

Since R is a 3-torsion free, we get from (33)

$$f(x)D(x)^2g(x) = 0, \ x \in R. \tag{34}$$

From (31), we arrive at

$$\begin{aligned} &B(x,y)f(x)D(x)f(x)+f(x)B(x,y)D(x)f(x)+f(x)^2D(y)f(x)\\ &+f(x)^2D(x)B(x,y)=0,\ x\in R, \end{aligned} \tag{35}$$

in the same manner that makes it possible to obtain (3) from (1) under the 5!-torsion-freeness of R.

Let $y=x^2$ in (35). Then it is clear that

$$\begin{aligned} &2f(x)xf(x)D(x)f(x)+2xf(x)^2D(x)f(x)+2f(x)^2xD(x)f(x)\\ &+2f(x)xf(x)D(x)f(x)+f(x)^2D(x)xf(x)+f(x)^2xD(x)f(x)\\ &+2f(x)^2D(x)f(x)x+2f(x)^2D(x)xf(x)=0,\ x\in R. \end{aligned} \tag{36}$$

Comparing (31) and (36),

$$4f(x)xf(x)D(x)f(x)+3f(x)^2xD(x)f(x)+3f(x)^2D(x)xf(x)=0,\ x\in R. \tag{37}$$

From (31) and (37),

$$\begin{aligned} &4g(x)f(x)D(x)f(x)+3f(x)g(x)D(x)f(x)\\ &+3g(x)f(x)D(x)f(x)-3f(x)^2D(x)g(x)\\ &=7g(x)f(x)D(x)f(x)+3f(x)g(x)D(x)f(x)-3f(x)^2D(x)g(x)=0,\ x\in R. \end{aligned} \tag{38}$$

Right multiplication of $f(x)$ in (39) leads to

$$\begin{aligned} &7g(x)f(x)D(x)f(x)^2+3f(x)g(x)D(x)f(x)^2\\ &-3f(x)^2D(x)g(x)f(x)=0,\ x\in R. \end{aligned} \tag{39}$$

Combining (30) with (39),

$$3(f(x)g(x)D(x)f(x)^2-f(x)^2D(x)g(x)f(x))=0,\ x\in R. \tag{40}$$

Since R is a 3-torsion free, we get from (40)

$$f(x)g(x)D(x)f(x)^2-f(x)^2D(x)g(x)f(x)=0,\ x\in R. \tag{41}$$

From (31), we obtain

$$\begin{aligned} &0=[f(x)^2D(x)f(x),x]\\ &=g(x)f(x)D(x)f(x)+f(x)g(x)D(x)f(x)+f(x)^4+f(x)^2D(x)g(x),\ x\in R. \end{aligned} \tag{42}$$

Right multiplication of $f(x)$ in (42) leads to

$$\begin{aligned} &g(x)f(x)D(x)f(x)^2+f(x)g(x)D(x)f(x)^2+f(x)^5\\ &+f(x)^2D(x)g(x)f(x)=0,\ x\in R. \end{aligned} \tag{43}$$

From (30) and (43), we obtain

$$f(x)g(x)D(x)f(x)^2 + f(x)^5 + f(x)^2D(x)g(x)f(x) = 0,\ x \in R. \tag{44}$$

From (41) and (44), we get

$$2f(x)g(x)D(x)f(x)^2 + f(x)^5 = 0,\ x \in R. \tag{45}$$

From (32), we have

$$\begin{aligned} 0 &= [g(x)D(x)^2f(x), x] \\ &= h(x)D(x)^2f(x) + g(x)f(x)D(x)f(x) + g(x)D(x)f(x)^2 + g(x)D(x)^2g(x), \\ &\quad x \in R. \end{aligned} \tag{46}$$

Right multiplication of $f(x)$ in (46) leads to

$$\begin{aligned} &h(x)D(x)^2f(x)^2 + g(x)f(x)D(x)f(x)^2 + g(x)D(x)f(x)^3 \\ &+g(x)D(x)^2g(x)f(x) = 0,\ x \in R. \end{aligned} \tag{47}$$

From (30) and (47), we obtain

$$h(x)D(x)^2f(x)^2 + g(x)D(x)f(x)^3 + g(x)D(x)^2g(x)f(x) = 0,\ x \in R. \tag{48}$$

From (32), we get

$$\begin{aligned} &([B(x,y),x] + [f(x),y])D(x)^2f(x) + g(x)D(y)D(x)f(x) \\ &+g(x)D(x)D(y)f(x) + g(x)D(x)^2B(x,y) = 0,\ x \in R \end{aligned} \tag{49}$$

in the same fashion that makes it possible to obtain (3) from (1) under the 5!-torsion-freeness of R.

Let $y = x^2$ in (49). Then it follows that

$$\begin{aligned} &3(g(x)x + xg(x))D(x)^2f(x) + g(x)(D(x)x + xD(x))D(x)f(x) \\ &+g(x)D(x)(D(x)x + xD(x))f(x) + 2g(x)D(x)^2(f(x)x + xf(x)) = 0,\ x \in R. \end{aligned} \tag{50}$$

Comparing (32) and (50), we obtain

$$\begin{aligned} &3g(x)xD(x)^2f(x) + 3xg(x)D(x)^2f(x) + g(x)D(x)xD(x)f(x) \\ &\quad +g(x)xD(x)D(x)f(x) + g(x)D(x)D(x)xf(x) + g(x)D(x)xD(x)f(x) \\ &\quad +2g(x)D(x)^2f(x)x + 2g(x)D(x)^2xf(x) \\ &= 4g(x)xD(x)^2f(x) + 3xg(x)D(x)^2f(x) + 2g(x)D(x)xD(x)f(x) \\ &\quad +3g(x)D(x)^2xf(x) + 2g(x)D(x)^2f(x)x = 0,\ x \in R. \end{aligned} \tag{51}$$

From (32) and (51), we have

$$\begin{aligned} &4h(x)D(x)^2f(x) + 2h(x)D(x)^2f(x) + 2g(x)f(x)D(x)f(x) - 3g(x)D(x)^2g(x) \\ &= 6h(x)D(x)^2f(x) + 2g(x)f(x)D(x)f(x) - 3g(x)D(x)^2g(x) = 0, \ x \in R. \end{aligned} \tag{52}$$

Right multiplication of $f(x)$ in (52) leads to

$$6h(x)D(x)^2f(x)^2 + 2g(x)f(x)D(x)f(x)^2 - 3g(x)D(x)^2g(x)f(x) = 0, \ x \in R. \tag{53}$$

Combining (30) with (53), we obtain

$$6h(x)D(x)^2f(x)^2 - 3g(x)D(x)^2g(x)f(x) = 0, \ x \in R. \tag{54}$$

Since R is a 7!-torsion free, we get from (54)

$$2h(x)D(x)^2f(x)^2 - g(x)D(x)^2g(x)f(x) = 0, \ x \in R. \tag{55}$$

From (48) and (55), we have

$$3h(x)D(x)^2f(x)^2 + g(x)D(x)f(x)^3 = 0, \ x \in R. \tag{56}$$

Comparing (54) and (56),

$$2g(x)D(x)f(x)^3 + 3g(x)D(x)^2g(x)f(x) = 0, \ x \in R. \tag{57}$$

Combining (45) with (57), we get

$$\begin{aligned} &f(x)(2g(x)D(x)f(x)^3) + f(x)^6 \\ &= -3f(x)g(x)D(x)^2g(x)f(x) + f(x)^6 = 0, \ x \in R. \end{aligned} \tag{58}$$

Substituting yx for y in (7), we obtain

$$\begin{aligned} &D(x)[y,x]xD(x)^2f(x) + 2f(x)yxD(x)^2f(x) + f(x)D(x)yxD(x)f(x) \\ &+g(x)D(y)xD(x)f(x) + g(x)yD(x)^2f(x) + g(x)D(x)D(y)xf(x) \\ &+g(x)D(x)yD(x)f(x) + f(x)^2D(y)xf(x) + f(x)^2yD(x)f(x) \\ &+f(x)D(x)^2yxf(x) + g(x)D(x)^2B(x,y)x + g(x)D(x)^2[y,x]D(x) \\ &+2g(x)D(x)^2yf(x) + f(x)^2D(x)B(x,y)x \\ &+f(x)^2D(x)[y,x]D(x) + 2f(x)^2D(x)yf(x) \\ &+f(x)D(x)f(x)B(x,y)x + f(x)D(x)f(x)[y,x]D(x) \\ &+2f(x)D(x)f(x)yf(x) + f(x)D(x)^3[y,x]x = 0, \ x,y \in R. \end{aligned} \tag{59}$$

Right multiplication of x in (7) leads to

$$\begin{aligned}&D(x)[y,x]D(x)^2f(x)x+2f(x)yD(x)^2f(x)x+f(x)D(x)yD(x)f(x)x\\&+g(x)D(y)D(x)f(x)x+g(x)D(x)D(y)f(x)x+f(x)^2D(y)f(x)x\\&+f(x)D(x)^2yf(x)x+g(x)D(x)^2B(x,y)x+f(x)^2D(x)B(x,y)x\\&+f(x)D(x)f(x)B(x,y)x+f(x)D(x)^3[y,x]x=0,\ x,y\in R.\end{aligned}\tag{60}$$

From (59) and (60),

$$\begin{aligned}&D(x)[y,x](f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))\\&+2f(x)y(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))\\&+f(x)D(x)y(f(x)^2+D(x)g(x))+g(x)D(y)(f(x)^2+D(x)g(x))\\&-g(x)yD(x)^2f(x)+g(x)D(x)D(y)g(x)-g(x)D(x)yD(x)f(x)\\&+f(x)^2D(y)g(x)-f(x)^2yD(x)f(x)+f(x)D(x)^2yg(x)\\&-g(x)D(x)^2[y,x]D(x)-2g(x)D(x)^2yf(x)-f(x)^2D(x)[y,x]D(x)\\&-2f(x)^2D(x)yf(x)-f(x)D(x)f(x)[y,x]D(x)-2f(x)D(x)f(x)yf(x)=0,\\&x,y\in R.\end{aligned}\tag{61}$$

Let $y=x$ in (61). Then we have

$$\begin{aligned}&2f(x)xf(x)D(x)f(x)+2f(x)xD(x)f(x)^2+2f(x)xD(x)^2g(x)\\&+f(x)D(x)xf(x)^2+f(x)D(x)xD(x)g(x)+g(x)D(x)f(x)^2+g(x)D(x)^2g(x)\\&-g(x)xD(x)^2f(x)+g(x)D(x)^2g(x)-g(x)D(x)xD(x)f(x)\\&+f(x)^2D(x)g(x)-f(x)^2xD(x)f(x)+f(x)D(x)^2xg(x)\\&-2g(x)D(x)^2xf(x)-2f(x)^2D(x)xf(x)-2f(x)D(x)f(x)xf(x)=0,\\&x,y\in R.\end{aligned}\tag{62}$$

Comparing (30), (31) and (62),

$$\begin{aligned}&2g(x)f(x)D(x)f(x)+2g(x)D(x)f(x)^2+2g(x)D(x)^2g(x)\\&+g(x)D(x)f(x)^2+f(x)^4+g(x)D(x)^2g(x)+f(x)^2D(x)g(x)\\&+g(x)D(x)f(x)^2+2g(x)D(x)^2g(x)\\&-h(x)D(x)^2f(x)+h(x)D(x)^2f(x)-g(x)f(x)D(x)f(x)\\&+f(x)^2D(x)g(x)-g(x)f(x)D(x)f(x)-f(x)g(x)D(x)f(x)\\&+g(x)D(x)^2g(x)+f(x)^2D(x)g(x)+f(x)D(x)f(x)g(x)\end{aligned}$$

$$+2g(x)D(x)^2g(x) + 2f(x)^2D(x)g(x) + 2f(x)D(x)f(x)g(x) = 0, x, y \in R. \tag{63}$$

(63) gives

$$\begin{aligned} &4g(x)D(x)f(x)^2 + 8g(x)D(x)^2g(x) + f(x)^4 + 5f(x)^2D(x)g(x) \\ &-f(x)g(x)D(x)f(x) + 3f(x)D(x)f(x)g(x) = 0, \ x, y \in R. \end{aligned} \tag{64}$$

Left multiplication of $f(x)$ in (64) leads to

$$\begin{aligned} &4f(x)g(x)D(x)f(x)^2 + 8f(x)g(x)D(x)^2g(x) + f(x)^5 \\ &+5f(x)^3D(x)g(x) - f(x)^2g(x)D(x)f(x) + 3f(x)^2D(x)f(x)g(x) = 0, \\ &x, y \in R. \end{aligned} \tag{65}$$

Combining (31) with (65),

$$\begin{aligned} &4f(x)g(x)D(x)f(x)^2 + 8f(x)g(x)D(x)^2g(x) + f(x)^5 + 5f(x)^3D(x)g(x) \\ &-f(x)^2g(x)D(x)f(x) = 0, \ x, y \in R. \end{aligned} \tag{66}$$

From (45) and (66),

$$\begin{aligned} &2f(x)g(x)D(x)f(x)^2 + 8f(x)g(x)D(x)^2g(x) + 5f(x)^3D(x)g(x) \\ &-f(x)^2g(x)D(x)f(x) = 0, \ x, y \in R. \end{aligned} \tag{67}$$

Right multiplication of $f(x)$ in (67) leads to

$$\begin{aligned} &2f(x)g(x)D(x)f(x)^3 + 8f(x)g(x)D(x)^2g(x)f(x) \\ &+5f(x)^3D(x)g(x)f(x) - f(x)^2g(x)D(x)f(x)^2 = 0, \ x \in R. \end{aligned} \tag{68}$$

Comparing (41) and (68),

$$\begin{aligned} &2f(x)g(x)D(x)f(x)^3 + 8f(x)g(x)D(x)^2g(x)f(x) \\ &\quad +5f(x)(f(x)^2D(x)g(x)f(x)) - f(x)^2g(x)D(x)f(x)^2 \\ &= 2f(x)g(x)D(x)f(x)^3 + 8f(x)g(x)D(x)^2g(x)f(x) \\ &\quad +4f(x)^2g(x)D(x)f(x)^2 = 0, \ x \in R. \end{aligned} \tag{69}$$

From (58) and (69),

$$6f(x)g(x)D(x)f(x)^3 + 12f(x)^2g(x)D(x)f(x)^2 + 8f(x)^6 = 0, \ x \in R. \tag{70}$$

Since R is a 2!-torsion free, (70) gives

$$3f(x)g(x)D(x)f(x)^3 + 6f(x)^2g(x)D(x)f(x)^2 + 4f(x)^6 = 0, \ x \in R. \tag{71}$$

Combining (57) with (58),

$$2f(x)g(x)D(x)f(x)^3 + f(x)^6 = 0, \ x \in R. \tag{72}$$

Comparing (71) and (72),

$$12f(x)^2g(x)D(x)f(x)^2 + 5f(x)^6 = 0,\ x \in R. \tag{73}$$

From (45) and (73),

$$\begin{aligned} &12f(x)^2g(x)D(x)f(x)^2 + 5f(x)^6 \\ &= 6f(x)(2f(x)g(x)D(x)f(x)^2) + 5f(x)^6 \\ &= 6f(x)(-f(x)^5) + 5f(x)^6 \\ &= -f(x)^6 = 0,\ x \in R. \end{aligned}$$

Therefore we have $f(x)^6 = 0$ for all $x \in R$. □

The following two theorems are proved by the same arguments as in the proof of J. Vukman's theorem [14].

Theorem 3.2. *Let A be a Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D(x)^2[D(x), x] \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. It suffices to prove the case that A is noncommutative. By the result of B.E. Johnson and A.M. Sinclair [5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [10] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P,\ \hat{x} = x + P$. Then we see that D_P is a linear Jordan derivation on A/P for each primitive ideals of A. Thus D_P is a derivation on A/P for each primitive ideals of A by Theorem 2.2. Also, by the assumption that $[D(x), x]D(x)^2[D(x), x] \in \text{rad}(A),\ x \in A$, we obtain $[D_P(\hat{x}), \hat{x}]D_P(\hat{x})^2[D_P(\hat{x}), \hat{x}] = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 3.1 are fulfilled. Let the factor prime Banach algebra A/P be noncommutative. Then we have $[D_P(\hat{x}), \hat{x}]^6 = 0,\ \hat{x} \in A/P$ by Theorem 3.1. So, by Theorem 2.3 we obtain $D_P(\hat{x}) = 0$ for all $\hat{x} \in A/P$. Thus we get $D(x) \in P$ for all $x \in A$ and all primitive ideals of A. Hence $D(A) \subseteq \text{rad}(A)$. And we consider the case that A/P is commutative. Then since A/P is a commutative Banach semisimple Banach algebra, from the result of B.E. Johnson and A.M. Sinclair [5], it follows that $D_P(\hat{x}) = 0,\ \hat{x} \in A/P$. And so, $D(x) \in P$ for all $x \in A$ and all primitive ideals of A. Hence $D(A) \subseteq \text{rad}(A)$. Therefore in any case we obtain $D(A) \subseteq \text{rad}(A)$. □

We have the following theorem from Theorem 3.2 and simple calculations.

Theorem 3.3. *Let A be a semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D(x)^2[D(x), x] = 0$$

for all $x \in A$. Then we have $D = 0$.

Proof. It suffices to prove the case that A is noncommutative. According to the result of B.E. Johnson and A.M. Sinclair [5] every linear derivation on a semisimple Banach algebra is continuous. A.M. Sinclair [10] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a linear Jordan derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. Then by Theorem 2.2, $D_P : A/P \longrightarrow A/P$ is a derivation on A/P for each primitive ideals of A. And so, from the given assumptions $[D(x), x]D(x)^2[D(x), x] = 0$, $x \in A$, it follows that $[D_P(\hat{x}), \hat{x}]D_P(\hat{x})^2[D_P(\hat{x}), \hat{x}] = 0$, $\hat{x} \in A/P$. And so, all the assumptions of Theorem 3.1 are fulfilled. For the prime factor algebra A/P is noncommutative, by Theorem 3.1 we have $D_P(\hat{x}) = 0$, $\hat{x} \in A/P$. Hence we get $D(A) \subseteq P$ for all primitive ideals P of A. Thus $D(A) \subseteq \text{rad}(A$. But since A is semisimple, $D = 0$. □

As a special case of Theorem 3.3 we get the following result which characterizes commutative semisimple Banach algebras.

Corollary 3.4. *Let A be a semisimple Banach algebra. Suppose*

$$[[y, x], x][y, x]^2[[y, x], x] = 0$$

for all $x, y \in A$. In this case, A is commutative.

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