INTUITIONISTIC FUZZY NORMAL SUBGROUP AND INTUITIONISTIC FUZZY ⊙ - CONGRUENCES

Kul Hur, So Ra Kim AND Pyung Ki Lim

Division of Mathematics and Informational Statistics, and Nanoscale Science and Tecchnology Institute, Wonkwang University, Iksan, Chonbuk, Korea 570-749

Abstract

We unite the two con concepts - normality We unite the two con concepts - normality and congruence - in an intuitionistic fuzzy subgroup setting. In particular, we prove that every intuitionistic fuzzy congruence determines an intuitionistic fuzzy subgroup. Conversely, given an intuitionistic fuzzy normal subgroup, we can associate an intuitionistic fuzzy congruence. The association between intuitionistic fuzzy normal sbgroups and intuitionistic fuzzy congruences is bijective and unigue. This leads to a new concept of cosets and a corresponding concept of guotient.

Keywords and phrases : intuitionistic fuzzy normal subgroup, intuitionistic fuzzy congruence.

1. Introduction

In 1986, Atanassov[1] introduced the notion of intuitionistic fuzzy sets as the generalization of fuzzy sets defined by Zodeh[23]. Since then others have studied intuitionistic fuzzy subrings [2], intuitionistic fuzzy subgroups [3,10-12,15], intuitionistic fuzzy topolgical spaces [5,6,13,22], and intuitionistic fuzzy topological groups[14] in various contexts. On the other hand, intuitionistic fuzzy relations [7,16] have also been investigated since a definition was introduced by Bustince and Burillo [4]. More recently, Hur et.al. studied intuitionistic fuzzy congruences on a lattice[17], on a groupoid [18], on a near-ring module[20] and on a semiring [21], respectively. In particular, Hur et.al.[19] investigated the lattice of intuitionistic fuzzy congruences.

In this paper, we unite the two concepts - normality and congruence - in an intuitionistic fuzzy subgroup setting. In particular, we prove that every intuitionistic fuzzy congruence determines an intuitionistic fuzzy subgroup. Conversely, given an intuitionistic fuzzy normal subgroup, we can associate an intuitionistic fuzzy congruence. The association between intuitionistic fuzzy normal sbgroups and intuitionistic fuzzy congruences is bijective and unigue. This leads to a

new concept of cosets and a corresponding concept of guotient.

2. Preliminaries

We will list some concepts and results needed in the later sections.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval[0,1] as I. Some of our results can be extended to more general lattices.

Definition 2.1 [1,5]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called *intuitionstic fuzzy set* (in short, IFS) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 2.2 [1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \supset A$.

(3) $A^c = (\nu_A, \mu_B)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\ ]A = (\mu_A, 1 - \mu_A), <> A = (1 - \nu_A, \nu_A)$.

Definition 2.3 [5]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 2.4 [10]. Let A be an IFS in a set X and let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then the set $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \mu\}$ is called a (λ, μ)- *level subset* of A.

Result 2.A [12, Proposition 2.2]. *Let A be an IFS in a set X and let*

$$(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in ImA.$$

If $\lambda_1 \leq \lambda_2$ and $\mu_1 \geq \mu_2$, then $A^{(\lambda_2,\mu_2)} \subset A^{(\lambda_1,\mu_1)}$, where Im A denotes the image of A.

Definition 2.5 [11]. Let G be a group and let $A \in$ IFS (G). Then A is called an *intuitionistic fuzzy subgroup* (in short, IFG) of G if it satisfies the following conditions :

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$.

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$ for each $x \in G$.

We will denote the set of all IFGs of G as IFG(G).

Result 2.B [11, Proposition 2.6]. *Let A be an IFG of a group G. Then $A(x^{-1}) = A(x)$ and $\mu_A(x) \leq \mu_A(e), \nu_A(x) \geq \nu_A(e)$ for each $x \in G$, where e is the identity element of G.*

Result 2.C [11, Proposition 2.17 and Proposition 2.18]. *Let A be an IFS of a group G. Then $A \in IFG(G)$ if and only if for each $(\lambda, \mu) \in ImA$, $A^{(\lambda,\mu)}$ a subgroup of G. In this case, $A^{(\lambda,\mu)}$ is called a level subgroup of G.*

Definition 2.6 [15]. Let A be an IFG of a group G and let $a \in G$ be fied. We define a complex mappings
$$Aa = (\mu_{Aa}, \nu_{Aa}) : G \to I \times I$$

and
$$aA = (\mu_{aA}, \nu_{aA}) : G \to I \times I$$
as follows respectively: for each $x \in G$,
$$Aa(x) = A(xa^{-1}) \quad and \quad aA(x) = A(a^{-1}x).$$

Then Aa [resp. aA] is called the *intuitionistic fuzzy right* [resp. *left*] *coset* of G determined by a and A.

Definition 2.7 [12]. Let A be an IFG of a group G. Then A is called an *intuitionistic fuzzy normal subgroup* (in short, $IFNG$) if $A(xy) = A(yx)$ for any $x, y \in G$.

We will denote the set of all IFNGs of G as IFNG(G).

Result 2.D [15, Proposition 2.9]. *Let A be an IFG of a group G. Then the followings are equivalent: for any $a, x \in G$,*

(1) $\mu_A(axa^{-1}) \geq \mu_A(x)$ and $\nu_A(axa^{-1}) \leq \nu_A(x)$.

(2) $A(aya^{-1}) = A(x)$.

(3) $A \in IFNG(G)$.

(4) $aA = Aa$.

(5) $aAa^{-1} = A$

Result 2.E [15, Proposition 2.13 and proposition 2.18]. *Let A be an IFG of a group G and let $(\lambda, \mu) \in ImA$. Then $A \in IFNG(G)$ if and only if $A^{(\lambda,\mu)} \lhd G$.*

Remarks. (1) Clearly there are intuitionistic fuzzy subgroups that are not intuitionistic fuzzy normal subgroup. For if G is any non-abelian group and H is any subgroup that is not normal, then we can get intuitionistic fuzzy subgroup out of H that are not intuitionistic normal subgroups.

(2) In condition (1) of Result 2.D, it is enough to require $\mu_A(a^{-1}xa) \geq \mu_A(x)$, and $\nu_A(a^{-1}xa) \leq \nu_A(x)$ for all $a \in G$ and $x \in sup\ p(A)$, where $sup\ p(A)$ is the support of A given by $\{x \in G : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}$.

3. Intuitionistic fuzzy corgruences

Definition 3.1 [4,7]. Let X be a set. Then complex mapping $R = (\mu_R, \nu_R) : X \times X \to I \times I$ is called an *intuitionistic fuzzy relation* (in short, IFR) on X if $\mu_R(x + y) + \nu_R(x, y) \leq 1$ for each $(x, y) \in X \times X$, i.e., $R \in IFS(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 3.2 [16]. Let X be a set and let $R \in$ IFR(X). For each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, let

$$R^{\lambda,\mu} = \{(a,b) \in X \times X : \mu_R(a,b)$$
$$\geq \lambda \text{ and } \nu_R(a,b) \leq \mu\}.$$

This set is called the (λ,μ)-level subset of R.

It is clear that $R^{\lambda,\mu}$ is a relation on X.

Let G be a group, let $R \in \text{IFR}(G)$ and let $(\lambda_0, \mu_0) = (\bigvee\limits_{x,y \in G} \mu_R(x,y), \bigwedge\limits_{x,y \in G} \nu_R(x,y))$. Then we observe that $(\lambda_0, \mu_0) \in I \times I$. $(\lambda_0, \mu_0) = (0,1)$ implies that we have the *empty* relation, namely, $R(x,y) = (0,1)$ for all $x, y \in G$. From now on, we assume $(\lambda_0, \mu_0) \in (0,1] \times [0,1)$.

We can define two operations on $\text{IFR}(G) \times \text{IFR}(G)$:

Definition 3.3 [4,7]. Let $P, Q \in \text{IFR}(G)$. Then the *composition* $Q \circ P$ of P and Q, is definde as follows: for any $x, y \in G$,

$$\mu_{Q \circ P}(x,y) = \bigvee_{z \in X} [\mu_P(x,y) \wedge \mu_Q(z,y)]$$

and

$$\nu_{Q \circ P}(x,y) = \bigwedge_{z \in X} [\nu_P(x,y) \vee \nu_Q(z,y)].$$

Definition 3.4. Let $P, Q \in \text{IFR}(G)$. Then the *multiplication* $P \odot Q$ of P and Q, is defined as follows: for any $x, y \in G$,

$$\mu_{P \odot Q}(x,y) = \bigvee_{x=x_1 x_2,\ y=y_1 y_2,\ x_1,x_2,y_1,y_2 \in G} [\mu_P(x_1,x_2) \wedge \mu_Q(y_1,y_2)]$$

and

$$\nu_{P \odot Q}(x,y) = \bigwedge_{x=x_1 x_2,\ y=y_1 y_2,\ x_1,x_2,y_1,y_2 \in G} [\nu_P(x_1,x_2) \vee \nu_Q(y_1,y_2)].$$

Definition 3.5 [20,21]. Let G be a group and let $0_\sim \neq R \in \text{IFR}(G)$. Then R is called an *intuitionistic fuzzy weak equivalence relation* (in short, $IFWER$) on G if

(1) R is *intuitionistic fuzzy weakly reflexive*,i.e., for each $x \in G$, $R(x,x) = (\lambda_0, \mu_0)$.
(2) R is *intuitionistic fuzzy symmetric*,
    i.e., $R(x,y) = R(y,x)$ for any $x, y \in G$.
(3) R is *intuitionistic fuzzy transitive*,
    i.e., $R \circ R \subset R$.

We will denote the set of all IFWERs on X as $\text{IFE}_W(X)$.

It is readily checked that if R is IFWER on a group, the R is idempotent for $\circ$. i.e., $R \circ R = R$ (see Proposition 2.9 of [16]). Furthermore, for each $(\lambda, \mu) \in [0, \lambda_0] \times [\mu_0, 1]$, $R^{(x,\mu)}$ is a crisp equivalence relation on G (See Theorem 2.17 of [16]). In particular, (λ_0, μ_0) - level subset $R^{(\lambda_0, \mu_0)}$ is a crisp aquivalence

relation on G and as such yields a partition of G in the crisp sense. The (λ_0, μ_0) - level classes of G under this partition are denoted by $\dot{x}, \dot{y}, \dot{e}$ etc., containing representative elements x, y, e respectively. For each (λ_0, μ_0) - level class $\dot{x}$ for $x \in G$, an intuitionistic fuzzy set $R_{\dot{x}} : G \to I \times I$ is chefined as $R_{\dot{x}}(a) = R(x,a)$ for each $a \in G$.

Now for each $(\lambda, \mu) \in [0, \lambda_0] \times [\mu_0, 1]$, the collection $\{R_{\dot{x}}^{(\lambda,\mu)} : x \in G\}$ is a crisp partition of G. The family of intuitioistic fuzzy sets $\{R_{\dot{x}}\}$ for $x \in G$, on G associated with an intuitionistic fuzzy weak equivalence relation R, is called the *intuitionistic fuzzy partition of* G with respect to R. It is uniquely determined by R, $\bigcup\limits_{x \in G} R_{\dot{x}} = 1_\sim$ and $R_{\dot{x}} \cap R_{\dot{y}}$ means that $\mu_{R\dot{x}}(a) \wedge \mu_{R\dot{y}}(a) \leq \alpha$ and $\nu_{R\dot{x}}(a) \vee \nu_{R\dot{y}}(a) \geq \beta$ for each $a \in G$ whenever $R(x,y) = (\alpha, \beta)$ (See Theorem 2.15 of [16]).

Definition 3.6 . Let G be a group and let $R \in \text{IFE}_W(G)$. Then R is called an *intuitionistic fuzzy $\odot$ - congruence* on G if $R \odot R \subset R$.

We will denote the set of all intuitionistic fuzzy $\odot$ - congruences on G as $\text{IFC}_\odot(G)$.

The relation $R \odot R \subset R$ can be thought of as a substitution property are is well known in crisp congruence relations on a group or a general algebra. Moreover, we can interpret in the crisp case a congruence as an equivalence relation E as a subset of $G \times G$ that is at the same time a subgroup of $G \times G$. This is indeed analogously the case in the intuitionistic fuzzy case. An IFWER that is at the same time an intuitionistic fuzzy subgroup of $G \times G$ is an *intuitionistic fuzzy $\odot$ - congruence on G*. It is easily seen that for each $(\lambda, \mu) \in [0, \lambda_0] \times [\mu_0, 1]$, $R^{(\lambda, \mu)}$ is a congruence if and only if R is an intuitionistic fuzzy $\odot$ - congruence on G (See Theorem 3.6 of [20]).

4. Main results

Now we turn our attention to the relationship between intuitionistic fuzzy $\odot$ - congruences on a group G on the one hand and intuitionistic fuzzy normal subgroups on the other.

Proposition 4.1 . *Let G be a group and let $A \in IFNG(G)$. We define a complex mapping $R_A = (\mu_{R_A}, \nu_{R_A}) : G \times G \to I \times I$ as follows: for any $x, y \in G$,*

$$R_A(x,y) = A(xy^{-1}).$$

Then $R_A \in IFC_{\odot}(G)$.

Proof . Let $x, y \in G$. Then
$$\mu_{R_A}(x, y) = \mu_A(xy^{-1}) \leq \mu_A(e) = \lambda_0$$
and
$$\nu_{R_A}(x, y) = \nu_A(xy^{-1}) \geq \nu_A(e) = \mu_0.$$

Then $R_A(x, x) = A(xx^{-1}) = A(e) = (\lambda_0, \mu_0)$, for each $x \in G$. So R_A is intuitionistic fuzzy weakly reflexive. We can easily see that R_A is both intuitionistic fuzzy symmetric and intuitionistic fuzzy transitive. Hence $R_A \in IFE_W(G)$.

Now let $x, y \in G$. Then
$$\mu_{R_A \odot R_A}(x, y) = \bigvee_{x = x_1 x_2, \ y = y_1 y_2, \ x_1, x_2, y_1, y_2 \in G} [\mu_{R_A}(x_1, x_2) \wedge \mu_{R_A}(y_1, y_2)]$$
and
$$\nu_{R_A \circ R_A}(x, y) = \bigwedge_{x = x_1 x_2, \ y = y_1 y_2, \ x_1, x_2, y_1, y_2 \in G} [\nu_{R_A}(x_1, x_2) \vee \nu_{R_A}(y_1, y_2)].$$
On the other hand, for each representation of $x = x_1 x_2$ and $y = y_1 y_2$,
$$R_A(x, y) = A(xy^{-1}) = A(x_1 x_2 y_2^{-1} y_1^{-1}).$$
Then
$$\mu_A(x_1 x_2 y_2^{-1} y_1^{-1}) = \mu_A(x_1 y_1^{-1} y_1 x_2 y_2^{-1} y_1^{-1})$$
$$\geq \mu_A(x_1 y_1^{-1})$$
$$\wedge \mu_A(y_1 x_2 y_2^{-1} y_1^{-1})$$
$$[\text{Since } A \in IFG(G)]$$
$$= \mu_A(x_1 y_1^{-1}) \wedge \mu_A(x_2 y_2^{-1})$$
$$[\text{Since } A \in IFNG(G)]$$
$$= \mu_{R_A}(x_1, y_1) \wedge \mu_{R_A}(x_2, y_2)$$
and
$$\nu_A(x_1 x_2 y_2^{-1} y_1^{-1}) = \nu_A(x_1 y_1^{-1} y_1 x_2 y_2^{-1} y_1^{-1}))$$
$$\leq \nu_A(x_1 y_1^{-1})$$
$$\vee \nu_A(y_1 x_2 y_2^{-1} y_1^{-1})$$
$$= \nu_A(x_1 y_1^{-1}) \vee \nu_A(x_2 y_2^{-1})$$
$$= \nu_{R_A}(x_1 y_1) \vee \nu_{R_A}(x_2, y_2).$$
Thus $\mu_{R_A \circ R_A}(x, y) \leq \mu_{R_A}(x, y)$ and $\nu_{R_A \circ R_A}(x, y) \geq \nu_{R_A}(x, y)$. So $R_A \in IFC_{\odot}(G)$. This completes the proof. $\square$

The following is a sort of converse of the above proposition:

Proposition 4.2 . *Let G be a group and let $R \in IFC_{\odot}(G)$. Then there is an $A_R \in IFNG(G)$ such that $A_R(xy^{-1}) = R(x, y)$ for any $x, y \in G$*

Proof. Since $R \in IFC_{\odot}(G)$, R is intuitionistic fuzzy weakly reflexive. Then $R(x, x) = (\lambda_0, \mu_0)$ for each $x \in G$. By an earlier remark (made just after Definition 3.5), $R^{(\lambda_0, \mu_0)}$ is a crisp congruence on G. Let $[e]_R(\lambda_0, \mu_0)$ be the class containing the identity e in the partition of G yielded by $R^{(\lambda_0, \mu_0)}$. We define a

complex mapping $A_R = (\mu_{A_R} \nu_{A_R}) : G \to I \times I$ as follows: for each $x \in G$,
$$A_R(x) = R(x, e). \quad \text{(i) } A_R \text{ is well - defined.}$$

(ii) $R(x, e) = R^{(x^{-1}, e)}$,i.e., $A_R(x) = A_R(x^{-1})$ for each $x \in G$.

Suppose $x \in [e]_R(\lambda_0, \mu_0)$. Then $xR^{(\lambda_0, \mu_0)}e$ and $x^{-1} R^{(\lambda_0, \mu_0)} x^{-1}$.

Thus $eR^{(\lambda_0, \mu_0)} x^{-1}$. So $R(x^{-1}, e) = (\lambda_0, \mu_0) = R(x, e)$.

Suppose $x \notin [e]_R(\lambda_0, \mu_0)$. Then $\mu_R(x, e) < \lambda_0$ or $\nu_R(x, e) > \mu_0$. Since $[e]_{R^{(\lambda_0, \mu_0)}}$ is a subgroup of G, $\mu_R(x^{-1}, e) < \lambda_0$ or $\nu_R(x^{-1}, e) > \mu_0$.

Let $t_1 = \mu_R(x, e)$, $t_2 = \mu_R(x^{-1}, e)$, $s_1 = \nu_R(x, e)$ and $s_2 = \nu_R(x^{-1}, e)$.

Assume that $t_1 < t_2$ and $s_1 > s_2$. Then $x \in [e]_{R^{(t_1, s_1)}}$ and $x \notin [e]_{R^{(t_2, s_2)}}$. So $x^{-1} \in [e]_{R^{(t_2, s_2)}}$. Since $[e]_{R^{(t_2, s_2)}}$ is a subgroup of G, $[e]_{R^{(t_2, s_2)}}$. This is a contradiction. A similar contradiction arises if we assume that $t_2 < t_1$ and $s_2 > s_1$. Thus $t_1 = t_2$ and $s_1 = s_2$. So in any cases, $R(x, e) = R(x^{-1}, e)$. Hence $A_R(x) = A_R(x^{-1})$ for each $x \in G$.

(iii) $\mu_{A_R}(xy) \geq \mu_{A_R}(x) \wedge \mu_{A_R}(y)$ and $\nu_{A_R}(xy) \leq \nu_{A_R}(x) \vee \nu_{A_R}(y)$ for any $x, y \in G$.

Let $x, y \in G$. Then
$$\mu_{A_R}(x, y) = \mu_R(xy, e)$$
$$\geq \mu_R(x, e) \wedge \mu_R(y, e)$$
$$[\text{Since } R \in IFC_{\odot}(G)]$$
$$= \mu_{A_R}(x) \wedge \mu_{A_R}(y)$$
and
$$\nu_{A_R}(xy) = \nu_R(xy, e) \leq \nu_R(x, e) \vee \nu_R(y, e)$$
$$= \nu_{A_R}(x) \vee \nu_{A_R}(y).$$
Hence, by (i),(ii) and (iii), $A_R \in IFG(G)$.

(iv) $R(x, e) = R(xy, y)$ for any $x, y \in G$.

Let $t_1 = \mu_R(xy, y)$, $t_2 = \mu_R(x, e)$, $s_1 = \nu_R(xy, y)$ and $s_2 = \nu_R(x, e)$.

Assume that $t_1 > t_2$ and $s_1 < s_2$. Then, by Result 1.A, $[e]_R^{(t_1, s_1)} \subset [e]_R^{(t_2, s_2)}$. Thus
$$xyR^{(t_1, s_1)}y \text{ and } y^{-1} R^{(t_1, s_1)} y^{-1} \Rightarrow xR^{(t_1, s_1)}e.$$
So $\mu_R(x, e) \geq t_1$ and $\nu_R(x, e) \leq s_1$ and hence $t_2 \geq t_1$ and $s_2 \leq s_1$. This is a contradiction. A similar contradiction arises if $t_1 < t_2$ and $s_1 > s_2$. Thus $R(x, e) = R(xy, y)$. So
$$R(xy^{-1}, e) = R(xy^{-1}y, y) = R(x, y), i.e.,$$
$$A_R(xy^{-1}) = R(x, y).$$
(v)$A_R \in IFNG(G)$.

Let $x, a \in G$. Then
$$A_R(a^{-1}xa) = R(a^{-1}xa, e) = R(aa^{-1}xa, a)$$
$$= R(xa, a) = R(x, e)$$
$$= A_R(x).$$
This completes the proof. $\square$

Proposition 4.3 . *Let G be a group and let $R \in IFC_{\odot}(G)$. Then the collection $\{R_{\dot{x}}\} : x \in G$ is an intuitionistic fuzzy partition of G in the sense that*

$$\bigcup_{x \in G} R_{\dot{x}} = \pi_G$$

and

$$R_{\dot{x}} \cap R_{\dot{y}} \subsetneqq C_{(\lambda_0, \mu_0)} \ \ for \ all \ \dot{x} \neq \dot{y}, i.e.,$$
$$\mu_{R_{\dot{x}}}(a) \wedge \mu_{R_{\dot{y}}}(a) < x_0 \ and \ \nu_{R_{\dot{x}}}(a)$$
$$\vee \nu_{R_{\dot{y}}}(a) > \mu_0 \ for \ each \ a \in G,$$

where $C_{(a_0, \mu_0)}(x) = (\lambda_0, \mu_0)$ and $\pi_G(x) = (\lambda_0, \mu_0)$ for each $x \in G$.

Proof. It is straightforward to check that $\bigcup_{x \in G} R_{\dot{x}} = \pi_G$. Assume that for any $\dot{x} \neq \dot{y}$ there exists $a \in G$ such that $R_{\dot{x}} \cap R_{\dot{y}}(a) = (\lambda_0, \mu_0)$. Then

$$\mu_{R_{\dot{x}}}(a) \wedge \mu_{R_{\dot{y}}}(a) = \lambda_0 \ \text{and} \ \nu_{R_{\dot{x}}}(a) \vee \nu_{R_{\dot{y}}}(a) = \mu_0.$$
$$\Rightarrow \mu_{R_{\dot{x}}}(a) = \lambda_0 = \mu_{R_{\dot{y}}}(a) \ \text{and} \ \nu_{R_{\dot{x}}}(a) = \mu_0$$
$$= \nu_{R_{\dot{y}}}(a)$$
$$\Rightarrow R(x, a) = R(y, a), \ x \in \dot{x} \ \text{and} \ y \in \dot{y}$$
$$\Rightarrow x R^{(\lambda_0, \mu_0)} a \ \text{and} \ y R^{(\lambda_0, \mu_0)} a$$
$$\Rightarrow x R^{(\lambda_0, \mu_0)} y$$

[Since $R^{(\lambda_0, \mu_0)}$ is an equivalence relation]
$$\Rightarrow y \in \dot{x} = [x]_{R^{(\lambda_0, \mu_0)}} \ \text{and} \ y \in \dot{y}.$$

Thus $\dot{x} \cap \dot{y} \neq \varnothing$. So this is a contradiction to the fact that $\dot{x} \cap \dot{y} = \varnothing$. Hence $R_{\dot{x}} \cap R_{\dot{y}} \subsetneqq C_{(\lambda_0, \mu_0)}$. This completes the proof. □

Proposition 4.4. *Let $\{R_{\dot{x}} : x \in G\}$ be the intuitionistic fuzzy partition of G given in Proposition 4.3. Then $\{R_{\dot{x}} : x \in G\}$ is a group under suitably defined binary operation. Furthermore an intuitionistic fuzzy set $R_{\dot{x}}$ in G precisely the intuitionistic fuzzy left coset $xR_{\dot{e}}$ of $R_{\dot{e}}$ associated with $x \in G$, where $\dot{e} = [e]_{R^{(\lambda_0, \mu_0)}}$.*

Proof. We define a binary operation in the collection $\{R_{\dot{x}} : x \in G\}$ as follows : for any $x, y \in G$, $R_{\dot{x}} R_{\dot{y}} = R_{\dot{xy}}$,
where $\dot{xy}$ is the class containing xy for $x \in \dot{x}$ and $y \in \dot{y}$.

Proof. (i) The multiplication · is well-defined.
Let $x, x_1 \in \dot{x}$ and $y, y_1 \in \dot{y}$. We must show that
$$R(xy, a) = R(x_1 y_1, a) \ \text{for each} \ a \in G.$$
Then clearly $x R^{(\lambda_0, \mu_0)} x_1$ and $y R^{(\lambda_0, \mu_0)} y_1$. Since $R^{(\lambda_0, \mu_0)}$ is a congruence on G, $(xy) R^{(\lambda_0, \mu_0)} (x_1 y_1)$. Thus

$$x_1 y_1 \in [xy]_{R^{(\lambda_0, \mu_0)}} = \dot{xy}$$
$$\Rightarrow \ \ R(xy, x_1 y_1) = (\lambda_0, \mu_0).$$

Case1 : If $a \in \dot{xy}$, then $R(xy, a) = (\lambda_0, \mu_0) = R(x_1 y_1, a)$.
Case2 : If $a \in \dot{xy}$, then
$$\mu_R(xy, a) < \lambda_0, \ \nu_R(xy, a) > \mu_0 \ \text{and}$$
$$\mu_R(x_1 y_1, a) < \lambda_0, \ \nu_R(x_1 y_1, a) > \mu_0.$$

Thus

$$\mu_R(xy, a) \geq \mu_R(xy, x_1 y_1) \wedge \mu_R(x_1 y_1, a)$$
[Since $R \in \text{IFE}_w(G)$]
$$= \mu_R(x_1 y_1, a)$$

and

$$\nu_R(xy, a) \leq \nu_R(xy, x_1 y_1) \vee \nu_R(x_1 y_1, a)$$
$$= \nu_R(x_1 y_1, a).$$

By the similar arguments,

$$\mu_R(x_1 y_1, a) \geq \mu_R(xy, a) \ \text{and} \ \nu_R(x_1 y_1, a)$$
$$\leq \nu_R(xy, a).$$

So, in any cases. $R(xy, a) = R(x_1 y_1, a)$, i.e., $R_{\dot{xy}}(a) = R(x_1 y_1, a)$.
Hence · is well-defined.

(ii) $\{R_{\dot{x}}\}$ is a group under the binary operation defined above.
Let $a \in G$. Then
$$R_{\dot{ex}}(a) = R(ex, a) = R(x, a) = R_{\dot{x}}(a).$$
Thus $R_{\dot{ex}} = R_{\dot{x}}$, i.e., $R_{\dot{e}} R_{\dot{x}} = R_{\dot{x}}$. Similarly, $R_{\dot{x}} R_{\dot{e}} = R_{\dot{x}}$.
So $R_{\dot{e}}$ is the identity of $\{R_{\dot{x}}.\}$
Let $\dot{x}^{-1} = [x^{-1}]_{R^{(\lambda_0, \mu_0)}}$ for each $x \in G$. Then
$$R_{\dot{x}} R_{\dot{x}^{-1}} = R_{\dot{xx}^{-1}} = R_{\dot{e}}. \ [\text{Since} \ e = xx^{-1} \in \dot{x}\dot{x}^{-1}]$$
Thus $(R_{\dot{x}})^{-1} = R_{\dot{x}^{-1}}$
The associativity $(R_{\dot{x}} R_{\dot{y}}) R_{\dot{z}} = R_{\dot{x}}(R_{\dot{y}} R_{\dot{z}})$ follows form the same property in G. Hence $(\{R_{\dot{x}}\}, \cdot)$ is a group.

Finally, we recall that the intuitionistic fuzzy left coset of R associated with $x \in G, xR$, is defined by
$$(xR)(y) = R(x^{-1}y) \ \text{for each} \ y \in G.$$
It is clear that $R_{\dot{e}} = A_R$, where A_R is the intuitionistic fuzzy set in G defined in Proposition 4.2. Since $A_R \in \text{IFNG}(G)$. $R_{\dot{e}} \in \text{IFNG}(G)$. We shall show that $xR_{\dot{e}} = R_{\dot{x}}$ for each $x \in G$. Let $y \in G$. Then

$$xR_{\dot{e}}(y) = R_{\dot{e}}(x^{-1}y) = R(x^{-1}y, e)$$
$$= R(xx^{-1}y, x)$$
$$= R(y, x)$$
$$= R(x, y)$$
$$= R_{\dot{x}}(y).$$

Hence $xR_{\dot{e}} = R_{\dot{x}}$ for each $x \in G$. This completes the proof. □

Finally, the following shows that every congruence arises form an intuitionistic fuzzy normal subgroup with the partition given by the intuitionistic fuzzy left cosets. The proof follows from Proposition 4.1 and is omitted.

Proposition 4.5. *Let G be a group and let $A \in IFNG(G)$. Then there exists an $R_A \in IFC_\odot(G)$ such that the intuitionistic fuzzy partition associated with R_A is the collection $\{xA : x \in G\}$ of intuitionistic fuzzy cosets of A.*

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Kul Hur
Professor of Wonkwang University
Research Area : Fuzzy Topology, Fuzzy Algebra, Hyperspace, Category Theory
E-mail : kulhur@wonkwang.ac.kr

So Ra Kim
Graduate Student : Fuzzy Topology, Fuzzy Algebra, Dynamic Theory
E-mail : soraking@naver.com

Pyung Ki Lim
Professor of Wonkwang University
Research Area : Fuzzy Topology, Dynamic Theory
E-mail : pklim@wonkwang.ac.kr