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Fuzzy Relations and Metrics

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Abstract

We investigate the properties of fuzzy relations, metrics and $\odot$-equivalence relation on a stsc quantale lattice L and a commutative cqm-lattice. In particular, pseudo-(quasi-) metrics induce $\odot$-(quasi)-equivalence relations.

Key words : stsc-quantales, t-norm, pseudo-(quasi-) metrics, $\odot$-equivalence relations

1. Introduction and Preliminaries

Quantales introduced by Mulvey [11,12] have arisen in an analysis of the semantics of linear logic systems developed by Girard [4], which supports part of foundation of theoretic computer science. Recently, Bělohlávek [3] investigated the properties of fuzzy relations and similarities on a residual lattice. De Baets and Mesiar [1,2] studied the relations between pseudo-metrics and $\odot$-equivalences relations.

In this paper, we investigate the properties of fuzzy relations, metrics and $\odot$-equivalence relation on a stsc-quantale lattice and a commutative cqm-lattice. Non-negative functions on $X \times X$ induce fuzzy relation and metrics. In particular, pseudo-(quasi-) metrics induce $\odot$-(quasi)-equivalence relations.

Definition 1.1. [11,12] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) if it satisfies the following conditions:

(Q1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(Q2) $(L, \odot)$ is a commutative semigroup;

(Q3) $a = a \odot 1$, for each $a \in L$;

(Q4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i \in \Gamma} a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).$$

Remark 1.2. [6-8] (1) The unit interval with a left-continuous t-norm $\odot$, $([0,1], \leq, \odot)$, is a stsc-quantale.

(2) Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y \in L$, we define $x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$.

Definition 1.3. [1-3], [6-8] Let X be a set and $(L, \leq, \odot)$ a stsc-quantale. A function $R : X \times X \to L$ is called a fuzzy relation. A fuzzy relation is called:

(R1) reflexive if $R(x,x) = 1$ for all $x \in X$,

(R2) symmetric if $R(x,y) = R(y,x)$, for all $x, y \in X$,

(R3) transitive if $R(x,y) \odot R(y,z) \leq R(x,z)$, for all $x, y, z \in X$.

If R satisfies (R1) and (R2), R is an $\odot$-quasi-equivalence relation. If an $\odot$-quasi-equivalence relation R satisfies (R2), then R is an $\odot$-equivalence relation.

Definition 1.4. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a quasi-metric on X if it satisfies;

(M1) $d(x,x) = 0$ for all $x \in X$,

(M2) $d(x,y) + d(y,z) \geq d(x,z)$, for all $x, y, z \in X$.

A quasi-metric d is called a pseudo-metric if it satisfies;

(M3) $d(x,y) = d(y,x)$, for all $x, y \in X$.

A pseudo-metric d is called a metric if it satisfies;

(M) if $d(x,y) = 0$, then $x = y$.

Theorem 1.5. [10] Let $R_1, R_2 \in L^{X \times X}$ be fuzzy relations. The compositions of R_1 and R_2 are defined as

$$R_1 \circ R_2(x,z) = \bigvee_{y \in X} R_1(x,y) \odot R_2(y,z)$$

$$(R_1 \Rightarrow R_2)(x,z) = \bigwedge_{y \in X} (R_1(x,y) \to R_2(y,z))$$

$$(R_1 \Leftarrow R_2)(x,z) = \bigwedge_{y \in X} (R_2(y,z) \to R_1(x,y))$$

$$(R_1 \Leftrightarrow R_2)(x,z) = \bigwedge_{y \in X} (R_1(x,y) \leftrightarrow R_2(y,z)).$$

$$R_1^s(y,x) = R_1(x,y).$$

Then we have the following properties.

(1) $(R_1 \circ R_2)^s = R_2^s \circ R_1^s$.

(2) $(R_1 \Rightarrow R_2)^s = R_2^s \Leftarrow R_1^s$ and $(R_1 \Leftarrow R_2)^s = R_2^s \Rightarrow R_1^s$.

(3) $(R_1 \Leftrightarrow R_2)^s = R_2^s \Leftrightarrow R_1^s$.

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Definition 1.6. [10] Let $(L, \odot)$ be a stsc-quantale. A function $T : L \to L$ is called an equivalence transformation map if it satisfies the following conditions:

(1) $T(1) = 1$,

(2) if $x \le y$, then $T(x) \le T(y)$,

(3) $T(x) \odot T(y) \le T(x \odot y)$.

Theorem 1.7. [10] Let R be an $\odot$-equivalence relation and T an equivalence transformation map. Then $T \circ R$ is an $\odot$-equivalence relation.

2. Fuzzy Relations and Metrics

Theorem 2.1. Let $d, d_1, d_2 \in [0, \infty]^{X \times X}$ be non-negative functions. We define $d_1 \uplus d_2 \in [0, \infty]^{X \times X}$ as follows:

$$d_1 \uplus d_2(x, z) = \bigwedge_{y \in X} (d_1(x, y) + d_2(y, z))$$

$$d_1 \rhd d_2(x, z) = \bigvee_{y \in X} ((d_1(x, y) - d_2(y, z)) \vee 0)$$

$$d_1 \lhd d_2(x, z) = \bigvee_{y \in X} ((d_2(y, z) - d_1(x, y)) \vee 0)$$

$$d_1 \diamond d_2(x, z) = \bigvee_{y \in X} |d_1(x, y) - d_2(y, z)|$$

$$\infty + a = a + \infty = \infty, \forall a \in [0, \infty].$$

$$\infty - \infty = 0.$$

We have the following properties.

(1) $(d_1 \uplus d_2)^s = d_2^s \uplus d_1^s$ where $d_i^s(x, y) = d_i(y, x)$ for each $x, y \in X$.

(2) $(d_1 \rhd d_2)^s = d_2^s \lhd d_1^s$ and $(d_1 \lhd d_2)^s = d_2^s \rhd d_1^s$.

(3) $d_1 \uplus (d_2 \rhd d_3) \ge (d_1 \uplus d_2) \rhd d_3$.

(4) $d_1 \lhd (d_2 \uplus d_3) \le (d_1 \lhd d_2) \uplus d_3$.

(5) $(d_1 \lhd d_2) \lhd d_3 \le d_1 \uplus (d_2 \lhd d_3)$.

(6) $d_1 \rhd (d_2 \rhd d_3) \le (d_1 \rhd d_2) \uplus d_3$.

(7) $(d_1 \lhd d_2) \rhd d_3 = d_1 \lhd (d_2 \rhd d_3)$.

(8) $d_1 \lhd (d_2 \lhd d_3) = (d_1 \uplus d_2) \lhd d_3$.

(9) $(d_1 \rhd d_2) \rhd d_3 = d_1 \rhd (d_2 \uplus d_3)$.

(10) If d_i is a (quasi-, pseudo-) metric for each $i = 1, 2$, then $d_1 \vee d_2$ and $d_1 + d_2$ are (quasi-, pseudo-) metric.

(11) Let d_1 and d_2 be quasi-metrics. $d_1 \wedge d_2$ is a quasi-metric iff $d_1 \uplus d_2 \ge d_1 \wedge d_2$ and $d_2 \uplus d_1 \ge d_1 \wedge d_2$.

(12) Let d_1 and d_2 be metrics. $d_1 \uplus d_2$ is a metric iff $d_1 \uplus d_2 = d_2 \uplus d_1$.

Proof. (1) and (2) are similarly proved as the following:

$$\begin{aligned} (d_1 \rhd d_2)^s(z, x) &= (d_1 \rhd d_2)(x, z) \\ &= \bigvee_{y \in X} ((d_1(x, y) - d_2(y, z)) \vee 0) \\ &= \bigvee_{y \in X} ((d_1^s(y, x) - d_2^s(z, y)) \vee 0) \\ &= (d_2^s \lhd d_1^s)(z, x). \end{aligned}$$

(3) Since $((a + b) - c) \vee 0 = (a + b - c) \vee 0$ and $a + ((b - c) \vee 0)) = (a + b - c) \vee a$ for all $a, b, c \ge 0$, we have $((a + b) - c) \vee 0 \le a + ((b - c) \vee 0)$. Thus

$$\begin{aligned} &((d_1 \uplus d_2) \rhd d_3)(x, w) \\ &= \bigvee_z \left(\bigwedge_y (((d_1(x, y) + d_2(y, z)) - d_3(z, w)) \vee 0 \right) \\ &= \bigvee_z \bigwedge_y \left(((d_1(x, y) + d_2(y, z)) - d_3(z, w)) \vee 0 \right) \\ &\le \bigwedge_y \bigvee_z \left(d_1(x, y) + ((d_2(y, z) - d_3(z, w)) \vee 0) \right) \\ &= (d_1 \uplus (d_2 \rhd d_3))(x, w). \end{aligned}$$

(4), (5) and (6) are similarly proved as in (3).

(7) Since $(((b-a) \vee 0) - c) \vee 0 = (((b-c) \vee 0) - a) \vee 0$,

$$\begin{aligned} &((d_1 \lhd d_2) \rhd d_3)(x, w) \\ &= \bigvee_z \left(\bigvee_y (((d_2(y, z) - d_1(x, y)) \vee 0) - d_3(z, w)) \vee 0 \right) \\ &= \bigvee_z \bigvee_y \left((((d_2(y, z) - d_1(x, y)) \vee 0) - d_3(z, w)) \vee 0 \right) \\ &= \bigvee_y \bigvee_z \left((((d_2(y, z) - d_3(z, w)) \vee 0) - d_1(x, y)) \vee 0 \right) \\ &= (d_1 \lhd (d_2 \rhd d_3))(x, w). \end{aligned}$$

(8) Since $(((c - b) \vee 0) - a) \vee 0 = ((c - a - b) \vee 0)$,

$$\begin{aligned} &(d_1 \lhd (d_2 \lhd d_3))(x, w) \\ &= \bigvee_y \left(\bigvee_z ((d_3(z, w) - d_2(y, z)) \vee 0) - d_1(x, y)) \vee 0 \right) \\ &= \bigvee_z \bigvee_y \left((d_3(z, w) - (d_1(x, y) + d_2(y, z))) \vee 0 \right) \\ &= ((d_1 \uplus d_2) \lhd d_3)(x, w). \end{aligned}$$

(9) Since $(((a - b) \vee 0) - c) \vee 0 = ((a - (b + c)) \vee 0)$, we similarly proved as same in (8).

(10) It is easily proved.

(11) $(\Rightarrow)$ Since $d_1 \wedge d_2$ is a quasi-metric,

$$\begin{aligned} (d_1 \uplus d_2)(x, z) &= \bigwedge_y (d_1(x, y) + d_2(y, z)) \\ &\ge \bigwedge_y ((d_1 \wedge d_2)(x, y) + (d_1 \wedge d_2)(y, z)) \\ &\ge (d_1 \wedge d_2)(x, z). \end{aligned}$$

$(\Leftarrow)$ We only show that $d_1 \wedge d_2$ satisfies (M2).

$$\begin{aligned} &(d_1 \wedge d_2)(x, y) + (d_1 \wedge d_2)(y, z) \\ &= (d_1(x, y) \wedge d_2(x, y)) + (d_1(y, z) \wedge d_2(y, z)) \\ &= (d_1(x, y) + d_1(y, z)) \wedge (d_2(x, y) + d_1(y, z)) \\ &\quad \wedge (d_1(x, y) + d_2(y, z)) \wedge (d_2(x, y) + d_2(y, z)) \\ &\ge d_1(x, z) \wedge (d_2 \uplus d_1)(x, z) \wedge (d_1 \uplus d_2)(x, z) \wedge d_2(x, z) \\ &\ge (d_1 \wedge d_2)(x, z). \end{aligned}$$

(12) $(\Rightarrow)$ Since $(d_1 \uplus d_2)$ is a metric,

$$\begin{aligned} (d_1 \uplus d_2)(x, z) &= (d_1 \uplus d_2)(z, x) \\ &= \bigwedge_{y \in X} (d_1(z, y) + d_2(y, x)) \\ &= \bigwedge_{y \in X} (d_1(y, z) + d_2(x, y)) \\ &= (d_2 \uplus d_1)(x, z). \end{aligned}$$

($\Leftarrow$)

$$
\begin{aligned}
&(d_1 \uplus d_2)(x,y) + (d_1 \uplus d_2)(y,z)\\
&= \bigwedge_{y_1\in X}[d_1(x,y_1) + d_2(y_1,y)]\\
&+ \bigwedge_{z_1\in X}[d_2(y,z_1) + d_1(z_1,z)]\\
&= \bigwedge_{y_1\in X}\bigwedge_{z_1\in X}\Big([d_1(x,y_1) + d_2(y_1,y)]\\
&+[d_2(y,z_1) + d_1(z_1,z)]\Big)\\
&= \bigwedge_{y_1\in X}\bigwedge_{z_1\in X}\Big([d_1(x,y_1)\\
&+(d_2(y_1,y) + d_2(y,z_1)) + d_1(z_1,z)]\Big)\\
&\geq \bigwedge_{y_1\in X}\bigwedge_{z_1\in X}\Big(d_1(x,y_1) + [d_2(y_1,z_1) + d_1(z_1,z)]\Big)\\
&= \bigwedge_{y_1\in X}\Big(d_1(x,y_1) + \bigwedge_{z_1\in X}[d_2(y_1,z_1) + d_1(z_1,z)]\Big)\\
&= \bigwedge_{y_1\in X}\Big(d_1(x,y_1) + (d_2 \uplus d_1)(y_1,z)\Big)\\
&= \bigwedge_{y_1\in X}\Big(d_1(x,y_1) + (d_1 \uplus d_2)(y_1,z)\Big)\\
&= \bigwedge_{y_1\in X}\Big(d_1(x,y_1) + \bigwedge_{z_2\in X}[d_1(y_1,z_2) + d_2(z_2,z)]\Big)\\
&= \bigwedge_{z_2\in X}\Big(\bigwedge_{y_1\in X}[d_1(x,y_1) + d_1(y_1,z_2)] + d_2(z_2,z)\Big)\\
&\geq \bigwedge_{z_2\in X}\Big(d_1(x,z_2) + d_2(z_2,z)\Big)\\
&= (d_1 \uplus d_2)(x,z).
\end{aligned}
$$

Other cases are easily proved.

□

Example 2.2. (1) We give an example $d_1 \uplus (d_2 \rhd d_3) \geq (d_1 \uplus d_2) \rhd d_3$ from:

$$\begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} \uplus \left[\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \rhd \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\right] = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\geq \left[\begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} \uplus \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\right] \rhd \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

(2) $d_1 \lhd (d_2 \uplus d_3) \leq (d_1 \lhd d_2) \uplus d_3$ from:

$$\begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} \lhd \left[\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \uplus \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\leq \left[\begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} \lhd \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\right] \uplus \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

(3) $(d_1 \lhd d_2) \rhd d_3 = d_1 \lhd (d_2 \rhd d_3)$ from:

$$\begin{pmatrix} 3 & 0 \\ 1 & 5 \end{pmatrix} \lhd \left[\begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} \rhd \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}\right] = \begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 3 & 0 \\ 1 & 5 \end{pmatrix} \lhd \begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix}\right] \rhd \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}$$

(4) Let d_1 and d_2 be quasi-metrics as follows:

$$d_1 = \begin{pmatrix} 0 & 2 & 3 \\ 3 & 0 & 4 \\ 5 & 2 & 0 \end{pmatrix} \quad d_2 = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 3 \\ 4 & 5 & 0 \end{pmatrix}$$

$$d_1 \uplus d_2 = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 3 \\ 3 & 2 & 0 \end{pmatrix}$$

$$d_1 \wedge d_2 = d_2 \uplus d_1 = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix}$$

Since $d_1 \uplus d_2 \not\geq d_1 \wedge d_2$, by Theorem 2.1(11), $d_1 \wedge d_2$ is not a quasi-metric because

$$3 = (d_1 \wedge d_2)(z,y) + (d_1 \wedge d_2)(y,x) \not\geq (d_1 \wedge d_2)(z,x) = 4.$$

(5) Let d_1 and d_2 be metrics as follows:

$$d_1 = \begin{pmatrix} 0 & 3 & 7 \\ 3 & 0 & 10 \\ 7 & 10 & 0 \end{pmatrix} \quad d_2 = \begin{pmatrix} 0 & 10 & 9 \\ 10 & 0 & 2 \\ 9 & 2 & 0 \end{pmatrix}$$

$$d_1 \uplus d_2 = \begin{pmatrix} 0 & 3 & 5 \\ 3 & 0 & 2 \\ 7 & 2 & 0 \end{pmatrix} \quad d_2 \uplus d_1 = \begin{pmatrix} 0 & 3 & 7 \\ 3 & 0 & 2 \\ 5 & 2 & 0 \end{pmatrix}$$

Since $d_1 \uplus d_2 \neq d_2 \uplus d_1$, $d_1 \uplus d_2$ is not a metric because

$$5 = (d_1 \uplus d_2)(x,z) \neq (d_1 \uplus d_2)(z,x) = 7,$$

$$5 = (d_1 \uplus d_2)(z,y) + (d_1 \uplus d_2)(y,x) \not\geq (d_1 \uplus d_2)(z,x) = 7.$$

(6) Let d_1 and d_2 be metrics as follows:

$$d_1 = \begin{pmatrix} 0 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 0 \end{pmatrix} \quad d_2 = \begin{pmatrix} 0 & 4 & 2 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$d_1 \uplus d_2 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \quad d_2 \uplus d_1 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

Since $d_1 \uplus d_2 = d_2 \uplus d_1 = d_1 \wedge d_2$, $d_1 \uplus d_2$ is a metric.

Theorem 2.3. Let $d \in [0,\infty]^{X\times X}$ be a non-negative function. We have the following properties.

(1) If $d(x,x) = 0$ for each $x \in X$, then $d \uplus d \leq d$, $d \leq (d \rhd d), d \leq (d \rhd d^s), d \leq (d \lhd d), d \leq (d^s \lhd d)$ and $d \uplus d(x,x) = 0$ for each $x \in X$.

(2) If d is symmetric, then $d \uplus d$ is symmetric, $(d \rhd d)(x,x) = 0$ for all $x \in X$, $(d \lhd d)^s = d \rhd d$, and $d \diamond d$ is symmetric and $d \diamond d(x,x) = 0$ for all $x \in X$.

(3) d is symmetric iff $(d \rhd d)(x,x) = 0$ for all $x \in X$ iff $(d \lhd d)(x,x) = 0$ for all $x \in X$.

(4) If $d \rhd d \leq d$, then $d \leq d^s \uplus d$. Moreover, if $d \lhd d \leq d$, then $d \leq d^s \uplus d$.

(5) $d(x,z) \leq d(x,y) + d(y,z)$ for each $x,y,z \in X$ iff $d \leq d \uplus d$ iff $d^s \lhd d \leq d$ iff $d \rhd d^s \leq d$

(6) If $d(x,x) = 0$ and $d(x,z) \leq d(x,y) + d(y,z)$ for each $x,y,z \in U$, then $d = d \uplus d = d^s \lhd d = d \rhd d^s$.

(7) $d \uplus d^s$ is symmetric.

(8) $d \uplus d^s \geq d$ and $d(x,x) = 0$ for each $x \in X$, iff d is is a pseudo-metric on X iff $d \rhd d \leq d$ and $d(x,x) =$

$(d \triangleright d)(x,x) = 0$ for each $x \in X$ iff $d \triangleleft d \leq d$ and $d(x,x) = (d \triangleleft d)(x,x) = 0$ for each $x \in X$.

(9) Let $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for each $x, y \in X$. We define

$$d^{\infty}(x,y) = \bigwedge_{n \in N} d^n(x,y)$$

Where $d^n = \overbrace{d \uplus d ... \uplus d}^{n}$. Then d^{∞} is a pseudo-metric on X.

(10) $d \diamond d^s$ is a pseudo-metric on X.

(11) If d is symmetric, then $d \diamond d$ is a pseudo-metric on X.

Proof. (1)

$$\begin{aligned}(d \triangleright d)(x,z) &= \bigvee_{y \in U}(d(x,y) - d(y,z)) \vee 0) \\ &\geq d(x,z) - d(z,z) = d(x,z).\end{aligned}$$

Other cases are similarly proved.

(2) By Theorem 2.1(2),

$$\begin{aligned}(d \diamond d)^s &= (d \triangleright d)^s \vee (d \triangleleft d)^s \\ &= (d \triangleleft d) \vee (d \triangleright d) = d \diamond d.\end{aligned}$$

(3) It easily proved because

$$\begin{aligned}&(d \triangleright d)(x,x) = \bigvee_{y \in X}(d(x,y) - d(y,x)) \vee 0 = 0 \\ &\text{iff } d(x,y) \leq d(y,x).\end{aligned}$$

(4) and (5) follow from:

$$\begin{aligned}&d(x,z) \leq d(x,y) + d(y,z) \\ &\text{iff } (d(x,z) - d^s(z,y)) \vee 0 \leq d(x,y) \\ &\text{iff } (d(x,z) - d^s(y,x)) \vee 0 \leq d(y,z).\end{aligned}$$

(6) Since $d(x,x) = 0$,

$$\begin{aligned}(d \triangleright d^s)(x,z) &= \bigvee_{y \in U}(d(x,y) - d^s(y,z)) \vee 0 \\ &\geq d(x,z) - d^s(z,z) \leq d(x,z).\end{aligned}$$

Since $d(x,y) + d(y,z) \leq d(x,z)$, we have $d(x,y) \leq d(x,z) - d(y,z)$. Thus

$$d(x,y) \leq \bigvee_{z \in U}(d(x,z) - d^s(z,y)) \vee 0.$$

Therefore $d = (d \triangleright d^s)$.

(7) It is proved from $(d \uplus d^s)^s = d \uplus d^s$.

(8) ($\Rightarrow$) Since $d(x,x) = 0, d \geq d \uplus d^s$. Thus $d \geq d \uplus d^s$. By (6), d is symmetric. Since $d = d^s$ and $d \uplus d = d$, $d(x,z) \leq d(x,y) + d(y,z)$.

($\Leftarrow$) It is easy.

(9) Suppose there exist $x, y, z \in X$ such that

$$d^{\infty}(x,y) + d^{\infty}(y,z) \not\geq d^{\infty}(x,z).$$

By the definition of $d^{\infty}(x,y)$, there exists $x_i \in X$ such that

$$d(x,x_1)+d(x_1,x_2)+...+d(x_n,y)+d^{\infty}(y,z) \not\geq d^{\infty}(x,z).$$

By the definition of $d^{\infty}(y,z)$, there exists $y_j \in X$ such that

$$d(x,x_1) + d(x_1,x_2) + ... + d(x_n,y)$$

$$+d(y,y_1) + d(y_1,y_2) + ... + d(y_n,z) \not\geq d^{\infty}(x,z).$$

It is a contradiction for the definition of $d^{\infty}(x,z)$.

(10)

$$\begin{aligned}&(d \triangleright d^s)(x_i,x_j) + (d \triangleright d^s)(x_j,x_k) \\ &\geq (d(x_i,y) - d^s(y,x_j) + (d(x_j,y) - d^s(y,x_k) \\ &= (d(x_i,y) - d(x_j,y) + (d(x_j,y) - d^s(y,x_k)) \\ &= d(x_i,y) - d^s(y,x_k)\end{aligned}$$

$$\begin{aligned}&(d \triangleleft d^s)(x_i,x_j) + (d \triangleleft d^s)(x_j,x_k) \\ &\geq (d^s(y,x_j) - d(x_i,y)) + (d^s(y,x_k - d(x_j,y)) \\ &= (d(x_j,y) - d(x_i,y)) + (d^s(y,x_k) - d(x_j,y)) \\ &= d^s(y,x_k) - d(x_i,y)\end{aligned}$$

$$\begin{aligned}&(d \diamond d^s)(x_i,x_j) + (d \diamond d^s)(x_j,x_k) \\ &\geq (d \triangleright d^s)(x_i,x_j) + (d \triangleright d^s)(x_j,x_k) \\ &\vee(d \triangleleft d^s)(x_i,x_j) + (d \triangleleft d^s)(x_j,x_k) \\ &\geq (d(x_i,y) - d^s(y,x_k)) \vee (d(x_i,y) - d^s(y,x_k))\end{aligned}$$

It implies

$$(d \diamond d^s)(x_i,x_j) + (d \diamond d^s)(x_j,x_k) \geq (d \diamond d^s)(x_i,x_k).$$

(11) It follows from (10). □

Example 2.4. Let d be a metric as follows:

$$d = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 7 \\ 3 & 7 & 0 \end{pmatrix} \quad d^{\infty} = d \uplus d = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 0 \end{pmatrix}$$

$$d = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{pmatrix} \quad d^{\infty} = d \uplus d = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 4 \\ 7 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix} \triangleright \begin{pmatrix} 0 & 7 & 1 \\ 3 & 0 & 0 \\ 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 \\ 7 & 0 & 6 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 4 \\ 7 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix} \triangleleft \begin{pmatrix} 0 & 7 & 1 \\ 3 & 0 & 0 \\ 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 7 & 1 \\ 3 & 0 & 1 \\ 3 & 6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 4 \\ 7 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix} \diamond \begin{pmatrix} 0 & 7 & 1 \\ 3 & 0 & 0 \\ 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 7 & 3 \\ 7 & 0 & 6 \\ 3 & 6 & 0 \end{pmatrix}$$

Definition 2.5. A strictly increasing function $s : [0, \infty] \to [0, \infty]$ is called a metric transformation map if it satisfies the following conditions:

(1) $s(0) = 0$,

(2) $s(x + y) \leq s(x) + s(y)$.

Lemma 2.6. Let $f : [0,1] \to [0, \infty]$ be a strictly decreasing continuous function with $f(1) = 0$ and $f^{-1} : [0, f(0)] \to [0,1]$. Then

(1) Define $x \odot y = f^{-1}(f(x) + f(y) \wedge f(0))$. Then $\odot$ is a continuous t-norm and $([0,1], \leq, \odot)$ is a stsc-quantale.

(2) $x \to y = f^{-1}((f(y) - f(x)) \vee 0)$.

Proof. (1) It follows from [13] and Remark 1.2(1).

(2)

$$\begin{aligned} x \to y &= \bigvee\{z \mid x \odot z \leq y\} \\ &= \bigvee\{z \mid f^{-1}((f(x) \odot f(z)) \wedge f(0)) \leq y\} \\ &= \bigvee\{z \mid f(z) \geq (f(y) - f(x)) \vee 0\} \\ &= f^{-1}((f(y) - f(x)) \vee 0). \end{aligned}$$

□

Theorem 2.7. Let d be a metric on X and s a metric transformation map. Then

(1) $s \circ d$ is a metric.

(2) If $f : [0,1] \to [0, \infty]$ is a strictly decreasing function with $f(1) = 0$ and $f^{-1} : [0, f(0)] \to [0,1]$ with $s(f(0)) \leq f(0)$, then $T = f^{-1} \circ s \circ f$ is an equivalence relation with respect to $x \odot y = f^{-1}(f(x) + f(y) \wedge f(0))$.

Proof. (1) It is easy.

(2)

$$\begin{aligned} T(x) \odot T(y) &= f^{-1} \circ s \circ f(x) \odot f^{-1} \circ s \circ f(y) \\ &= f^{-1}(f(0) \wedge (s(f(x)) + s(f(y)))) \\ &\leq f^{-1}(f(0) \wedge (s(f(x) + f(y))). \end{aligned}$$

$$\begin{aligned} T(x \odot y) &= (f^{-1} \circ s \circ f)(f^{-1}(f(0) \wedge (f(x) + f(y)))) \\ &= f^{-1}(s(f(0)) \wedge s(f(x) + f(y))) \end{aligned}$$

Since $s(f(0)) \leq f(0)$, $T(x) \odot T(y) \leq T(x \odot y)$.

□

Example 2.8. Let $s(x) = \frac{2x}{x+1}$ and $f(x) = 1 - x$. Then $x \odot y = (x + y - 1) \vee 0$ and $T(x) = \frac{x}{2-x}$. Put d as follows:

$$d = \begin{pmatrix} 0 & 0.2 & 0.3 \\ 0.2 & 0 & 0.4 \\ 0.3 & 0.4 & 0 \end{pmatrix} f^{-1}(d) = \begin{pmatrix} 1 & 0.8 & 0.7 \\ 0.8 & 1 & 0.6 \\ 0.7 & 0.6 & 1 \end{pmatrix}$$

Then we obtain $f^{-1}(s \circ d) = T(f^{-1}(d))$:

$$s \circ d = \begin{pmatrix} 0 & \frac{4}{12} & \frac{6}{13} \\ \frac{4}{12} & 0 & \frac{8}{14} \\ \frac{6}{13} & \frac{8}{14} & 0 \end{pmatrix} f^{-1}(s \circ d) = \begin{pmatrix} 1 & \frac{8}{12} & \frac{7}{13} \\ \frac{8}{12} & 1 & \frac{6}{14} \\ \frac{7}{13} & \frac{6}{14} & 1 \end{pmatrix}$$

From the following theorem, non-negative functions on $X \times X$ induce fuzzy relations.

Theorem 2.9. Let $d_1, d_2 \in [0, \infty]^{X \times X}$ be non-negative functions. Let $f : [0,1] \to [0, \infty]$ be a strictly decreasing continuous function with $f(1) = 0$ and $f^{-1} : [0, f(0)] \to [0,1]$. We define $f^{-1}(d_i) \in [0,1]^{X \times X}$ as $f^{-1}(d_i)(x,y) = f^{-1}(d_i(x,y))$ and $x \odot y = f^{-1}((f(x) + f(y)) \wedge f(0))$.

We have the following properties.

(1) If d_i is a pseudo-metric on X with where $f(0) \geq \bigvee_{x,y,z}(d_i(x,y) + d_i(y,z))$ for each $i = 1,2$, then $f^{-1}(d_i)$ is an $\odot$-equivalence relation on X for each $i = 1,2$ where $x \odot y = f^{-1}(f(x) + f(y))$. Furthermore, $d_1 \vee d_2$ is a pseudo-metric such that $f^{-1}(d_1 \vee d_2) = f^{-1}(d_1) \wedge f^{-1}(d_2)$.

(2) If d_i is a pseudo-metric on X with where $f(0) \geq \bigvee_{x,y}(d_1(x,y) + d_2(x,y))$ for each $i = 1,2$, then $f^{-1}(d_1) \odot f^{-1}(d_2)$ is an $\odot$-equivalence relation on X such that $f^{-1}(d_1 + d_2) = f^{-1}(d_1) \odot f^{-1}(d_2)$.

(3) If $f(0) \geq \bigvee_{x,z}(\bigwedge_y(d_1(x,y) + d_2(y,z)))$, then $f^{-1}(d_1 \uplus d_2) = f^{-1}(d_1) \circ f^{-1}(d_2)$.

(4) $f^{-1}(d_1 \rhd d_2) = f^{-1}(d_1) \Leftarrow f^{-1}(d_2)$.

(5) $f^{-1}(d_1 \lhd d_2) = f^{-1}(d_1) \Rightarrow f^{-1}(d_2)$.

(6) $f^{-1}(d_1 \diamond d_2) = f^{-1}(d_1) \Leftrightarrow f^{-1}(d_2)$.

Proof. (1) It follows from:

$$\begin{aligned} & f^{-1}(d_1)(x,y) \odot f^{-1}(d_1)(y,z) \\ &= f^{-1}(f(0) \wedge (d_1(x,y) + d_1(y,z))) \\ &= f^{-1}(d_1(x,y) + d_1(y,z)) \\ &\geq f^{-1}(d_1)(x,z). \end{aligned}$$

(2) It follows from:

$$\begin{aligned} & f^{-1}(d_1)(x,y) \odot f^{-1}(d_2)(x,y) \\ &= f^{-1}(f(0) \wedge (d_1(x,y) + d_2(x,y))) \\ &= f^{-1}(d_1(x,y) + d_2(x,y)) \\ &\geq f^{-1}(d_1 + d_2)(x,y). \end{aligned}$$

(3)

$$\begin{aligned} & f^{-1}((d_1 \uplus d_2)(x,z)) \\ &= f^{-1}((\textstyle\bigwedge_{y \in Y}(d_1(x,y) + d_2(y,z)))) \\ &= \textstyle\bigvee_{y \in X} f^{-1}(d_1(x,y) + d_2(y,z)) \\ & (f^{-1}(d_1) \circ f^{-1}(d_2))(x,z) \\ &= \textstyle\bigvee_{y \in X}(f^{-1}(d_1)(x,y) \odot f^{-1}(d_2)(y,z)) \\ &= \textstyle\bigvee_{y \in X} f^{-1}(f(0) \wedge (d_1(x,y) + d_2(y,z))). \\ &= \textstyle\bigvee_{y \in X} f^{-1}(d_1(x,y) + d_2(y,z)). \end{aligned}$$

(4)

$$\begin{aligned} & f^{-1}((d_1 \rhd d_2)(x,z))) \\ &= f^{-1}(\textstyle\bigvee_{y \in Y}((d_1(x,y) - d_2(y,z)) \vee 0)) \\ &= \textstyle\bigwedge_{y \in X} f^{-1}((d_1(x,y) - d_2(y,z)) \vee 0)) \\ & (f^{-1}(d_1) \Leftarrow f^{-1}(d_2))(x,z) \\ &= \textstyle\bigwedge_{y \in X}(f^{-1}(d_2)(y,z) \to f^{-1}(d_1)(x,y)) \\ &= \textstyle\bigwedge_{y \in X} f^{-1}((d_1(x,y) - d_2(y,z)) \vee 0). \end{aligned}$$

(5) and (6) are similarly proved as in (4).

□

Example 2.10. Put $d_1 \uplus d_2$ as follows:

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 0 & 4 \\ 3 & 2 & 1 \end{pmatrix} \uplus \begin{pmatrix} 3 & 2 & 4 \\ 1 & 7 & 6 \\ 0 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 5 \\ 1 & 4 & 6 \\ 1 & 5 & 7 \end{pmatrix}$$

Let $f(x) = 10 - 10x$, $f^{-1}(x) = (1 - \frac{x}{10}) \wedge 10$ and $f(0) > 7$. We obtain $f^{-1}(d_1 \uplus d_2) = f^{-1}(d_1) \circ f^{-1}(d_2)$ as

$$\begin{pmatrix} \frac{9}{10} & \frac{7}{10} & \frac{5}{10} \\ \frac{8}{10} & 1 & \frac{6}{10} \\ \frac{7}{10} & \frac{8}{10} & \frac{9}{10} \end{pmatrix} \circ \begin{pmatrix} \frac{7}{10} & \frac{8}{10} & \frac{6}{10} \\ \frac{9}{10} & \frac{3}{10} & \frac{4}{10} \\ 1 & \frac{6}{10} & \frac{2}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{10} & \frac{7}{10} & \frac{5}{10} \\ \frac{9}{10} & \frac{6}{10} & \frac{4}{10} \\ \frac{9}{10} & \frac{5}{10} & \frac{3}{10} \end{pmatrix}$$

We obtain $f^{-1}(d_1 \triangleright d_2) = f^{-1}(d_1) \Leftarrow f^{-1}(d_2)$ as

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 0 & 4 \\ 3 & 2 & 1 \end{pmatrix} \triangleright \begin{pmatrix} 3 & 2 & 4 \\ 1 & 7 & 6 \\ 0 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 0 \\ 4 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{9}{10} & \frac{7}{10} & \frac{5}{10} \\ \frac{8}{10} & 1 & \frac{6}{10} \\ \frac{7}{10} & \frac{8}{10} & \frac{9}{10} \end{pmatrix} \Leftarrow \begin{pmatrix} \frac{7}{10} & \frac{8}{10} & \frac{6}{10} \\ \frac{9}{10} & \frac{3}{10} & \frac{4}{10} \\ 1 & \frac{6}{10} & \frac{2}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{10} & \frac{9}{10} & 1 \\ \frac{6}{10} & 1 & 1 \\ \frac{9}{10} & \frac{9}{10} & 1 \end{pmatrix}$$

From Theorem 2.9, we obtain the following corollary.

Corollary 2.11. Let $d, d_1, d_2, d_3 \in [0, \infty]^{X \times X}$ be a fuzzy relation. Let f and $\odot$ be defined as in Theorem 2.9. We have the following properties.

(1) If $f(0) \geq \bigvee_{x,z}(\bigwedge_y(d_1(z,y) + d_2(y,x)))$, $f^{-1}((d_1 \uplus d_2)^s) = f^{-1}(d_2^s) \circ f^{-1}(d_1^s)$.

(2) $f^{-1}((d_1 \triangleright d_2)^s) = (f^{-1}(d_2^s) \Rightarrow f^{-1}(d_1^s))$ and $f^{-1}((d_1 \triangleleft d_2)^s) = (f^{-1}(d_2^s) \Leftarrow f^{-1}(d_1^s))$.

(3) If $f(0) \geq \bigvee_{x,z}(\bigwedge_y(d_1(x,y) + d_2(y,z)))$, $f^{-1}(d_1) \circ (f^{-1}(d_2) \Leftarrow f^{-1}(d_3)) \leq ((f^{-1}(d_1) \circ f^{-1}(d_2)) \Leftarrow f^{-1}(d_3))$.

(4) If $f(0) \geq \bigvee_{y,w}(\bigwedge_y(d_2(y,z) + d_3(z,w)))$, $f^{-1}(d_1) \Rightarrow (f^{-1}(d_2) \circ f^{-1}(d_3)) \geq (f^{-1}(d_1) \Rightarrow f^{-1}(d_2)) \circ f^{-1}(d_3)$.

(5) If $f(0) \geq \bigvee_{x,z}(\bigwedge_y(d_1(x,y) + d_2(y,z)))$, $(f^{-1}(d_1) \Rightarrow f^{-1}(d_2)) \Rightarrow f^{-1}(d_3) \geq f^{-1}(d_1) \circ (f^{-1}(d_2) \Rightarrow f^{-1}(d_3))$.

(6) $(f^{-1}(d_1) \Rightarrow f^{-1}(d_2)) \Leftarrow f^{-1}(d_3) = f^{-1}(d_1) \Rightarrow (f^{-1}(d_2) \Leftarrow f^{-1}(d_3))$.

(7) $f^{-1}(d_1) \Leftarrow (f^{-1}(d_2) \Leftarrow f^{-1}(d_3)) \geq (f^{-1}(d_1) \Leftarrow f^{-1}(d_2)) \circ f^{-1}(d_3)$.

(8) If $f(0) \geq \bigvee_{x,z}(\bigwedge_y(d_1(x,y) + d_2(y,z)))$, $f^{-1}(d_1) \Rightarrow (f^{-1}(d_2) \Rightarrow f^{-1}(d_3)) = (f^{-1}(d_1) \circ f^{-1}(d_2)) \Rightarrow f^{-1}(d_3)$.

(9) If $f(0) \geq \bigvee_{y,w}(\bigwedge_y(d_2(y,z) + d_3(z,w)))$, $(f^{-1}(d_1) \Leftarrow f^{-1}(d_2)) \Leftarrow f^{-1}(d_3) = (f^{-1}(d_1) \Leftarrow (f^{-1}(d_2) \circ f^{-1}(d_3)))$.

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