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L-filters and L-filter convergence

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Abstract

In this paper, we study the relations between L-fuzzy topologies and L-filters on a strictly two-sided, commutative quantale lattice L. We define an L-fuzzy neighborhood filter and introduce the notion of L-filter convergence in L-fuzzy topological spaces.

Key words : Quantale lattice, L-fuzzy topologies, L-filters, L-fuzzy neighborhood filter, fuzzy cluster (limit) point

1. Introduction and preliminaries

Šostak [12] introduced the notion of L-fuzzy topological spaces as a generalization of L-topological spaces [9]. Höhle and Šostak [6] substitute a complete quasi-monoidal lattice (or GL-monoid) instead of a completely distributive lattice or an unit interval. In [4,5,9], it was introduced the concepts of fuzzy cluster and fuzzy limit points in L-topological spaces. It has been developed in many directions [1-3, 8,11]. Höhle and Sostak [6] introduced the concept of L-filters for a complete quasi-monoidal lattice L.

In this paper, we study the properties of L-filters on a strictly two-sided, commutative quantale lattice L as an extension of a completely distributive lattice or an unit interval. We define an L-fuzzy neighborhood filter, fuzzy cluster points and fuzzy limit points. Also, we study some properties of them and give examples. In particular, we investigate relations among LF-continuous maps, filter convergence, L-fuzzy neighborhood filters and L-filter maps.

Definition 1.1. [10] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following conditions:

(Q1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(Q2) $(L, \odot)$ is a commutative semigroup;

(Q3) $a = a \odot 1$, for each $a \in L$;

(Q4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

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Remark 1.2. [10](1) A completely distributive lattice is a stsc-quantale. In particular, the unit interval $([0,1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(2) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a stsc-quantale.

(3) Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y \in L$, we define

$$x \rightarrow y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,
$x \odot y \leq z$ iff $x \leq (y \rightarrow z)$.

Lemma 1.3. [13] Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$.

(4) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(5) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$.

(6) $(y \rightarrow z) \leq (x \odot y) \rightarrow (x \odot z)$.

(7) $(y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$ and $(y \rightarrow x) \leq (x \rightarrow z) \rightarrow (y \rightarrow z)$.

(8) $(x_i \rightarrow y_i) \leq (\bigwedge_{i\in\Gamma} x_i) \rightarrow (\bigwedge_{i\in\Gamma} y_i)$.

(9) $(x_i \rightarrow y_i) \leq (\bigvee_{i\in\Gamma} x_i) \rightarrow (\bigvee_{i\in\Gamma} y_i)$.

Definition 1.4. [6] A mapping $\tau : L^X \rightarrow L$ is called an *L-fuzzy topology* on X if it satisfies the following conditions:

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$ where $\alpha \in L$, $\overline{\alpha}(x) = \alpha$ for each $x \in X$.

(O2) $\tau(\mu_1 \odot \mu_2) \geq \tau(\mu_1) \odot \tau(\mu_2)$, for any $\mu_1, \mu_2 \in L^X$.

(O3) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma} \subset L^X$.

An L-fuzzy topology is called *enriched* if

(E) $\tau(\alpha \odot \lambda) \geq \tau(\lambda)$ for each $\lambda \in L^X$ and $\alpha \in L$.

The pair (X, τ) is called an (resp. enriched) *L-fuzzy topological space*.

Let τ_1 and τ_2 be L-fuzzy topologies on X. We say τ_1 is *finer* than τ_2 (or τ_2 is *coarser* than τ_1), denoted by $\tau_2 \leq \tau_1$, iff $\tau_2(\lambda) \leq \tau_1(\lambda)$ for all $\lambda \in L^X$. Let (X, τ_1) and (Y, τ_2) be two L-fuzzy topological spaces. A mapping $\psi : X \to Y$ is said to be *LF-continuous* iff $\tau_2(\mu) \leq \tau_1(\psi^{\leftarrow}(\mu))$ for each $\mu \in L^Y$.

2. The properties of L-filters

Definition 2.1. [6] A mapping $\mathcal{F} : L^X \to L$ is called an *L-filter* on X if it satisfies the following conditions:

(F1) $\mathcal{F}(\bar{0}) = 0$ and $\mathcal{F}(\bar{1}) = 1$.

(F2) $\mathcal{F}(\lambda \odot \mu) \geq \mathcal{F}(\lambda) \odot \mathcal{F}(\mu)$, for each $\lambda, \mu \in L^X$.

(F3) If $\lambda \leq \mu$, $\mathcal{F}(\lambda) \leq \mathcal{F}(\mu)$.

An L-filter is called *stratified* if

(S) $\mathcal{F}(\alpha \odot \lambda) \geq \alpha \odot \mathcal{F}(\lambda)$ for each $\lambda \in L^X$ and $\alpha \in L$.

The pair $(X, \mathcal{F})$ is called an (resp. stratified) *L-filter space*.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be L-filters on X. We say $\mathcal{F}_1$ is *finer* than $\mathcal{F}_2$ (or $\mathcal{F}_2$ is *coarser* than $\mathcal{F}_1$), denoted by $\mathcal{F}_2 \leq \mathcal{F}_1$, iff $\mathcal{F}_2(\lambda) \leq \mathcal{F}_1(\lambda)$ for all $\lambda \in L^X$. Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be L-filter spaces. A mapping $\psi : X \to Y$ is said to be an *L-filter map* iff $\mathcal{F}_2(\mu) \leq \mathcal{F}_1(\psi^{\leftarrow}(\mu))$ for each $\mu \in L^Y$.

Theorem 2.2. Let $F = \{\mathcal{F}^x \mid x \in X\}$ be a family of L-filters for each $x \in X$. We define a map $\tau_F : L^X \to L$ as follows:

$$\tau_F(\lambda) = \bigwedge_{x \in X} (\lambda(x) \to \mathcal{F}^x(\lambda))$$

Then (1) τ_F is an L-fuzzy topology.

(2) If $\mathcal{F}^x$ is a stratified L-filter, then τ_F is an enriched L-fuzzy topology.

(3) Let $G = \{\mathcal{G}^y \mid y \in Y\}$ be a family of L-filters for each $y \in Y$ and $\psi : X \to Y$ be a map. Then each $x \in X$, $\rho \in L^Y$,

$$(\mathcal{G}^{\psi(x)}(\rho) \to \mathcal{F}^x(\psi^{\leftarrow}(\rho)) \leq \tau_G(\rho) \to \tau_F(\psi^{\leftarrow}(\rho))$$

In particular, if $\psi : (X, \mathcal{F}^x) \to (Y, \mathcal{G}^y)$ is an L-filter map each $x \in X$, then $\psi : (X, \tau_F) \to (Y, (X, \tau_G)$ is LF-continuous.

Proof. (1) (T1)

$$\begin{aligned} \tau_F(0) &= \textstyle\bigwedge_{x \in X}(0(x) \to \mathcal{F}^x(0)) = 1 \\ \tau_F(1) &= \textstyle\bigwedge_{x \in X}(1(x) \to \mathcal{F}^x(1)) = 1 \end{aligned}$$

(T2)

$$\begin{aligned} &\tau_F(\lambda \odot \mu) \\ &= \textstyle\bigwedge_{x \in X}((\lambda \odot \mu)(x) \to \mathcal{F}^x(\lambda \odot \mu)) \\ &\geq \textstyle\bigwedge_{x \in X}((\lambda(x) \odot \mu(x)) \to \mathcal{F}^x(\lambda) \odot \mathcal{F}^x(\mu)) \\ &\quad \text{(by Lemma 1.3.(5))} \\ &\geq \textstyle\bigwedge_{x \in X}\Big((\lambda(x) \to \mathcal{F}^x(\lambda)) \odot (\mu(x) \to \mathcal{F}^x(\mu))\Big) \\ &\geq \textstyle\bigwedge_{x \in X}(\lambda(x) \to \mathcal{F}^x(\lambda)) \odot \bigwedge_{x \in X}(\mu(x) \to \mathcal{F}^x(\mu)) \\ &\geq \tau_F(\lambda) \odot \tau_F(\mu). \end{aligned}$$

(T3)

$$\begin{aligned} \tau_F(\textstyle\bigvee_i \lambda_i) &= \textstyle\bigwedge_{x \in X}((\bigvee_i \lambda_i(x) \to \mathcal{F}^x(\bigvee_i \lambda_i)) \\ &\geq \textstyle\bigwedge_{x \in X}((\bigvee_i \lambda_i(x) \to \bigvee_i \mathcal{F}^x(\lambda_i)) \\ &\quad \text{(by Lemma 1.3.(9))} \\ &\geq \textstyle\bigwedge_{x \in X} \bigwedge_i (\lambda_i(x) \to \mathcal{F}^x(\lambda_i)) \\ &\geq \textstyle\bigwedge_i \bigwedge_{x \in X} (\lambda_i(x) \to \mathcal{F}^x(\lambda_i)) \\ &= \textstyle\bigwedge_i \tau_F(\lambda_i) \end{aligned}$$

(2)

$$\begin{aligned} \tau_F(\alpha \odot \lambda) &= \textstyle\bigwedge_{x \in X}(\alpha \odot \lambda(x) \to \mathcal{F}^x(\alpha \odot \lambda)) \\ &\geq \textstyle\bigwedge_{x \in X}((\alpha \odot \lambda(x)) \to (\alpha \odot \mathcal{F}^x(\lambda))) \\ &\geq \textstyle\bigwedge_{x \in X}(\lambda(x) \to \mathcal{F}^x(\lambda)) \text{ (by Lemma 1.3.(6))} \\ &\geq \tau_F(\lambda). \end{aligned}$$

(3)

$$\begin{aligned} &\tau_G(\rho) \to \tau_F(\psi^{\leftarrow}(\rho)) \\ &\geq \textstyle\bigwedge_{y \in Y}(\rho(y) \to \mathcal{G}^y(\rho)) \to \bigwedge_{x \in X}(\psi^{\leftarrow}(\rho)(x) \to \mathcal{F}^x(\psi^{\leftarrow}(\rho)) \\ &\geq \textstyle\bigwedge_{x \in X}(\psi^{\leftarrow}(\rho)(x) \to \mathcal{G}^{\psi(x)}(\rho)) \to \\ &\quad \textstyle\bigwedge_{x \in X}(\psi^{\leftarrow}(\rho)(x) \to \mathcal{F}^x(\psi^{\leftarrow}(\rho))) \text{ (by Lemma 1.3.(8))} \\ &\geq \Big((\psi^{\leftarrow}(\rho)(x) \to \mathcal{G}^{\psi(x)}(\rho)) \to \\ &\quad (\psi^{\leftarrow}(\rho)(x) \to \mathcal{F}^x(\psi^{\leftarrow}(\rho))\Big) \\ &\geq \mathcal{G}^{\psi(x)}(\rho) \to \mathcal{F}^x(\psi^{\leftarrow}(\rho))). \text{ (by Lemma 1.3.(7))} \end{aligned}$$

□

Corollary 2.3. Let $F = \{\mathcal{F}^x = \mathcal{F} \mid x \in X\}$ be a family of an L-filter. We define a map $\tau_F : L^X \to L$ as follows:

$$\tau_F(\lambda) = \bigwedge_{x \in X} (\lambda(x) \to \mathcal{F}(\lambda))$$

Then (1) τ_F is an L-fuzzy topology.

(2) If $\mathcal{F}$ is a stratified L-filter, then τ_F is an enriched L-fuzzy topology.

(3) If $\psi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is an L-filter map, then $\psi : (X, \tau_F) \to (Y, \tau_G)$ is LF-continuous.

Example 2.4. Define $\mathcal{F} : L^X \to L$ as $\mathcal{F}(\lambda) = \inf \lambda$. Since $\mathcal{F}(\lambda \odot \mu) = \inf(\lambda \odot \mu) \geq \inf(\lambda) \odot \inf(\mu)$, then $\mathcal{F}$ is an L-filter. We obtain a $[0, 1]$-fuzzy topology τ_F as follows:

$$\tau_F(\lambda) = \bigwedge_{x \in X} (\lambda(x) \to \inf \lambda)$$

Theorem 2.5. Let $\mathcal{F}$ and $\mathcal{G}$ be L-filters satisfying condition $\mathcal{F}(\lambda)\odot\mathcal{G}(\mu)=0$ for each $\lambda\odot\mu=0$. We define $\mathcal{F}\odot\mathcal{G}:L^X\to L$ as follows:

$$\mathcal{F}\odot\mathcal{G}(\rho)=\bigvee\{\mathcal{F}(\lambda)\odot\mathcal{G}(\mu)\mid\lambda\odot\mu\le\rho\}.$$

Let τ_1,τ_2 be L-fuzzy topologies on X. We define $\tau_1\odot\tau_2:L^X\to L$ as follows:

$$(\tau_1\odot\tau_2)(\rho)=\bigvee\{\tau_1(\lambda)\odot\tau_2(\mu)\mid\lambda\odot\mu=\rho\}.$$

Then (1) $\mathcal{F}\odot\mathcal{G}$ is the coarsest L-filter on X which is finer than $\mathcal{F}$ and $\mathcal{G}$.

(2) If $\mathcal{F}$ or $\mathcal{G}$ is a stratified L-filter, then $\mathcal{F}\odot\mathcal{G}$ is a stratified L-filter on X.

(3) $\tau_1\odot\tau_2$ is the coarsest L-fuzzy topology on X which is finer than τ_1 and τ_2.

(4) If τ_1 or τ_1 is an enriched L-fuzzy topology, then $\tau_1\odot\tau_2$ is an enriched L-fuzzy topology on X.

(5) $\tau_{\mathcal{F}\odot\mathcal{G}}\ge\tau_{\mathcal{F}}\odot\tau_{\mathcal{G}}$.

Proof. (1) $\mathcal{F}\odot\mathcal{G}$ is an L-filter from:

$$\begin{aligned}
&(\mathcal{F}\odot\mathcal{G})(\lambda)\odot(\mathcal{F}\odot\mathcal{G})(\mu)\\
&=\bigvee_{\lambda_1\odot\lambda_2\le\lambda}(\mathcal{F}(\lambda_1)\odot\mathcal{G}(\lambda_2))\\
&\quad\odot\bigvee_{\mu_1\odot\mu_2\le\mu}(\mathcal{F}(\mu_1)\odot\mathcal{G}(\mu_2))\\
&=\bigvee_{\lambda_1\odot\lambda_2\le\lambda,\ \mu_1\odot\mu_2\le\mu}(\mathcal{F}(\lambda_1)\odot\mathcal{G}(\lambda_2))\\
&\quad\odot(\mathcal{F}(\mu_1)\odot\mathcal{G}(\mu_2))\\
&\le\bigvee_{\lambda_1\odot\lambda_2\le\lambda,\ \mu_1\odot\mu_2\le\mu}(\mathcal{F}(\lambda_1\odot\mu_1)\odot\mathcal{G}(\lambda_2\odot\mu_2))\\
&\le(\mathcal{F}\odot\mathcal{G})(\lambda\odot\mu).
\end{aligned}$$

For $\lambda=\lambda\odot\bar{1}$, since $\mathcal{F}\odot\mathcal{G}(\lambda)\ge\mathcal{F}(\lambda)\odot\mathcal{G}(\bar{1})=\mathcal{F}(\lambda)$, $\mathcal{F}\odot\mathcal{G}\ge\mathcal{F},\mathcal{G}$. If $\mathcal{F}\le\mathcal{H}$ and $\mathcal{G}\le\mathcal{H}$, then $\mathcal{F}\odot\mathcal{G}\le\mathcal{H}$.

(2)

$$\begin{aligned}
&\alpha\odot(\mathcal{F}\odot\mathcal{G})(\rho)\\
&=\alpha\odot\bigvee\{\mathcal{F}(\lambda)\odot\mathcal{G}(\mu)\mid\lambda\odot\mu\le\rho\}\\
&=\bigvee\{\alpha\odot(\mathcal{F}(\lambda)\odot\mathcal{G}(\mu))\mid\lambda\odot\mu\le\rho\}\\
&\le\bigvee\{(\alpha\odot\mathcal{F}(\lambda))\odot\mathcal{G}(\mu)\mid\lambda\odot\mu\le\rho\}\\
&\le\bigvee\{\mathcal{F}(\alpha\odot\lambda)\odot\mathcal{G}(\mu)\mid\alpha\odot\lambda\odot\mu\le\alpha\odot\rho\}\\
&\le(\mathcal{F}\odot\mathcal{G})(\alpha\odot\rho).
\end{aligned}$$

(3) It is similarly proved as (1).

(4)

$$\begin{aligned}
&(\tau_1\odot\tau_2)(\alpha\odot h)\\
&=\bigvee\{\tau_1(h_1)\odot\tau_2(h_2)\mid h_1\odot h_2=\alpha\odot h\}\\
&\ge\bigvee\{\tau_1(\alpha\odot f)\odot\tau_2(g)\mid\alpha\odot\lambda\odot\mu=\alpha\odot h\}\\
&\ge\bigvee\{\tau_1(f)\odot\tau_2(g)\mid\alpha\odot\lambda\odot\mu=h\}\\
&=(\tau_1\odot\tau_2)(h).
\end{aligned}$$

(5)

$$\begin{aligned}
&\tau_{\mathcal{F}\odot\mathcal{G}}(\rho)\\
&=\bigwedge_{x\in X}(\rho(x)\to(\mathcal{F}\odot\mathcal{G})(\rho))\\
&\ge\bigwedge_{x\in X}(\bigvee_{\lambda\odot\mu\le\rho}(\lambda\odot\mu)(x)\\
&\quad\to\bigvee_{\lambda\odot\mu\le\rho}(\mathcal{F}(\lambda)\odot\mathcal{G}(\mu)))\\
&\ge\bigwedge_{x\in X}(\bigvee_{\lambda\odot\mu\le\rho}(\lambda(x)\odot\mu(x))\to\bigvee_{\lambda\odot\mu\le\rho}(\mathcal{F}(\lambda)\odot\mathcal{G}(\mu)))\\
&\ge\bigwedge_{x\in X}(\bigvee_{\lambda\odot\mu\le\rho}(\lambda(x)\odot\mu(x))\to\bigvee_{\lambda\odot\mu\le\rho}(\mathcal{F}(\lambda)\odot\mathcal{G}(\mu)))\\
&\ge\bigwedge_{x\in X}\bigvee_{\lambda\odot\mu\le\rho}((\lambda(x)\to\mathcal{F}(\lambda))\odot(\mu(x)\to\mathcal{G}(\mu)))\\
&\ge\bigvee_{\lambda\odot\mu\le\rho}(\bigwedge_{x\in X}(\lambda(x)\to\mathcal{F}(\lambda))\\
&\odot\bigwedge_{x\in X}(\mu(x)\to\mathcal{G}(\mu))))\\
&=(\tau_{\mathcal{F}}\odot\tau_{\mathcal{G}})(\rho).
\end{aligned}$$

□

Theorem 2.6. Let (X,τ) be an L-fuzzy topological space. We define a map $\mathcal{N}^x_\tau:L^X\to L$ as follows:

$$\mathcal{N}^x_\tau(\lambda)=\bigvee_{\mu\le\lambda}(\mu(x)\odot\tau(\mu))$$

Then (1) $\mathcal{N}^x_\tau$ is an L-filter.

(2) If τ is an enriched L-fuzzy topology, then $\mathcal{N}^x_\tau$ is a stratified L-filter

(3) Let (X,τ_X) and (Y,τ_Y) be L-fuzzy topological spaces and $\psi:X\to Y$ be a map. Then for $\mu\in L^Y$,

$$\tau_Y(\mu)\to\tau_X(\psi^{\leftarrow}(\mu))\le\mathcal{N}^{\psi(x)}_{\tau_Y}(\mu)\to\mathcal{N}^x_{\tau_Y}(\psi^{\leftarrow}(\mu))$$

In particular, if ψ is LF continuous , then ψ is an L-filter map.

(4) $\tau_{\mathcal{N}^x_\tau}\ge\tau$ and $\mathcal{N}^x_{\tau_{\mathcal{F}}}\le\mathcal{F}$.

(5) $\mathcal{N}^x_{\tau_1\odot\tau_2}\ge\mathcal{N}^x_{\tau_1}\odot\mathcal{N}^x_{\tau_2}$.

Proof. (1) and (2) follow from Theorem 3.6 in [14].

(3)

$$\begin{aligned}
&\mathcal{N}^{\psi(x)}_{\tau_Y}(\mu)\to\mathcal{N}^x_{\tau_X}(\psi^{\leftarrow}(\mu))\\
&=\Big(\bigvee_{\rho\le\mu}\rho(y)\odot\tau_Y(\rho))\to\bigvee_{\nu\le\psi^{\leftarrow}(\mu)}(\nu(x)\odot\tau_X(\nu)\Big)\\
&\ge\Big(\bigvee_{\rho\le\mu}\rho(\psi(x))\odot\tau_Y(\rho))\to\\
&\quad\bigvee_{\psi^{\leftarrow}(\rho)\le\psi^{\leftarrow}(\mu)}(\psi^{\leftarrow}(\rho)(x)\odot\tau_X(\psi^{\leftarrow}(\rho))\Big)\\
&\ge\Big(\bigvee_{\rho\le\mu}\rho(\psi(x))\odot\tau_Y(\rho))\to\\
&\quad\bigvee_{\psi^{\leftarrow}(\rho)\le\psi^{\leftarrow}(\mu)}(\rho(\psi(x))\odot\tau_X(\psi^{\leftarrow}(\rho))\Big)\\
&\ge(\mu(\psi(x))\odot\tau_Y(\mu))\to(\mu(\psi(x))\odot\tau_X(\psi^{\leftarrow}(\mu)))\\
&\ge\tau_Y(\mu)\to\tau_X(\psi^{\leftarrow}(\mu))
\end{aligned}$$

(4)

$$\begin{aligned}
\tau_{\mathcal{N}^x_\tau}(\lambda)&=\bigwedge\Big(\lambda(x)\to\bigvee_{\mu\le\lambda}(\mu(x)\odot\tau(\mu))\Big)\\
&\ge\bigwedge\Big(\lambda(x)\to(\lambda(x)\odot\tau(\lambda))\Big)\\
&\ge\tau(\lambda)
\end{aligned}$$

$$\begin{aligned}\mathcal{N}_{\tau_{\mathcal{F}}}(\lambda) &= \bigvee_{\mu\leq\lambda}\Big(\mu(x)\odot\bigwedge_{x\in X}(\mu(x)\rightarrow\mathcal{F}(\mu))\Big)\\ &\leq \bigvee_{\mu\leq\lambda}\Big(\mu(x)\odot(\mu(x)\rightarrow\mathcal{F}(\mu))\Big)\\ &\leq \bigvee_{\mu\leq\lambda}\mathcal{F}(\mu)=\mathcal{F}(\lambda).\end{aligned}$$

(5)

$$\begin{aligned}&(\mathcal{N}^x_{\tau_1}\odot\mathcal{N}^x_{\tau_2})(\rho)\\ &=\bigvee_{\lambda\odot\mu\leq\rho}(\mathcal{N}^x_{\tau_1}(\lambda)\odot\mathcal{N}^x_{\tau_2}(\mu))\\ &=\bigvee_{\lambda\odot\mu\leq\rho}\{(\bigvee_{\lambda_1\leq\lambda}(\lambda_1(x)\odot\tau_1(\lambda_1))\\ &\quad\odot(\bigvee_{\mu_1\leq\mu}(\mu_1(x)\odot\tau_2(\mu_1))\}\\ &\leq\bigvee_{\lambda\odot\mu\leq\rho}\{(\bigvee_{\lambda_1\odot\mu_1\leq\lambda\odot\mu}(\lambda_1\odot\mu_1)(x)\\ &\quad\odot(\tau_1(\lambda_1)\odot\tau_2(\mu_1))\}\\ &\leq\bigvee\{(\bigvee(\lambda\odot\mu)(x)\\ &\quad\odot(\tau_1\odot\tau_2)(\lambda\odot\mu)\}\\ &\leq\mathcal{N}^x_{\tau_1\odot\tau_2}(\rho).\end{aligned}$$

□

Example 2.7. Let $X=\{x,y\}$ be a set and $L=[0,1]$ an unit interval. Define a binary operation $\odot$ (called Lukasiewicz structure) on $[0,1]$ by

$$x\odot y=\max\{0,x+y-1\}.$$

Then $([0,1],\vee,\wedge,\odot,0,1)$ is a scq-lattice (ref.[8,10]).

(1) Let $\mu\in[0,1]^X$ as follows:

$$\mu(x)=0.6,\ \mu(y)=0.5.$$

Then $\mu\odot\mu(x)=0.2$, $\mu\odot\mu(y)=0$. We define an $[0,1]$-fuzzy topology $\tau:[0,1]^X\rightarrow[0,1]$ as follows:

$$\tau(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{0}\text{ or }\overline{1},\\ 0.6, & \text{if }\lambda=\mu,\\ 0.3, & \text{if }\lambda=\mu\odot\mu,\\ 0, & \text{otherwise.}\end{cases}$$

We obtain $[0,1]$-filters $\mathcal{N}^x_\tau,\mathcal{N}^y_\tau:[0,1]^X\rightarrow[0,1]$ as follows:

$$\mathcal{N}^x_\tau(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ 0.2, & \text{if }\mu\leq\lambda\neq\overline{1},\\ 0, & \text{otherwise.}\end{cases}$$

$$\mathcal{N}^y_\tau(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ 0.1, & \text{if }\mu\leq\lambda\neq\overline{1},\\ 0, & \text{otherwise.}\end{cases}$$

We obtain $[0,1]$-fuzzy topologies as follows:

$$\tau_{\mathcal{N}^x_\tau}(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ 1.2-\lambda(x), & \text{if }\mu\leq\lambda\neq\overline{1},\\ 1-\lambda(x), & \text{otherwise.}\end{cases}$$

$$\tau_{\mathcal{N}^y_\tau}(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ 1.2-\lambda(y), & \text{if }\mu\leq\lambda\neq\overline{1},\\ 1-\lambda(y), & \text{otherwise.}\end{cases}$$

Then $\tau_{\mathcal{N}^x_\tau}\geq\tau$ and $\tau_{\mathcal{N}^y_\tau}\geq\tau$.

Let $F=\{\mathcal{N}^x_\tau,\mathcal{N}^y_\tau\}$ be a family of $[0,1]$-filters. We obtain a $[0,1]$-fuzzy topology as follows:

$$\tau_F(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ (1.2-\lambda(x))\wedge(1.2-\lambda(y)), & \text{if }\mu\leq\lambda\neq\overline{1},\\ (1-\lambda(x))\wedge(1-\lambda(y)), & \text{otherwise.}\end{cases}$$

(2) Let $\mu\in[0,1]^X$ as follows:

$$\mu(x)=0.3,\ \mu(y)=0.5.$$

We define $[0,1]$-filter $\mathcal{F}:[0,1]^X\rightarrow[0,1]$ as follows:

$$\mathcal{F}(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ 0.4, & \text{if }\mu\leq\lambda\neq\overline{1},\\ 0, & \text{otherwise.}\end{cases}$$

We obtain $[0,1]$-fuzzy topology $\tau_{\mathcal{F}}:[0,1]^X\rightarrow[0,1]$ as follows:

$$\tau_{\mathcal{F}}(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ (1.4-\lambda(x))\wedge(1.4-\lambda(y)), & \text{if }\mu\leq\lambda\neq\overline{1},\\ (1-\lambda(x))\wedge(1-\lambda(y)), & \text{otherwise.}\end{cases}$$

We obtain $[0,1]$-filters $\mathcal{N}^x_{\tau_{\mathcal{F}}},\mathcal{N}^y_{\tau_{\mathcal{F}}}:[0,1]^X\rightarrow[0,1]$ as follows:

$$\mathcal{N}^x_{\tau_{\mathcal{F}}}(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ 0.4\wedge(\lambda(x)-\lambda(y)+0.4), & \text{if }\mu\leq\lambda\neq\overline{1},\\ 0, & \text{otherwise.}\end{cases}$$

$$\mathcal{N}^y_{\tau_{\mathcal{F}}}(\lambda)=\begin{cases}1, & \text{if }\lambda=\overline{1},\\ 0.4\wedge(\lambda(y)-\lambda(x)+0.4), & \text{if }\mu\leq\lambda\neq\overline{1},\\ 0, & \text{otherwise.}\end{cases}$$

Definition 2.8. Let (X,τ) be an L-fuzzy topological space, $\mathcal{F}$ an L-filter , $\lambda,\mu\in L^X$ and $x\in X$.

(1) $\mathcal{N}^x_\tau$ is called the *L-fuzzy neighborhood filter* at x.

(2) x is called a *cluster point* of $\mathcal{F}$, denoted by $\mathcal{F}\propto x$, if for every $\mathcal{N}^x_\tau(\mu)\odot\mathcal{F}(\lambda)\neq\overline{0}$, we have $\mu\odot\lambda\neq\overline{0}$.

(3) x_t is called a *limit point* of $\mathcal{F}$, denoted by $\mathcal{F}\rightarrow x$, if $\mathcal{N}^x_\tau\leq\mathcal{F}$.

We denote

$$clu_\tau(\mathcal{F})=\bigcup\{x\in X\mid x\text{ is a cluster point of }\mathcal{F}\},$$

$$lim_\tau(\mathcal{F})=\bigcup\{x\in X\mid x\text{ is a limit point of }\mathcal{F}\}.$$

Theorem 2.9. Let (X,τ) be an L-fuzzy topological space. Let $\mathcal{F}$ and $\mathcal{G}$ be L-filters on X which $\mathcal{F}$ is coarser than $\mathcal{G}$. For each $x\in X$, the following properties hold:

(1) If $\mathcal{F}\rightarrow x$, then $\mathcal{F}\propto x$.

(2) $lim_\tau(\mathcal{F})\leq clu_\tau(\mathcal{F})$.

(3) If $\mathcal{F}\rightarrow x$, then $\mathcal{G}\rightarrow x$.

(4) $lim_\tau(\mathcal{F})\leq lim_\tau(\mathcal{G})$.

(5) If $\mathcal{G}\propto x$, then $\mathcal{F}\propto x$.

(6) $clu_\tau(\mathcal{G})\leq clu_\tau(\mathcal{F})$.

Proof. (1) Let $\mathcal{F} \to x$. For every $\mathcal{N}_\tau^x(\mu) \odot \mathcal{F}(\lambda) \neq \overline{0}$, since $\mathcal{N}_\mathcal{T}^x \leq \mathcal{F}$,

$$\mathcal{F}(\mu \odot \lambda) \geq \mathcal{F}(\mu) \odot \mathcal{F}(\lambda) \geq \mathcal{N}_\mathcal{T}^x(\mu) \odot \mathcal{F}(\lambda) \neq 0$$

Hence $\mathcal{F}(\mu \odot \lambda) \neq 0$. It implies $\mu \odot \lambda \neq \overline{0}$. Thus, $\mathcal{F} \infty x$.

(2) From (1), it is clear.

(3) It is easily proved from $\mathcal{N}_\tau^x \leq \mathcal{F} \leq \mathcal{G}$.

(4) From (3), it is clear.

(5) For every $\mathcal{N}_\tau^x(\mu) \odot \mathcal{F}(\lambda) \neq \overline{0}$, since $\mathcal{F} \leq \mathcal{G}$,

$$\mathcal{N}_\tau^x(\mu) \odot \mathcal{G}(\lambda) \geq \mathcal{N}_\tau^x(\mu) \odot \mathcal{F}(\lambda) \neq 0.$$

Since $\mathcal{G} \infty x$, we have $\mu \odot \lambda \neq \overline{0}$. Thus, $\mathcal{F} \infty x$.

(6) From (5), it is clear.

□

Example 2.10. Let $\mathcal{N}_\tau^x$ and $\mathcal{N}_\tau^x$ be given as same in Example 2.7 (1).

(1) Let $\mathcal{F}_1(\lambda) = \inf \lambda$ be given. If $\mu \leq \lambda \neq \overline{1}$, then $\mathcal{F}_1(\lambda) = \inf \lambda \geq \inf \mu = 0.5$, $\mathcal{N}_\tau^x(\lambda) = 0.2$ and $\mathcal{N}_\tau^y(\lambda) = 0.1$. Hence $\mathcal{N}_\tau^x, \mathcal{N}_\tau^y \leq \mathcal{F}_1$. So, $\mathcal{F}_1 \to x, y$. Thus, $lim_\tau(\mathcal{F}_1) = clu_\tau(\mathcal{F}_1) = \{x, y\}$.

(2) Let $\mathcal{F}_2(\lambda) = 3 \inf \lambda$ be given. For $\mathcal{N}_\tau^x(\mu) \odot \mathcal{F}_2(\overline{0.3}) = 0.1 \neq \overline{0}$, we have $\mu \odot \overline{0.3} = \overline{0}$. So, $\mathcal{F}_2 \not\infty x$. For every $\mathcal{N}_\tau^y(\mu) \odot \mathcal{F}_2(\overline{0.4}) \neq \overline{1}$, we have $\mu \odot \overline{0.4} = \overline{0}$. So, $\mathcal{F}_2 / \infty y$. From Theorem 2.9 (1-2), $lim_\tau(\mathcal{F}_1) = clu_\tau(\mathcal{F}_1) = \emptyset$.

(3) Let $[0,1]$-filter $\mathcal{F}_3 : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{F}_3(la) = \begin{cases} 1, & \text{if } \lambda = \overline{1}, \\ \frac{1}{3} \inf \lambda, & \text{otherwise,} \end{cases}$$

Since $0.2 = \mathcal{N}_\tau^x(\mu) \not\leq \mathcal{F}_3(\mu) = \frac{5}{3}$, then $\mathcal{F}_3 \not\to x$. Since $\mathcal{N}_\tau^y \leq \mathcal{F}_3$, then $\mathcal{F}_3 \to y$. For every $\mathcal{N}_\tau^x(\mu) \odot \mathcal{F}_3(\lambda) \neq \overline{0}$, we have $\mu \odot \overline{1} \neq \overline{0}$. So, $\mathcal{F}_3 \infty x$. Hence $lim_\tau(\mathcal{F}_3) = \{y\}$, $clu_\tau(\mathcal{F}_3) = \{x, y\}$.

Theorem 2.11. Let (X, τ) be an L-fuzzy topological space and $\mathcal{F}$ an L-filter . Then $\mathcal{F} \infty x$ iff $\mathcal{F}$ has a finer L-filter $\mathcal{G}$ such that $\mathcal{G} \to x$.

Proof. Since $\mathcal{F} \infty x$, for every $\mathcal{N}_\tau^x(\mu) \odot \mathcal{F}(\lambda) \neq \overline{0}$, we have $\mu \odot \lambda \neq \overline{0}$. From Theorem 2.5, there exists an L-fuzzy filter $\mathcal{G} = \mathcal{N}_\tau^x \odot \mathcal{F}$ such that $\mathcal{N}_\tau^x \leq \mathcal{G}$ and $\mathcal{F} \leq \mathcal{G}$.

Conversely, since $\mathcal{N}_\tau^x \leq \mathcal{G}$ and $\mathcal{F} \leq \mathcal{G}$, for each $\mathcal{N}_\tau^x(\mu) \odot \mathcal{F}(\lambda) \neq \overline{0}$,

$$\mathcal{G}(\mu \odot \lambda) \geq \mathcal{G}(\mu) \odot \mathcal{G}(\lambda) \geq \mathcal{N}_\mathcal{T}^x(\mu) \odot \mathcal{F}(\lambda) \neq \overline{0}$$

Hence $\mathcal{G}(\mu \odot \lambda) \neq 0$ implies $\mu \odot \lambda \neq \overline{0}$. It follows $\mathcal{F} \infty x_t$.

□

Theorem 2.12. Let $\psi : X \to Y$ be a map and $\mathcal{F}$ an L-filter on X. We define a map $\psi(\mathcal{F}) : L^Y \to L$ as $\psi(\mathcal{F})(\mu) = \mathcal{F}(\psi^{\leftarrow}(\mu))$. Then $\psi(\mathcal{F})$ is the finest L-filter on Y for which each $\psi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is an L-filter map.

Proof. (F1) and (F3) are easy.

(F2) For $\lambda, \mu \in L^Y$, we have

$$\begin{aligned} \psi(\mathcal{F})(\lambda \odot \mu) &= \mathcal{F}(\psi^{\leftarrow}(\lambda \odot \mu)) \\ &= \mathcal{F}(\psi^{\leftarrow}(\lambda) \odot \psi^{\leftarrow}(\mu)) \\ &\geq \mathcal{F}(\psi^{\leftarrow}(\lambda)) \odot \mathcal{F}(\psi^{\leftarrow}(\mu)) \\ &= \psi(\mathcal{F})(\lambda) \odot \psi(\mathcal{F})(\mu). \end{aligned}$$

Hence $\psi(\mathcal{F})$ is an L-filter. Also, $\psi : (X, \mathcal{F}) \to (Y, \psi(\mathcal{F}))$ is an L-filter map. Since $\mathcal{G}(\mu) \leq \mathcal{F}(\psi^{\leftarrow}(\mu)) = \psi(\mathcal{F})(\mu)$, then $\psi(\mathcal{F})$ is the finest L-filter on Y for which each $\psi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is an L-filter map.

□

Theorem 2.13. Let (X, τ_1) and (Y, τ_2) be L-fuzzy topological spaces. Let $\psi : (X, \tau_1) \to (Y, \tau_2)$ be LF-continuous. For each L-filter $\mathcal{F}$ in X, $x \in X$ and $\mu \in L^Y$, we have the following statements:

(1) $\mathcal{N}_{\tau_2}^{\psi(x)}(\mu) \leq \mathcal{N}_{\tau_1}^x(\psi^{\leftarrow}(\mu))$.

(2) If $\mathcal{F} \to x$, then $\psi(\mathcal{F}) \to \psi(x)$.

(3) $\psi(lim_{\mathcal{T}_1}(\mathcal{F})) \leq lim_{\mathcal{T}_2}(\psi(\mathcal{F}))$.

(4) If $\mathcal{F} \infty x$, then $\psi(\mathcal{F}) \infty f(x)$.

(5) $\psi(clu_{\tau_1}(\mathcal{F})) \leq clu_{\tau_2}(f(\mathcal{F}))$.

Proof. (1) It follows from Theorem 2.6(3).

(2) Let $\mathcal{F}$ be an L-filter and $x \in X$ such that $\mathcal{F} \to x$. Since $\mathcal{F} \to x$, we have $\mathcal{N}_{\tau_1}^x \leq \mathcal{F}$. Since $\mathcal{N}_{\tau_2}^{\psi(x)}(\mu) \leq \mathcal{N}_{\tau_1}^x(\psi^{\leftarrow}(\mu))$ from (1), we have

$$\mathcal{N}_{\tau_2}^{\psi(x)}(\mu) \leq \mathcal{N}_{\tau_1}^x(\psi^{\leftarrow}(\mu)) \leq \mathcal{F}(\psi^{\leftarrow}(\mu)) = \psi(\mathcal{F})(\mu).$$

Thus, $\psi(\mathcal{F}) \to \psi(x)$.

(3) It is trivial from (2).

(4) Let $\mathcal{F} \infty x$. From Theorem 2.11, there exists an L-filter $\mathcal{G}$ such that $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \to x$. Since ψ is LF-continuous, by (3), $\psi(\mathcal{G}) \to \psi(x)$. By Theorem 2.9(1), $\psi(\mathcal{G}) \infty \psi(x)$. Since $\mathcal{F} \leq \mathcal{G}$ implies $\psi(\mathcal{F}) \leq \psi(\mathcal{G})$, by Theorem 2.9(5), $\psi(\mathcal{F}) \infty \psi(x)$.

(5) It is trivial from (4).

□

The converses of Theorem 2.13 are not true from the following example.

Example 2.14. Let $X = \{x, y\}$ be a set. We define $[0,1]$-fuzzy topologies $\tau_1, \tau_2 : [0,1]^X \to [0,1]$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ 0.6, & \text{if } \lambda = \overline{0.6}, \\ 0.3, & \text{if } \lambda = \overline{0.2}, \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ 0.6, & \text{if } \lambda = \overline{0.6}, \\ 0.5, & \text{if } \lambda = \overline{0.2}, \\ 0, & \text{otherwise} \end{cases}$$

We obtain $[0,1]$-filters $\mathcal{N}_{\tau_i}^x = \mathcal{N}_{\tau_i}^y : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{N}_{\tau_1}^x(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{1}, \\ 0.2, & \text{if } \overline{0.6} \leq \lambda \neq \overline{1}, \\ 0, & \text{otherwise,} \end{cases}$$

An identity map $id_X : (X, \tau_1) \to (X, \tau_2)$ is not LF-continuous because $0.5 = \tau_2(\overline{0.2}) \not\leq \tau_1(\overline{0.2}) = 0.3$. We have the following properties:

(1) $\mathcal{N}^x_{\tau_2} = \mathcal{N}^x_{\tau_1} = \mathcal{N}^y_{\tau_2} = \mathcal{N}^y_{\tau_1}$ for each $x, y \in X$.

(2) $\mathcal{F} \infty x$ iff $id_X(\mathcal{F}) = \mathcal{F} \infty x$.

(3) $\mathcal{F} \to x$ iff $id_X(\mathcal{F}) = \mathcal{F} \to x$.

(4) $lim_{\tau_1}(\mathcal{F}) = lim_{\tau_2}(\mathcal{F})$ and $clu_{\tau_1}(\mathcal{F}) = clu_{\tau_2}(\mathcal{F})$

Hence the converses of Theorem 2.13 is not true.

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