International Journal of Fuzzy Logic and Intelligent Systems, Vol. 9, No. 1, March 2009 pp. 20-29

FUZZY EQUIVALENCE RELATIONS AND FUZZY FUNCTIONS

Keon Chang Lee

Department of Computer Science, Dongshin University, Naju, Cheonnam, Korea 520-714

Abstract

In this paper, by using the definition of fuzzy equivalence relations introduced by Dib and Youssef, we obtain fuzzy analogues of many results concerning ordinary equivalence relations. Moreover, we investigate fuzzy analogues of many results concerning relationships between ordinary equivalence relations and ordinary functions. In particular, we obtain the fuzzy canonical decomposition of a fuzzy function.

Key Words :fuzzy Cartesian product, fuzzy relation, fuzzy equivalence relation, fuzzy cutting, fuzzy equivalence class, fuzzy partition.

1. Introduction

In the usual set theory, functions are special types of relations and relations are subsets of Cartesian product. Thus the concept of Cartesian product plays an important role in the usual theory of relations and functions. Almost all authors have worked with fuzzy relations without referring to what may be called *fuzzy Cartesian product* (See [1,3,6,7]).

However, in 1991, by using J-fuzzy sets, Dib and Youssef introduced the notion of fuzzy Cartesian product and they defined a fuzzy relation as a subset of the fuzzy Cartesian product. This definition is different from all known definitions of fuzzy relations. Also they defined a fuzzy function as a special type of a fuzzy relation. We can see that this definition generalizes Zadeh's definition and is different from those in [3]. In particular, Hur et al. [5] obtained fuzzy analogues of many results concerning ordinary equivalence relations and partitions.

In section 2, by using the definition of fuzzy equivalence relations introduced by Dib and Youssef, we obtain fuzzy analogues of many results concerning ordinary equivalence relations.

In section 3, we investigate fuzzy analogues of many results concerning relationships between ordinary equivalence relations and ordinary functions. In particular, we obtain the fuzzy canonical decomposition of a fuzzy function.

2. Preliminaries

In this section, are list some definitions and some results needed in the next sections.

The totally ordered set $I = [0, 1]$ is a distributive but not complemented lattice under the operations of infimum $\wedge$ and supremum $\vee$. On $J = I \times I$ we define a partial order $\leq$, in terms of the partial order on I, as follows: For every $(r_1, r_2), (s_1, s_2) \in J$,

(i) $(r_1, r_2) \leq (s_1, s_2)$ if and only if $r_1 \leq s_1$, $r_2 \leq s_2$ whenever $s_1 \neq 0$ and $s_2 \neq 0$,

(ii) $(0, 0) = (s_1, s_2)$ whenever $s_1 = 0$ or $s_2 = 0$.

It is clear that J is a distributive but not complemented vector lattice. The operations of infimum and supremum in J are given respectively by: For every $(r_1, r_2), (s_1, s_2) \in J$,

$$(r_1, r_2) \wedge (s_1, s_2) = (r_1 \wedge s_1, r_2 \wedge s_2)$$

and

$$(r_1, r_2) \vee (s_1, s_2) \leq (r_1 \vee s_1, r_2 \vee s_2),$$

where the equality holds in the last relation when $r_i \neq 0 \neq s_i$.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping if* $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings, where $f(x) = (f_1(x), f_2(x))$ for each $x \in X$.

Definition 2.1[5]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \eta_A) : X \to J$ is called a *J-fuzzy set* (in short, *fuzzy set*) in X, where

Manuscript received Dec. 3, 2008; revised Feb. 2, 2009.
2000 Mathematics Subject Classification of **AMS**: 03E72, 04A72, 14C10.
This research was supported by the Dongshin University research grants in 2008.

$A(x) = (\mu_A(x), \eta_A(x))$ for each $x \in X$. In particular, $\varnothing$ and X denote the *J-fuzzy empty set* and *J-fuzzy whole set* in X defined by $\varnothing(x) = (0,0)$ and $X(x) = (1,1)$ for each $x \in X$, respectively.

The notation $\{(x, A(x)) : x \in X\}$ or simply $\{(x,r)\}$, where $r = A(x)$, will be used to denote a fuzzy set in X (see [7]). Similarly, a J-fuzzy set in X, a fuzzy set in $X \times Y$ and a J-fuzzy set in $X \times Y$ will be denoted respectively by $\{(x,(r_1,r_2))\}$, $\{((x,y),r)\}$ and $\{((x,y),(r_1,r_2))\}$. To each fuzzy set $\{(x,r_1)\}$ in X and fuzzy set $\{(y,r_2)\}$ in Y there corresponds a J-fuzzy set $\{((x,y),(r_1,r_2))\}$ in $X \times Y$. Throughout this paper, the notation $(x,r) \in A$ means that $A(x) = r$, where $A \in I^X$, and X, Y, Z, etc denote ordinary sets.

Definition 2.2[2]. Let X and Y be two ordinary sets. Then the collection of all J-fuzzy sets in $X \times Y$ is called the *fuzzy Cartesian product* of X and Y and is denoted by $X\underline{\times}Y$. Hence $X\underline{\times}Y = J^{X\times Y}$.

The *fuzzy Cartesian product* of a fuzzy set $A = \{(x,r)\}$ in X and a fuzzy set $B = \{(y,s)\}$ in Y is the J-fuzzy set $A\underline{\times}B$ in $X \times Y$ defined by:

$$A\underline{\times}B = \{((x,y),(r,s)) : x \in X, y \in Y\}$$

$$\equiv \{((x,y),(r,s))\}.$$

It is clear that $A\underline{\times}B \in X\underline{\times}Y$ for each $A \in I^X$ and $B \in I^Y$. The above definitions can be generalized for any finite number of sets. Furthermore, the above definitions can be generalized in an obvious way by replacing the unit interval I by an arbitrary completely distributive lattice.

Definition 2.3[2]. ρ is called a *fuzzy relation from X to Y* if $\rho \subset X\underline{\times}Y$. In particular, ρ is called a *fuzzy relation in X* if $\rho \subset X\underline{\times}X$.

It is clear that $X\underline{\times}Y$ is itself a fuzzy relation from X to Y. Any collection of $A\underline{\times}B$, where $A \in I^X$ and $B \in I^Y$, is a fuzzy relation from X to Y.

The fuzzy Cartesian product $X\underline{\times}X$ is called the *universal fuzzy relation in X*. The fuzzy relation $\emptyset\underline{\times}\emptyset = \emptyset$ is called the *empty fuzzy relation*. Between these two extreme cases, lies the *identity fuzzy relation*, denoted by Δ_X, where Δ_X is the fuzzy relation in X whose members are the J-fuzzy sets $\{((x,x),(r,r)) : x \in X \text{ and } r \in I\}$.

Definition 2.4[2]. Let $\rho_1, \rho_2 \subset X\underline{\times}Y$.

(1) We say that ρ_1 *is contained in* ρ_2 if whenever $((x,y),(r_1,r_2)) \in A \in \rho_1$, there exists $B \in \rho_2$ such that $((x,y),(r_1,r_2)) \in B$. In this case, we write $\rho_1 \subset \rho_2$.

(2) We say that ρ_1 and ρ_2 *are equal* if $\rho_1 \subset \rho_2$ and $\rho_2 \subset \rho_1$. In this case, we write $\rho_1 = \rho_2$.

To each J-fuzzy set $C = \{((x,y),(r,s))\}$ in $X \times Y$ we associate a J-fuzzy set C^{-1} in $Y \times X$ defined by $C^{-1} = \{((y,x),(s,r))\}$

Definition 2.5[2]. Let $\rho \subset X\underline{\times}Y$. Then the *inverse* of ρ, denoted ρ^{-1}, is the fuzzy relation from Y to X defined by $\rho^{-1} = \{C^{-1} : C \in \rho\}$.

Definition 2.6[2]. Let $\rho \subset X\underline{\times}Y$ and let $\sigma \subset Y\underline{\times}Z$. Then the *composition* of ρ and σ, denoted $\sigma \circ \rho$, is the fuzzy relation from X to Z whose constituting J-fuzzy sets $C \in X\underline{\times}Z$ are defined as follows:

$((x,z),(r_1,r_3)) \in C$ if and only if there exists $(y,r_2) \in Y \times I$ such that $((x,y),(r_1,r_2)) \in A$ and $((y,z),(r_2,r_3)) \in B$ for some $A \in \rho$ and $B \in \sigma$. Hence $\sigma \circ \rho = \{C \in X\underline{\times}Z : C \text{ is as defined above }\}$.

It is clear that if $\rho \subset X\underline{\times}X$, then $\Delta_X \circ \rho \subset \rho$ and $\rho \circ \Delta_X \subset \rho$.

Result 2.A[2, Proposition in p.303]. Let $\rho, \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2$ be any fuzzy relations defined on the appropriate sets. Then:

(1) $(\rho_1 \circ \rho_2) \circ \rho_3 = \rho_1 \circ (\rho_2 \circ \rho_3)$.
(2) $\rho_1 \subset \rho_2$ and $\sigma_1 \subset \sigma_2 \Rightarrow \rho_1 \circ \sigma_1 \subset \rho_2 \circ \sigma_2$.
(3) $\rho_1 \circ (\rho_2 \cup \rho_3) = (\rho_1 \circ \rho_2) \cup (\rho_1 \circ \rho_3)$.
(4) $\rho_1 \circ (\rho_2 \cap \rho_3) \subset (\rho_1 \circ \rho_2) \cap (\rho_1 \circ \rho_3)$.
(5) $\rho_1 \subset \rho_2 \Rightarrow \rho_1^{-1} \subset \rho_2^{-1}$.
(6) $(\rho^{-1})^{-1} = \rho$ and $(\rho_1 \circ \rho_2)^{-1} = \rho_2^{-1} \circ \rho_1^{-1}$.
(7) $(\rho_1 \cup \rho_2)^{-1} = \rho_1^{-1} \cup \rho_2^{-1}$.
(8) $(\rho_1 \cap \rho_2)^{-1} = \rho_1^{-1} \cap \rho_2^{-1}$.

Definition 2.7[2]. Let $\rho \subset X\underline{\times}X$. Then ρ is said to be:

(1) *reflexive* in X if for each $x \in X$ and $r \in I$, there exists $A \in \rho$ such that $((x,x),(r,r)) \in A$, i.e., $\Delta_X \subset \rho$.

(2) *symmetric* in X if whenever $((x,y),(r,s)) \in A \in \rho$, there exists $B \in \rho$ such that $((y,x),(s,r)) \in B$, i.e., $\rho^{-1} = \rho$.

(3) *transitive* in X if whenever $((x,y),(r,s)) \in A \in \rho$ and $((y,z),(s,t)) \in B \in \rho$, there exists $C \in \rho$ such that $((x,z),(r,t)) \in C$, i.e., $\rho \circ \rho \subset \rho$.

(4) a *fuzzy equivalence relation* in X if it is reflexive, symmetric and transitive. We will denote the set of all fuzzy equivalence relations in X as $\mathrm{F}Rel_E(X)$.

It is clear that $X\underline{\times}X$, $\Delta_X \in \mathrm{F}Rel_E(X)$.

Result 2.B[2, Theorem 1]. Let $\rho \in \mathrm{F}Rel_E(X)$. Then

(1) For each $x_0 \in X$, ρ induces an (ordinary) equivalence relation $\rho_I(x_0)$ in I defined by:

$\rho_I(x_0) = \{(r,s) \in J : \exists A \in \rho$ s.t. $((x_0,x_0),(r,s)) \in A\}$.

(2) For each $r_0 \in I$, ρ induces an (ordinary) equivalence relation $\rho_X(r_0)$ in X defined by:

$\rho_X(r_0) = \{(x,y) \in X \times X : \exists A \in \rho$ s.t. $((x,y),(r_0,r_0)) \in A\}$.

Result 2.C[2, Corollary in p.304]. To each $\rho \in \mathrm{F}Rel_E(X)$ there are associated an ordinary equivalence relation ρ_I in I and an ordinary equivalence relation ρ_X in X. In fact,

$$\rho_I = \bigcap_{x\in X}\rho_I(x) \quad \text{and} \quad \rho_X = \bigcap_{r\in I}\rho_X(r).$$

In this case, ρ_I [resp. ρ_X] is called the *equivalence relation in I [resp. X] associated to the fuzzy equivalence relation* ρ.

Definition 2.8[2]. Let X be a nonempty set. Then H is called a *fuzzy cutting on* X if H is a function of X into the power set of I, i.e., $H : X \to P(I)$. In this case, we will denote H as $\{(x, H_x) : x \in X\}$ or simply $\{(x, H_x)\}$, where $H_x = H(x)$ is the subset of I corresponding to $x \in X$.

A fuzzy cutting on X is said to be *empty* and is denoted by $\emptyset$ if $\emptyset_x = \emptyset$ for each $x \in X$. A fuzzy cutting on X is said to be *universal* and is denoted by $\bigcup$ if $\bigcup_x = I$ for each $x \in X$, i.e., $\bigcup = \{(x, I)\}$.

A fuzzy set $A \in I^X$ is said to *be contained in* the fuzzy cutting H of X, symbolically $A \subset H$, if $A(x) \in H_x$ for each $x \in X$ for which $H_x \neq \emptyset$ and $A(x) = 0$ whenever $H_x = \emptyset$. Note that $\emptyset$ is the fuzzy set in X contained in the empty fuzzy cutting, and that $\bigcup$ contains all fuzzy sets in X.

Let $H = \{(x, H_x)\}$ and $H' = \{(x, H_x{}')\}$ be fuzzy cuttings on X. Then H is said to *be contained in* H', symbolically $H \subset H'$, if $H_x \subset H_x{}'$ for each $x \in X$. Clearly, $\emptyset \subset H \subset \bigcup$ for each fuzzy cutting H of X. The union $H \cup H'$ of H and H' is the fuzzy cutting defined by $H \cup H' = \{(x, H_x \cup H_x{}')\}$. The operations of intersection, complements, etc. on fuzzy cuttings are similarly defined. The fuzzy cuttings H and H' are said to be *disjoint* if $H_x \cap H_x{}' = \emptyset$ for each $x \in X$. A collection of fuzzy cuttings is said to be *disjoint* if each pair of this collection is disjoint.

Let $\rho \in \mathrm{F}Rel_E(X)$. Then, by Result 1.B(1), ρ induces an equivalence relation $\rho_I(x)$ in I, for each $x \in X$. Let the equivalence class of $r \in I$ with respect to $\rho_I(x)$ be denoted by $[r]_x$ or $r/\rho_I(x)$. In fact.

$$\begin{aligned}[r]_x &= \{s \in I : (r,s) \in \rho_I(x)\} \\ &= \{s \in I : \exists A \in \rho \text{ s.t. } ((x,x),(r,s)) \in A\}.\end{aligned}$$

For each $x_0 \in X$ and $r_0 \in I$, we define a fuzzy cutting $H(x_0, r_0)$ on X as follows: For each $y \in X$, if there exists $r \in I$ such that $((x_0,y),(r_0,r)) \in A$ for some $A \in \rho$. we set $H(x_0,r_0)(y) = [r]_y$ and it such r does not exist, we set $H(x_0,r_0)(y) = \emptyset$. It is clear that the function $H(x_0,r_0) : X \to P(I)$ is well-defined (see [2]).

Result 2.D[2, Proposition in p.305]. Let $\rho \in \mathrm{F}Rel_E(X)$. For every $x, x_1, x_2 \in X$ and let $r, r_1, r_2 \in I$. we have:

(1) $(x, [r]_x) \in H(x,r)$. Hence $H(x,r) \neq \emptyset$.

(2) $((x_1,x_2),(r_1,r_2)) \in A \in \rho$ if and only if $H(x_1,r_1) = H(x_2,r_2)$.

(3) $(r, r_1) \in \rho_I(x)$ if and only if $H(x,r) = H(x,r_1)$.

(4) If $H(x_1,r_1) \cap H(x_2,r_2) \neq \emptyset$, then $H(x_1,r_1) = H(x_2,r_2)$.

(5) $\bigcup_{x\in X, r\in I} H(x,r) = X$.

Definition 2.9[2]. Let $\rho \in \mathrm{F}Rel_E(X)$, let $x \in X$ and let $r \in I$. Then the fuzzy cutting $H(x,r)$ is called the *fuzzy equivalence class of* (x,r) *with respect to* ρ (or the ρ*-fuzzy equivalence class of* (x,r)) and is denoted by $[(x,r)]_\rho$ or simply $[(x,r)]$ if there is no ambiguity.

Let ρ be a fuzzy equivalence relation in X. Then we will denote the set of all ρ-fuzzy equivalence classes as X/ρ and call it the *fuzzy quotient set of* X *by* ρ.

Result 2.E[5,Proposition 3.9]. Let $\rho \in \mathrm{FRel_E}(X)$, let $x, y \in X$ and let $r, s \in I$. Then $[(x,r)]_\rho = [(y,s)]_\rho$ if and only if $[r]_x = [s]_y$.

3. Pre-image, restriction and quotient of fuzzy equivalence relations

Definition 3.1[2]. Let X and Y be nonempty sets. Then a fuzzy relation $\mathbb{F}$ from X to Y is called a *fuzzy function from* X *to* Y if $\mathbb{F} : I^X \to I^Y$ is a function characterized by the ordered pair $(F, \{f_x\}_{x\in X})$, where $F : X \to Y$ is a function and $\{f_x\}_{x\in X}$ is a family of functions $f_x : I \to I$ satisfying the conditions:

(α) f_x is nondecreasing on I,

(β) $f_x(0) = 0$ and $f_x(1) = 1$,

such that the image of any fuzzy set A in X under $\mathbb{F}$ is the fuzzy set in Y defined by: For each $y \in Y$,

$$\mathbb{F}(A)y = \begin{cases} \bigvee_{x\in F^{-1}(y)} f_x(A(x)) & \text{if} \quad F^{-1}(y) \neq \emptyset, \\ 0 & \text{if} \quad F^{-1}(y) = \emptyset. \end{cases}$$

In this case, we write $\mathbb{F} = (F, \{f_x\}_{x\in X}) : X \to Y$ or simply $\mathbb{F} = (F, f_x) : X \to Y$ and we call the functions f_x, $x \in X$, the *comemberhip function* to $\mathbb{F}$.

Example 3.1. Let $X = \{x, y, z\}$ and let $Y = \{a, b, c, d, e\}$. Consider the mappings $F : X \to Y$ and $f_x : I \to I$, $x \in X$, are respectively given by:

$$F(x) = a,\ F(y) = b,\ F(z) = c$$

and

$$f_x = id_I \text{ for each } x \in X.$$

Then we can easily see that $\mathbb{F} = (F, f_x) : X \to Y$ is a fuzzy function.

Definition 3.1 can be generalized in an obvious way by replacing the unit interval I by an arbitrary complete and completely distributive lattice.

Theorem 3.2. Let X and Y be nonempty sets, let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function and let $\rho \in \mathrm{F}Rel_E(Y)$. We define the fuzzy relation $\mathbb{F}^{-1}(\rho)$ in X as follows:

$$\mathbb{F}^{-1}(\rho) = \{\{((x,y),(r,s))\} \in X \underline{\times} X : \exists A \in \rho \quad \text{s.t.} \quad ((F(x), F(y)), (f_x(r), f_y(s))) \in A\}.$$

Then $\mathbb{F}^{-1}(\rho) \in \mathrm{F}Rel_E(X)$. In this case, $\mathbb{F}^{-1}(\rho)$ is called the *pre-image* of ρ under $\mathbb{F}$.

Proof. Let $x \in X$ and let $r \in I$. Since ρ is reflexive in Y, there exists $A \in \rho$ such that $((F(x), F(x)), (f_x(r), f_x(r))) \in A$. Then there exists $B \in \mathbb{F}^{-1}(\rho)$ such that $((x,x),(r,r)) \in B$. Thus $\mathbb{F}^{-1}(\rho)$ is reflexive in X. Suppose $((x,y),(r,s)) \in A \in \mathbb{F}^{-1}(\rho)$. Then there exists $B \in \rho$ such that $((F(x), F(y)), (f_x(r), f_y(s))) \in B$. Since ρ is symmetric in Y, there exists $C \in \rho$ such that $((F(y), F(x)), (f_y(s), f_x(r))) \in C$. Thus there exists $D \in \mathbb{F}^{-1}(\rho)$ such that $((y,x),(s,r)) \in D$. So $\mathbb{F}^{-1}(\rho)$ is symmetric in X. Suppose $((x,y),(r,s)) \in A \in \mathbb{F}^{-1}(\rho)$ and $((y,z),(s,t)) \in B \in \mathbb{F}^{-1}(\rho)$. Then there exist $C, D \in \rho$ such that $((F(x), F(y)), (f_x(r), f_y(s))) \in C$ and $((F(y), F(z)), (f_y(s), f_z(t))) \in D$. Since ρ is transitive in Y, there exists $E \in \rho$ such that $((F(x), F(z)), (f_x(r), f_z(t))) \in E$. Thus there exists $E' \in \mathbb{F}^{-1}(\rho)$ such that $((x,z),(r,t)) \in E'$. So $\mathbb{F}^{-1}(\rho)$ is transitive in X. Hence $\mathbb{F}^{-1}(\rho)$ is a fuzzy equivalence relation in X. □

Example 3.2. Let $\mathbb{F} = (F, f_x) : X \to Y$ be the fuzzy function in Example 2.1 and let $\rho = \Delta_Y \cup \{A, A^{-1}\}$ be the fuzzy relation in Y defined as follows :

A	a	b	c	d	e
a	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
b	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
c	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
d	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
e	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)

where $(r_0, t_0) \in J$ is fixed and $r_0 \neq t_0$.

Then ρ is a fuzzy equivalence relation in Y and $\mathbb{F}^{-1}(\rho) = \Delta_X \cup \{B, B^{-1}\}$, where

B	x	y	z
x	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
y	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)
z	(r_0, t_0)	(r_0, t_0)	(r_0, t_0)

Moreover it is clear that $\rho \in \mathrm{F}Rel_E(X)$. □

Definition 3.3. Let $\rho \in \mathrm{F}Rel_E(X)$ and let $Y \subset X$. Then the *fuzzy restriction of* ρ to Y, denoted by $\rho_{[Y]}$, is a fuzzy relation in Y defined as follows:

$$\rho_{[Y]} = \{C|_{Y \times Y} \in Y \underline{\times} Y : C \in \rho\}.$$

The following is the immediate result of Definition 3.3.

Proposition 3.4. $\rho_{[Y]}$ is a fuzzy equivalence relation in Y.

Proposition 3.5. Let $\rho \in \mathrm{F}Rel_E(X)$ and let $Y \subset X$. Then for each $x \in Y$ and $r \in I$,

$$[(x,r)]_{\rho_{[Y]}} = [(x,r)]_\rho|_Y.$$

Proof For each $y \in Y$, suppose there exists $s \in I$ such that $((x,y),(r,s)) \in A$ for some $A \in \rho$. Then $[(x,r)]_\rho|_Y(y) = s/\rho_I(y)$. Since $x, y \in Y$, $A|_{Y \times Y} \in \rho_{[Y]}$. Thus $[(x,r)]_{\rho_{[Y]}}(y) = s/\rho_{[Y]I}(y)$. Moreover, we can easily see that $s/\rho_I(y) = s/\rho_{[Y]I}(y)$. Suppose such $s \in I$ does not exist. Then $[(x,r)]_{\rho_{[Y]}}(y) = \varnothing = [(x,r)]_\rho|_Y(y)$. Hence $[(x,r)]_{\rho_{[Y]}} = [(x,r)]_\rho|_Y$. □

Proposition 3.6. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$ and let $\rho \subset \sigma$. If $(z, r/\sigma_I(z)) \in [(x,t)]_\sigma$, then $[(z,r)]_\rho \subset [(x,t)]_\sigma$.

Proof. Suppose $(z, r/\sigma_I(z)) \in [(x,t)]_\sigma$. Then $[(x,t)]_\sigma(z) = r/\sigma_I(z)$. Thus there exists $A \in \sigma$ such that $((x,z),(t,r)) \in A$. Let $(y, s/\rho_I(y)) \in [(z,r)]_\rho$. Then $[(z,r)]_\rho(y) = s/\rho_I(y)$. Thus there exists $B \in \rho$ such that $((z,y),(r,s)) \in B$. Since $\rho \subset \sigma$, there exists $C \in \sigma$ such that $((z,y),(r,s)) \in C$. Since σ is transitive, there exists $D \in \sigma$ such that $((x,y),(t,s)) \in D$. Thus $(y, s/\sigma_I(y)) \in [(x,t)]_\sigma$, i.e., $[(x,t)]_\sigma(y) = s/\sigma_I(y)$. Since $\rho \subset \sigma$, $s/\rho_I(y) \subset s/\sigma_I(y)$. So $[(z,r)]_\rho(y) \subset [(x,t)]_\sigma(y)$. Hence $[(z,r)]_\rho \subset [(x,t)]_\sigma$. □

Corollary 3.6. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$. If $\rho \subset \sigma$, then $[(x,r)]_\rho \subset [(x,r)]_\sigma$ for each $(x,r) \in X \times I$.

Theorem 3.7. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$ and let $\rho \subset \sigma$. We define a fuzzy relation σ/ρ in X/ρ as follows:

$$\sigma/\rho = \{\{(([(x,r)]_\rho, [(y,s)]_\rho), (u,v))\} \in X/\rho \,\overline{\times}\, X/\rho : \exists A \in \sigma \text{ s.t. } ((x,y),(r,s)) \in A,\ u = \textstyle\bigwedge_{t \in r/\rho_I(x)} t \text{ and } v = \bigwedge_{t \in s/\rho_I(y)} t\}.$$

Then σ/ρ is a fuzzy equivalence relation in X/ρ. In this case, σ/ρ is called the *fuzzy quotient of* σ *by* ρ.

Proof. Let $(x,u) \in X \times I$. Since $\rho \in \mathrm{F}Rel_E(X)$, there exists $r \in I$ such that $[(x,r)]_\rho \in X/\rho$ and $u = \bigwedge_{t \in r/\rho_I(x)} t$. Since σ is reflexive in X, there exists $A \in \sigma$ such that $((x,x),(r,r)) \in A$. Then there exists $B \in \sigma/\rho$ such that $(([(x,r)]_\rho, [(x,r)]_\rho), (u,u)) \in B$. Thus σ/ρ is reflexive in X/ρ. Suppose $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in A \in \sigma/\rho$. Then there exists $A \in \sigma$ such that $((x,y),(r,s)) \in A$, $u = \bigwedge_{t \in r/\rho_I(x)} t$ and $v = \bigwedge_{t \in s/\rho_I(y)} t$. Since σ is symmetric in X, there exists $B \in \sigma$ such that $((y,x),(s,r)) \in B$, $v = \bigwedge_{t \in s/\rho_I(y)} t$ and $u = \bigwedge_{t \in r/\rho_I(x)} t$. Thus there exists $C \in \sigma/\rho$ such that $(([(y,s)]_\rho, [(x,r)]_\rho), (v,u)) \in C$. So σ/ρ is symmetric in X/ρ. Suppose $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in A \in \sigma/\rho$ and $(([(y,s)]_\rho, [(z,t)]_\rho), (v,w)) \in B \in \sigma/\rho$. Then there exist $C, D \in \sigma$ such that $((x,y),(r,s)) \in C$, $u = \bigwedge_{a \in r/\rho_I(x)} a$, $v = \bigwedge_{a \in s/\rho_I(y)} a$ and $((y,z),(s,t)) \in D$, $v = \bigwedge_{a \in s/\rho_I(y)} a$ and $w = \bigwedge_{a \in t/\rho_I(z)} a$. Since σ is transitive in X, there exists $E \in \sigma$ such that $((x,z),(r,t)) \in E$, $u = \bigwedge_{a \in r/\rho_I(x)} a$ and $w = \bigwedge_{a \in t/\rho_I(z)} a$. Thus there exists $F \in \sigma/\rho$ such that $(([(x,r)]_\rho, [(z,t)]_\rho), (u,w)) \in F$. So σ/ρ is transitive in X/ρ. Hence σ/ρ is a fuzzy equivalence relation in X/ρ. □

Proposition 3.8. Let $\rho, \sigma, \pi \in \mathrm{F}Rel_E(X)$. If $\rho \subset \sigma$ and $\sigma \subset \pi$, then $\sigma/\rho \subset \pi/\rho$.

Proof. Let $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in A \in \sigma/\rho$. Then there exists $B \in \sigma$ such that $((x,y),(r,s)) \in B$, $u = \bigwedge_{t \in r/\rho_I(x)} t$ and $v = \bigwedge_{t \in s/\rho_I(y)} t$. Since $\sigma \subset \pi$, there exists $C \in \pi$ such that $((x,y),(r,s)) \in B$. Thus there exists $D \in \pi/\rho$ such that $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in D$. Hence $\sigma/\rho \subset \pi/\rho$. □

Theorem 3.9. Let $\rho, \sigma, \pi \in \mathrm{F}Rel_E(X)$, let $\rho \subset \sigma$ and let $\sigma \subset \pi$.

(1) $\rho \subset \sigma \circ \pi$.

(2) If $\sigma \circ \pi \in \mathrm{F}Rel_E(X)$, then $(\sigma/\rho) \circ (\pi/\rho) = (\sigma \circ \pi)/\rho$.

(3) $(\sigma/\rho) \circ (\pi/\rho)$ is a fuzzy equivalence relation in X/ρ.

Proof. (1) Let $((x,y),(r,s)) \in A \in \rho$. Since $\rho \subset \sigma$, there exists $B \in \sigma$ such that $((x,y),(r,s)) \in B$. Since $\sigma \subset \pi$, there exists $C \in \pi$ such that $((x,y),(r,s)) \in C$. Since σ is reflexive in X, there exists $D \in \sigma$ such that $((y,y),(s,s)) \in D$. Thus there exists $E \in \sigma \circ \pi$ such that $((x,y),(r,s)) \in E$. So $\rho \subset \sigma \circ \pi$.

(2) Suppose $\sigma \circ \pi \in \mathrm{F}Rel_E(X)$. Let $(([(x,r)]_\rho, [(z,t)]_\rho), (u,w)) \in A \in (\sigma \circ \pi)/\rho$. Then there exists $B \in \sigma \circ \pi$ such that $((x,z),(r,t)) \in B$, $u = \bigwedge_{a \in r/\rho_I(x)} a$ and $w = \bigwedge_{a \in t/\rho_I(z)} a$. Thus there exists $(y,s) \in X \times I$ such that $((x,y),(r,s)) \in C$ and $((y,z),(s,t)) \in D$ for some $C \in \pi$ and $D \in \sigma$. Let $v = \bigwedge_{a \in s/\rho_I(y)} a$. Since $\rho \subset \sigma \subset \pi$, there exist $E \in \pi/\rho$ and $F \in \sigma/\rho$ such that $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in E$ and $(([(y,s)]_\rho, [(z,t)]_\rho), (v,w)) \in F$. Thus there exists $G \in (\sigma/\rho) \circ (\pi/\rho)$ such that $(([(x,r)]_\rho, [(z,t)]_\rho), (u,w)) \in G$. So $(\sigma \circ \pi)/\rho \subset (\sigma/\rho) \circ (\pi/\rho)$. Now let $(([(x,r)]_\rho, [(z,t)]_\rho), (u,w)) \in A \in (\sigma/\rho) \circ (\pi/\rho)$. Then there exists $([(y,s)]_\rho, v) \in X/\rho \times I$ such that $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in B$ and $(([(y,s)]_\rho, [(z,t)]_\rho), (v,w)) \in C$ for some $B \in \pi/\rho$ and $C \in \sigma/\rho$. Thus there exist $D \in \pi$ and $E \in \sigma$ such that $((x,y),(r,s)) \in D$, $((y,z),(s,t)) \in E$, $u = \bigwedge_{a \in r/\rho_I(x)} a$, $v = \bigwedge_{a \in s/\rho_I(y)} a$ and $w = \bigwedge_{a \in t/\rho_I(z)} a$. So there exists $F \in \sigma \circ \pi$ such that $((x,z),(r,t)) \in F$. Since $\sigma \circ \pi$ is a fuzzy equivalence relation in X and $\rho \subset \sigma \circ \pi$, there exists $G \in (\sigma \circ \pi)/\rho$ such that $(([(x,r)]_\rho, [(z,t)]_\rho), (u,w)) \in G$. Thus $(\sigma/\rho) \circ (\pi/\rho) \subset (\sigma \circ \pi)/\rho$. Hence $(\sigma/\rho) \circ (\pi/\rho) = (\sigma \circ \pi)/\rho$.

(3) For each $(x,r) \in X \times I$, let $u = \bigwedge_{a \in r/\rho_I(x)} a$. Since σ/ρ and π/ρ is reflexive in X/ρ, there exist $A \in \pi/\rho$ and $B \in \sigma/\rho$ such that $(([(x,r)]_\rho, [(x,r)]_\rho), (u,u)) \in A$ and $(([(x,r)]_\rho, [(x,r)]_\rho), (u,u)) \in B$. Thus there exists $C \in (\sigma/\rho) \circ (\pi/\rho)$ such that $(([(x,r)]_\rho, [(x,r)]_\rho), (u,u)) \in C$. So $(\sigma/\rho) \circ (\pi/\rho)$ is reflexive in X/ρ. Now suppose $(([(x,r)]_\rho, [(z,t)]_\rho), (u,w)) \in A \in (\sigma/\rho) \circ (\pi/\rho)$. Then there exists $([(y,s)]_\rho, v) \in X/\rho \times I$ such that $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in D$ and $(([(y,s)]_\rho, [(z,t)]_\rho), (v,w)) \in E$ for some $D \in \pi/\rho$ and for some $E \in \sigma/\rho$. Thus there exist $D' \in \pi$ and $E' \in \sigma$ such that $((x,y),(r,s)) \in D'$, $((y,z),(s,t)) \in E'$, $u = \bigwedge_{a \in r/\rho_I(x)} a$, $v = \bigwedge_{a \in s/\rho_I(y)} a$ and $w = \bigwedge_{a \in t/\rho_I(z)} a$. Since π and σ are symmetric in X, there exist $D'' \in \pi$ and $E'' \in \sigma$ such that $((y,x),(s,r)) \in D''$ and $((z,y),(t,s)) \in E''$. Then there exist $G \in \pi/\rho$ and $F \in \sigma/\rho$ such that $(([(y,s)]_\rho, [(x,r)]_\rho), (v,u)) \in F$ and $(([(z,t)]_\rho, [(y,s)]_\rho), (w,v)) \in G$. Thus there exists $H \in (\sigma/\rho) \circ (\pi/\rho)$ such that $(([(z,t)]_\rho, [(x,r)]_\rho), (w,u)) \in H$. So $(\sigma/\rho) \circ (\pi/\rho)$ is symmetric in X/ρ. Finally, suppose $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in A \in (\sigma/\rho) \circ (\pi/\rho)$ and $(([(y,s)]_\rho, [(z,t)]_\rho), (v,w)) \in B \in (\sigma/\rho) \circ (\pi/\rho)$. Then there exists $([(x',r')]_\rho, u') \in X/\rho \times I$

such that $(([(x,r)]_\rho,[(x',r')]_\rho),(u,u')) \in C$ and $(([(x',r')]_\rho,[(y,s)]_\rho),(u',v)) \in D$ for some $C \in \pi/\rho$ and $D \in \sigma/\rho$, and there exists $([(y',s')]_\rho, v') \in X/\rho \times I$ such that $(([(y,s)]_\rho,[(y',s')]_\rho),(v,v')) \in E$ and $(([(y',s')]_\rho,[(z,t)]_\rho),(v',w)) \in F$ for some $E \in \pi/\rho$ and $F \in \sigma/\rho$. Moreover, $u = \bigwedge_{a\in r/\rho_I(x)} a$, $u' = \bigwedge_{a\in r'/\rho_I(x')} a$, $v = \bigwedge_{a\in s/\rho_I(y)} a$, $v' = \bigwedge_{a\in s'/\rho_I(y')} a$ and $w = \bigwedge_{a\in t/\rho_I(z)} a$. Thus there exist $C' \in \pi$ and $D' \in \sigma$ such that $((x,x'),(r,r')) \in C'$ and $((x',y),(r',s)) \in D'$, and there exist $E' \in \pi$ and $F' \in \sigma$ such that $((y,y'),(s,s')) \in D'$ and $((y',z),(s',t)) \in F'$. Since $\sigma \subset \pi$, there exists $D'' \in \pi$ such that $((x',y),(r',s)) \in D''$. Since π is transitive in X, there exists $G \in \pi$ such that $((x,y'),(r,s')) \in G$. Then there exist $G' \in \pi/\rho$ and $F'' \in \sigma/\rho$ such that $(([(x,r)]_\rho,[(y',s')]_\rho),(u,v')) \in G'$ and $(([(y',s')]_\rho,[(z,t)]_\rho),(v',w)) \in F''$. Thus there exists $H \in (\sigma/\rho)\circ(\pi/\rho)$ such that $(([(x,r)]_\rho,[(z,t)]_\rho),(u,w)) \in H$. So $(\sigma/\rho)\circ(\pi/\rho)$ is transitive in X/ρ. Hence $(\sigma/\rho)\circ(\pi/\rho)$ is a fuzzy equivalence relation in X/ρ. □

Theorem 3.10. Let $\rho \in \mathrm{F}Rel_E(X)$ and let $\sigma \in \mathrm{F}Rel_E(Y)$. Let $\rho\cdot\sigma$ be the fuzzy relation in $X\times Y$ defined as follows:

$$\rho\cdot\sigma = \{\{([(x,w),(y,z)],(r,s))\} \in (X\times Y)\underline{\times}(X\times Y) : \exists A \in \rho \text{ and } B \in \sigma \text{s.t. } ((x,y),(r,s)) \in A \text{ and } ((w,z),(r,s)) \in B\}.$$

Then $\rho\cdot\sigma \in \mathrm{F}Rel_E(X\times Y)$. In this case, $\rho\cdot\sigma$ is called the *fuzzy product* of ρ and σ.

Proof. Let $[(x,y),(x,y)] \in X\times Y$ and let $r \in I$. Since ρ is reflexive in X and σ is reflexive in Y, there exist $A \in \rho$ and $B \in \sigma$ such that $((x,x),(r,r)) \in A$ and $((y,y),(r,r)) \in B$. Thus there exists $C \in \rho\cdot\sigma$ such that $([(x,y),(x,y)],(r,r)) \in C$. So $\rho\cdot\sigma$ is reflexive in $X\times Y$. Now suppose $([(x,w),(y,z)],(r,s)) \in A \in \rho\cdot\sigma$. Then there exist $B \in \rho$ and $C \in \sigma$ such that $((x,y),(r,s)) \in B$ and $((w,z),(r,s)) \in C$. Since ρ and σ are symmetric, there exist $B' \in \rho$ and $C' \in \sigma$ such that $((y,x),(s,r)) \in B'$ and $((z,w),(s,r)) \in C'$. Thus there exists $A' \in \rho\cdot\sigma$ such that $([(y,z),(x,w)],(s,r)) \in A'$. So $\rho\cdot\sigma$ is symmetric in $X\times Y$. Finally suppose $([(x,w),(y,z)],(r,s)) \in A \in \rho\cdot\sigma$ and $([(y,z),(u,v)],(s,t)) \in B \in \rho\cdot\sigma$. Then there exist $B, C \in \rho$ and $D, E \in \sigma$ such that $((x,y),(r,s)) \in B$, $((w,z),(r,s)) \in D$ and $((y,u),(s,t)) \in C$, $((z,v),(s,t)) \in E$. Since ρ and σ are transitive, there exist $F \in \rho$ and $G \in \sigma$ such that $((x,u),(r,t)) \in F$ and $((w,v),(r,t)) \in G$. Thus there exists $H \in \rho\cdot\sigma$ such that $([(x,w),(u,v)],(r,t)) \in H$. So $\rho\cdot\sigma$ is transitive in $X\times Y$. Hence $\rho\cdot\sigma$ is a fuzzy equivalence relation in $X\times Y$. □

Proposition 3.11. Every fuzzy equivalence relation in a set X is the pre-image of a fuzzy equivalence relation in $X\times X$.

Proof. Let $\mathbb{F} = (F, id_I) : X \to X\times X$ be the fuzzy function defined as follows: $F(x) = (x,x)$ for each $x \in X$. Let ρ be a fuzzy equivalence relation in X. Consider the fuzzy product $\rho\cdot\rho$ in $X\times X$. Then clearly $\rho\cdot\rho$ is a fuzzy equivalence relation in $X\times X$ by Theorem 3.10. We shall show that $\rho = \mathbb{F}^{-1}(\rho\cdot\rho)$. Let $((x,y),(r,s)) \in A \in \mathbb{F}^{-1}(\rho\cdot\rho)$. Then, by Theorem 3.2, there exists $B \in \rho\cdot\rho$ such that $((F(x),F(y)),(id_I(r),id_I(s))) \in B$. Since $F(x) = (x,x)$, $([(x,x),(y,y)],(r,s)) \in B$. Thus, by the definition of fuzzy product, there exist $C, D \in \rho$ such that $((x,y),(r,s)) \in C$ and $((x,y),(r,s)) \in D$. So $\mathbb{F}^{-1}(\rho\cdot\rho) \subset \rho$. Now let $((x,y),(r,s)) \in A \in \rho$. Then there exists $B \in \rho\cdot\rho$ such that $([(x,x),(y,y)],(r,s)) \in B$. By the definition of $\mathbb{F}$, $((F(x),F(y)),(id_I(r),id_I(s))) \in B$. Thus there exists $C \in \mathbb{F}^{-1}(\rho\cdot\rho)$ such that $((x,y),(r,s)) \in C$. So $\rho \subset \mathbb{F}^{-1}(\rho\cdot\rho)$. Hence $\rho = \mathbb{F}^{-1}(\rho\cdot\rho)$. □

Proposition 3.12. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function and let $\rho \in \mathrm{F}Rel_E(Y)$. Then $\mathbb{F}^{-1}(\rho) = \mathbb{F}^{-1}\circ\rho\circ\mathbb{F}$.

Proof. Let $((x,y),(r,s)) \in A \in \mathbb{F}^{-1}(\rho)$. Then there exists $B \in \rho$ such that $((F(x),F(y)),(f_x(r),f_y(s)) \in B$. Since $\mathbb{F}(x,r) = (F(x), f_x(r))$, there exists $C \in \mathbb{F}$ such that $((x,F(x)),(r,f_x(r)) \in C$. Thus there exists $D \in \rho\circ\mathbb{F}$ such that $((x,F(y)),(r,f_y(s)) \in D$. Since $\mathbb{F}(y,s) = (F(y), f_y(s))$, there exists $C' \in \mathbb{F}^{-1}$ such that $((F(y),y),(f_y(s),s)) \in \mathbb{F}^{-1}$. Thus there exists $D' \in \mathbb{F}^{-1}\circ\rho\circ\mathbb{F}$ such that $((x,y),(r,s)) \in D'$. So $\mathbb{F}^{-1}(\rho) \subset \mathbb{F}^{-1}\circ\rho\circ\mathbb{F}$. By the similar arguments, we can see that $\mathbb{F}^{-1}\circ\rho\circ F \subset \mathbb{F}^{-1}(\rho)$. This completes the proof. □

4. Fuzzy equivalence relations and fuzzy functions

Two fuzzy functions $\mathbb{F} = (F, f_x)$ and $\mathbb{G} = (G, g_x)$ from X to Y are said to be *equal*, symbolically $\mathbb{F} = \mathbb{G}$, if $\mathbb{F}(A) = \mathbb{G}(A)$ for each $A \in I^X$.

Result 4.A[2, Theorem 5]. Two fuzzy functions $\mathbb{F} = (F, f_x)$ and $\mathbb{G} = (G, g_x)$ from X to Y are equal if and only if $F = G$ and $f_x = g_x$ for each $x \in X$.

Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. The *inverse image* under $\mathbb{F}$ of a fuzzy set B in Y, written $\mathbb{F}^{-1}(B)$, is a fuzzy set in X defined by:

$$\mathbb{F}^{-1}(B) = \bigcup\{C \in I^Y : \mathbb{F}(C) \subset B\}.$$

Result 4.B[2, Proposition in p.311]. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function whose comembership functions f_x are surjective. Then for each fuzzy set B in Y,

$$\mathbb{F}^{-1}(B) = \bigvee f_x{}^{-1}[B(F(x))].$$

where the supremum is taken over the set of values $f_x{}^{-1}[B(F(x))] \subset I$.

Result 4.C[2, Theorem 6]. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. Let $A, B \in I^X$, $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$ and let $C, D \in I^Y$, $\{C_\alpha\}_{\alpha\in\Gamma} \subset I^Y$. Then:

(1) $\mathbb{F}(\emptyset) = \emptyset$.

(2) $\mathbb{F}(X) = Y$ if F is surjective.

(3) If $A \subset B$, then $\mathbb{F}(A) \subset \mathbb{F}(B)$.

(4) $\mathbb{F}(A \cup B) = \mathbb{F}(A) \cup \mathbb{F}(B)$.

(5) $\mathbb{F}(A \cap B) \subset \mathbb{F}(A) \cap \mathbb{F}(B)$ (equality holds if F is injective).

(6) $\mathbb{F}^{-1}(Y) = X$.

(7) If $C \subset D$, then $\mathbb{F}^{-1}(C) \subset \mathbb{F}^{-1}(D)$.

(8) $\mathbb{F}^{-1}(\mathbb{F}(A)) \supset A$ (equality holds if F is injective).

If f_x is surjective for each $x \in X$, then:

(9) $\mathbb{F}(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} \mathbb{F}(A_\alpha)$.

(10) $\mathbb{F}(\bigcap_{\alpha\in\Gamma} A_\alpha) \subset \bigcap_{\alpha\in\Gamma} \mathbb{F}(A_\alpha)$ (equality holds if F is injective).

(11) $\mathbb{F}^{-1}(\bigcup_{\alpha\in\Gamma} C_\alpha) = \bigcup_{\alpha\in\Gamma} \mathbb{F}^{-1}(C_\alpha)$.

(12) $\mathbb{F}^{-1}(\bigcap_{\alpha\in\Gamma} C_\alpha) = \bigcap_{\alpha\in\Gamma} \mathbb{F}^{-1}(C_\alpha)$.

(13) $\mathbb{F}(\mathbb{F}^{-1}(C)) \subset C$ (equality holds if F is surjective).

If $f_x(1-r) \geq 1 - f_x(r)$ for each $x \in X$, $r \in I$, then:

(14) $\mathbb{F}(A^c) \supset (\mathbb{F}(A))^c$ if F is surjective (Equality holds if F is bijective and $f_x(1-r) = 1 - f_x(r)$).

If f_x is bijective and if $f_x(1-r) = 1 - f_x(r)$, then

(15) $\mathbb{F}^{-1}(D^c) = (\mathbb{F}^{-1}(D))^c$.

The *composition* of two fuzzy functions $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_x) : Y \to Z$ is the fuzzy function $\mathbb{G} \circ \mathbb{F} : X \to Z$ defined by $(\mathbb{G} \circ \mathbb{F})(A) = \mathbb{G}(\mathbb{F}(A))$.

Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. $\mathbb{F}$ is said to be *injective* if for any $A_1, A_2 \in I^X$ with $\mathbb{F}(A_1) = \mathbb{F}(A_2)$, $A_1 = A_2$. Surjective and bijective fuzzy functions can be defined similarly in an obvious manner. It is not difficult to prove that $\mathbb{F}$ is injective [resp. surjective] if and only if F and f_x, $x \in X$, are injective [resp. surjective].

A fuzzy function $\mathbb{F} = (F, f_x) : X \to Y$ is said to be *invertible* if there exists a fuzzy function $\mathbb{G} = (G, g_y) : Y \to X$ such that $\mathbb{G} \circ \mathbb{F} = \mathbf{id}_X$ and $\mathbb{F} \circ \mathbb{G} = \mathbf{id}_Y$, where $\mathbf{id}_X = (id_X, id_I)$. The fuzzy function $\mathbb{G}$ is called the *inverse* of $\mathbb{F}$ and is denoted by $\mathbb{F}^{-1}$.

Result 4.D[2, Theorem 7]. Let $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to Z$ be fuzzy functions. Let g_y be surjective for each $y \in Y$. Then:

(1) The composition $\mathbb{G} \circ \mathbb{F} : X \to Z$ of $\mathbb{F}$ and $\mathbb{G}$ is given by

$$\mathbb{G} \circ \mathbb{F} = (G \circ F, g_{F(x)} \circ f_x).$$

(2) $\mathbb{F} = (F, f_x)$ is injective [resp. surjective] if and only if F and f_x, $x \in X$, are injective [resp. surjective].

(3) $\mathbb{F} = (F, f_x)$ is invertible if and only if F and f_x are invertible. The inverse $\mathbb{F}^{-1}$ of $\mathbb{F}$ is given by $\mathbb{F}^{-1} = (F^{-1}, f_x{}^{-1})$.

Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. We define a fuzzy relation ρ in X as follows:

$$\rho = \{\{((x,y),(r,s))\} \in X \underline{\times} X : F(x) = F(y) \text{ and } f_x(r) = f_y(s)\}.$$

Then it is easy to show that ρ is a fuzzy equivalence relation in X. In this case, ρ is called the *fuzzy equivalence relation determined by* $\mathbb{F}$ or the *fuzzy Kernel* of $\mathbb{F}$ and will be denoted by $\rho_{\mathbb{F}}$ or Ker $\mathbb{F}$.

Theorem 4.1. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a surjective fuzzy function and let $\rho \in \mathrm{F}Rel_E(X)$ such that $\rho_{\mathbb{F}} \subset \rho$. We define the *image of* ρ *under* $\mathbb{F}$ as follows:

$$\mathbb{F}(\rho) = \{\{((F(x), F(y)),(t,u))\} \in Y \underline{\times} Y : \exists A \in \rho \text{ and } r, s \in I \text{ s.t.} ((x,y),(r,s)) \in A,\ t = f_x(r) \text{ and } u = f_y(s)\}.$$

Then $\mathbb{F}(\rho) \in \mathrm{F}Rel_E(Y)$.

Proof. Let $(y,t) \in Y \times I$. Since $\mathbb{F} = (F, f_y) : X \to Y$ is surjective, by Result 4.D(2), $F : X \to Y$ and $f_y : I \to I$ are surjective. Then there exists $(x, r) \in X \times I$ such that $t = f_y(r)$ and $y = F(x)$. Since ρ is reflexive, there exists $A \in \rho$ such that $((x,x),(r,r)) \in A$. Thus there exists $B \in \mathbb{F}(\rho)$ such that $((F(x),F(x)),(t,t)) = ((y,y),(t,t)) \in B$. So $\Delta_Y \subset \mathbb{F}(\rho)$, i.e., $\mathbb{F}(\rho)$ is reflexive in Y. Suppose $((F(x),F(y)),(t,u)) \in A \in \mathbb{F}(\rho)$. Then there exist $B \in \rho$ and $r, s \in I$ s.t. $((x,y),(r,s)) \in B$, $t = f_x(r)$ and $u = f_y(s)$. Since ρ is symmetric, there exists $C \in \rho$ such that $((y,x),(s,r)) \in C$, $u = f_y(s)$ and $t = f_x(r)$. Thus there exists $D \in \mathbb{F}(\rho)$ such that $((F(y),F(x)),(u,t)) \in D$. So $\mathbb{F}(\rho)$ is symmetric in Y. Suppose $((F(x),F(y)),(t,u)) \in A \in \mathbb{F}(\rho)$ and $((F(y),F(z)),(u,v)) \in B \in \mathbb{F}(\rho)$. Then there exist $A', B' \in \rho$ and $r_1, r_2, r_3 \in I$ such that $((x,y),(r_1,r_2)) \in A'$, $((y,z),(r_2,r_3)) \in B'$, $t = f_x(r_1)$, $u = f_y(r_2)$ and $v = f_z(r_3)$. Since ρ is transitive, there exists $C \in \rho$ such that $((x,z),(r_1,r_3)) \in C$. Thus there exists $D \in \mathbb{F}(\rho)$ such that $((F(x),F(z)),(t,v)) \in D$. So $\mathbb{F}(\rho)$ is transitive in Y. Hence $\mathbb{F}(\rho) \in FRel_E(Y)$. □

Theorem 4.2. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a surjective fuzzy function and let ρ be any fuzzy equivalence relation in Y. Then:

(1) $\rho_{\mathbb{F}} \subset \mathbb{F}^{-1}(\rho)$.

(2) $\sigma = \mathbb{F}^{-1}(\rho)$ if and only if $\rho = \mathbb{F}(\sigma)$.

Hence there exists a one-to-one correspondence between the fuzzy equivalence relations in Y and the fuzzy equivalence relations in X containing $\rho_{\mathbb{F}}$.

Proof. (1) Let $((x, y), (r, s)) \in A \in \rho_{\mathbb{F}}$. Then $F(x) = F(y)$ and $f_x(r) = f_y(s)$. Since ρ is reflexive, there exists $B \in \rho$ such that $((F(x), F(y)), (f_x(r), f_y(s))) \in B$. Thus there exists $C \in \mathbb{F}^{-1}(\rho)$ such that $((x, y), (r, s)) \in C$. Hence $\rho_{\mathbb{F}} \subset \mathbb{F}^{-1}(\rho)$.

(2) ($\Rightarrow$): Suppose $\sigma = \mathbb{F}^{-1}(\rho)$. Then, by (1), $\rho_{\mathbb{F}} \subset \sigma$. Thus, by Theorem 4.1, $\mathbb{F}(\sigma) \in FRel_E(Y)$. Now let $((y, y'), (t, u)) \in A \in \rho$. Since $\mathbb{F} = (F, f_x) : X \to Y$ is a surjective fuzzy function by Result 4.D(2), $F : X \to Y$ and $f_x : I \to I$ are surjective. Thus there exist $x, x' \in X$ and $r, s \in I$ such that $y = F(x)$, $y' = F(x')$ and $t = f(r)$, $u = f(s)$. So $((F(x), F(x')), (f(r), f(s))) \in A \in \rho$. By the definition of $\mathbb{F}^{-1}(\rho)$, there exists $B \in \mathbb{F}^{-1}(\rho)$ such that $((x, x'), (r, s)) \in B$. Since $\sigma = \mathbb{F}^{-1}(\rho)$, by the definition of $\mathbb{F}(\sigma)$, there exists $C \in \mathbb{F}(\sigma)$ such that $((F(x), F(x')), (f(r), f(s))) = ((y, y'), (t, u)) \in C$. So $\rho \subset \mathbb{F}(\sigma)$. Similarly, we can see that $\mathbb{F}(\sigma) \subset \rho$. Hence $\rho = \mathbb{F}(\sigma)$.

($\Leftarrow$): Suppose $\rho = \mathbb{F}(\sigma)$. Then, by the similar arguments, we can easily see that $\sigma = \mathbb{F}^{-1}(\rho)$. This completes the proof. □

Proposition 4.3. If $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function, then $\rho_{\mathbb{F}} = \mathbb{F}^{-1} \circ \mathbb{F}$.

Proof. Let $((x, y), (r, s) \in A \in \rho_{\mathbb{F}}$. Then $F(x) = F(y)$ and $f_x(r) = f_y(s)$. Since $\mathbb{F}(x, r) = (F(x), f_x(r))$ and $\mathbb{F}(y, s) = (F(y), f_y(s))$, there exist $B \in \mathbb{F}$ and $C \in \mathbb{F}^{-1}$ such that $((x, F(x)), (r, f_x(r)) \in B$ and $((F(y), y), (f_y(s), s)) \in C$. In particular, $((F(x), y), (f_x(r), s)) \in C$. Thus there exists $D \in \mathbb{F}^{-1} \circ \mathbb{F}$ such that $((x, y), (r, s)) \in D$. So $\rho_{\mathbb{F}} \subset \mathbb{F}^{-1} \circ \mathbb{F}$. By the similar arguments, we can see that $\mathbb{F}^{-1} \circ \mathbb{F} \subset \rho_{\mathbb{F}}$. Hence $\rho_{\mathbb{F}} = \mathbb{F}^{-1} \circ \mathbb{F}$. □

Proposition 4.4. Let $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to Z$ be fuzzy functions and let g_y be surjective for each $y \in Y$. Then $\mathbb{F}^{-1}(\rho_{\mathbb{G}}) = \rho_{\mathbb{G} \circ \mathbb{F}}$.

Proof. Let $((x, y), (r, s)) \in A \in \mathbb{F}^{-1}(\rho_{\mathbb{G}})$. Then there exists $B \in \rho_{\mathbb{G}}$ such that
$((F(x), F(y)), (f_x(r), f_y(s))) \in B$. Thus

$$(G \circ F)(x) = G(F(x)) = G(F(y)) = (G \circ F)(y)$$

and

$$(g_{F(x)} \circ f_x)(r) = g_{F(x)}(f_x(r)) = g_{F(y)}(f_y(s)) = (g_{F(y)} \circ f_y)(s).$$

Moreover, by Result 4.D(1), $\mathbb{G} \circ \mathbb{F} = (G \circ F, g_{F(x)} \circ f_x)$. So there exists $C \in \rho_{\mathbb{G} \circ \mathbb{F}}$ such that $((x, y), (r, s)) \in C$, i.e., $\mathbb{F}^{-1}(\rho_{\mathbb{G}}) \subset \rho_{\mathbb{G} \circ \mathbb{F}}$. By the similar arguments, we can see that $\rho_{\mathbb{G} \circ \mathbb{F}} \subset \mathbb{F}^{-1}(\rho_{\mathbb{G}})$. Hence $\mathbb{F}^{-1}(\rho_{\mathbb{G}}) = \rho_{\mathbb{G} \circ \mathbb{F}}$. □

Let ρ be a fuzzy equivalence relation in a set X. We define a fuzzy relation $\mathbb{F}$ from X to X/ρ as follows: For each $(x, r) \in X \times I$,

$$\mathbb{F} = \{\{(x, F(x), r, f_x(r))\} \in X \underline{\times} X/\rho\},$$

where $F : X \to X/\rho$ is the mapping given by $F(x) = [(x, r)]_\rho$ and $f_x : I \to I$ is the mapping given by $f_x(r) = \bigwedge_{t \in r/\rho_I(x)} t$. Then it is easy to see that $\mathbb{F} = (F, f_x) : X \to X/\rho$ is a fuzzy function. In this case, $\mathbb{F}$ is called the *canonical fuzzy function from X to X/ρ*.

Theorem 4.5. Let ρ be a fuzzy equivalence relation in a set X. If $\mathbb{F} = (F, f_x) : X \to X/\rho$ is the canonical fuzzy function, then $\rho = \rho_{\mathbb{F}}$.

Proof. Let $((x, y), (r, s)) \in A \in \rho$. Then, by Result 2.D(2), $[(x, r)]_\rho = [(y, s)]_\rho$. Thus, by Result 1.E, $r/\rho_I(x) = s/\rho_I(y)$. So, by the definition of $\mathbb{F}$, $F(x) = F(y)$ and $f_x(r) = f_y(s)$. Thus there exists $B \in \rho_{\mathbb{F}}$ such that $((x, y), (r, s)) \in B$ and hence $\rho \subset \rho_{\mathbb{F}}$. Now let $((x, y), (r, s)) \in A \in \rho_{\mathbb{F}}$. Then, by the definition of $\rho_{\mathbb{F}}$, $F(x) = F(y)$ and $f_x(r) = f_y(s)$. Thus $[(x, r)]_\rho = [(y, s)]_\rho$. So, by Result 1.D(2), there exists $C \in \rho$ such that $((x, y), (r, s)) \in C$ and hence $\rho_{\mathbb{F}} \subset \rho$. Therefore $\rho = \rho_{\mathbb{F}}$. □

Let X and Y be sets and let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. Then we will define three fuzzy functions $\mathbb{R}$, $\mathbb{S}$, $\mathbb{T}$, obtained from $\mathbb{F}$, which play an important in many mathematical arguments. Let ρ be a fuzzy equivalence relation in X determined by $\mathbb{F}$.

$\mathbb{R} = (R, r_x) : X \to X/\rho$ is the canonical fuzzy function from X to X/ρ.

$\mathbb{S} = (S, s_x) : X/\rho \to \mathbb{F}(X)$ is the fuzzy function defined as follows:
For each $([(x, r)]_\rho, t) \in X/\rho \times I$,

$$\mathbb{S}([(x, r)]_\rho, t) = (S([(x, r)]_\rho), s_x(t)),$$

where $S : X/\rho \to \mathbb{F}(X)$ is the mapping given by $S([(x, r)]_\rho) = F(x)$ and $s_x : I \to I$ is the mapping given by $s_x = id_I$.

$\mathbb{T} = (T, t_x) : \mathbb{F}(X) \to Y$ is the fuzzy function defined as follows:
For each $(y, s) \in \mathbb{F}(X) \times I$,

$$\mathbb{T}(y, s) = (T(y), t_x(s)),$$

where $T : F(X) \to Y$ is the inclusion mapping of $F(X)$ in Y and $t_x : I \to I$ is the mapping given by

$t_x = id_I$.

The following is the immediate result of above definitions and Result 4.D.

Theorem 4.6. Let X and Y be sets, let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function, let ρ be the fuzzy equivalence relation determined by $\mathbb{F}$ and let $\mathbb{R}, \mathbb{S}, \mathbb{T}$ be the fuzzy functions defined above. Then $\mathbb{R}$ is surjective, $\mathbb{S}$ is bijective, $\mathbb{T}$ is injective and $\mathbb{F} = \mathbb{T} \circ \mathbb{S} \circ \mathbb{R}$.

We may sum up the foregoing results by saying that any fuzzy function $\mathbb{F} = (F, f_x) : X \to Y$ can be expressed as a composite of three fuzzy functions $\mathbb{R}$, $\mathbb{S}$, $\mathbb{T}$ which are, respectively, surjective, bijective and injective. This is referred to as the *fuzzy canonical decomposition of* $\mathbb{F}$, and it will be customarily exhibited in a diagram such as the following:

$$X \overset{\mathbb{R}}{\underset{\text{surj}}{\to}} X/\rho \overset{\mathbb{S}}{\underset{\text{bij}}{\hookrightarrow}} \mathbb{F}(X) \overset{\mathbb{T}}{\underset{\text{inj}}{\hookrightarrow}} Y.$$

One of the results of Theorem 3.6 is especially useful; namely, that if $\mathbb{F} = (F, f_x) : X \to Y$ is a fuzzy function, then $X/\rho_{\mathbb{F}}$ and $\mathbb{F}(X)$ are in one-to-one correspondence. This will be customarily expressed by writing $X/\rho_{\mathbb{F}} \approx \mathbb{F}(X)$.

The following is the immediate result of Result 4.C(2) and Theorem 4.6.

Corollary 4.6. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. If $\mathbb{F}$ is surjective, then $X/\rho_{\mathbb{F}} \approx Y$.

Proposition 4.7. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$ such that $\rho \subset \sigma$. If $\mathbb{F} = (F, f_x) : X \to X/\rho$ is the canonical fuzzy function, then $\sigma/\rho = \mathbb{F}(\sigma)$.

Proof. Let $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in A \in \sigma/\rho$. Then there exists $B \in \sigma$ such that $((x,y),(r,s)) \in B$, $u = \bigwedge_{t \in r/\rho_I(x)} t$ and $v = \bigwedge_{t \in s/\rho_I(y)} t$. Since $\mathbb{F} = (F, f_x) : X \to X/\rho$ is the canonical fuzzy function, $F(x) = [(x,r)]_\rho$, $F(y) = [(y,s)]_\rho$, $u = f_x(r)$ and $v = f_y(s)$. Thus there exists $C \in \mathbb{F}(\sigma)$ such that $((F(x), F(y)), (u,v)) \in C$. So $\sigma/\rho \subset \mathbb{F}(\sigma)$. By the similar arguments, we can see that $\mathbb{F}(\sigma) \subset \sigma/\rho$. Hence $\sigma/\rho = \mathbb{F}(\sigma)$. □

Proposition 4.8. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$ such that $\rho \subset \sigma$. If $\mathbb{F} = (F, f_x) : X \to X/\rho$ is the canonical fuzzy function, then $\sigma = \mathbb{F}^{-1}(\sigma/\rho)$.

Proof. Let $((x,y),(r,s)) \in A \in \sigma$. Since $\mathbb{F} = (F, f_x) : X \to X/\rho$ is the canonical fuzzy function, $F(x) = [(x,r)]_\rho$, $F(y) = [(y,s)]_\rho$, $f_x(r) = \bigwedge_{t \in r/\rho_I(x)} t$ and $f_y(s) = \bigwedge_{t \in s/\rho_I(y)} t$. Then there exists $B \in \sigma/\rho$ such that $((F(x), F(y)), (f_x(r), f_y(s))) \in B$. Thus there exists $C \in \mathbb{F}^{-1}(\sigma/\rho)$ such that $((x,y),(r,s)) \in C$. So $\sigma \subset \mathbb{F}^{-1}(\sigma/\rho)$. By the similar arguments, we can easily see that $\mathbb{F}^{-1}(\sigma/\rho) \subset \sigma$. Hence $\sigma = \mathbb{F}^{-1}(\sigma/\rho)$. □

We can easily see that the following holds.

Proposition 4.9. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function and let ρ be any fuzzy equivalence relation in X such that $\rho \subset \rho_{\mathbb{F}}$. We define a fuzzy relation $\mathbb{F}/\rho$ from X/ρ to Y as follows: For each $([(x,r)]_\rho, u) \in X/\rho \times I$,

$$\mathbb{F}/\rho = \{\{([(x,r)]_\rho, F(x), u, v)\} \in X/\rho \underline{\times} Y\},$$

where $F/\rho : X/\rho \to Y$ is the mapping given by $(F/\rho)([(x,r)]_\rho) = F(x)$ and $(f/\rho)_{[(x,r)]_\rho} : I \to I$ is the mapping given by $(f/\rho)_{[(x,r)]_\rho}(u) = f_x(r) = v$ and $u = \bigwedge_{t \in r/\rho_I(x)} t$. Then $\mathbb{F}/\rho = (F/\rho, (f/\rho)_{[(x,r)]_\rho}) : X/\rho \to Y$ is a fuzzy function from X/ρ to Y. In this case, $\mathbb{F}/\rho$ is called the *fuzzy quotient of* $\mathbb{F}$ *by* ρ.

The following is the immediate result of Proposition 4.9 and Result 4.D(2).

Proposition 4.10. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$ such that $\rho \subset \sigma$. If $\mathbb{F} = (F, f_x) : X \to X/\sigma$ is the canonical fuzzy function, then $\mathbb{F}/\rho : X/\rho \to X/\sigma$ is surjective.

Theorem 4.11. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function and let $\rho \in \mathrm{F}Rel_E(X)$. If $\rho \subset \rho_{\mathbb{F}}$, then $\rho_{\mathbb{F}}/\rho = \rho_{\mathbb{F}/\rho}$.

Proof. Let $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in A \in \rho_{\mathbb{F}}/\rho$. Then there exists $B \in \rho_{\mathbb{F}}$ such that $((x,y),(r,s)) \in B$, $u = \bigwedge_{t \in r/\rho_I(x)} t$ and $v = \bigwedge_{t \in s/\rho_I(y)} t$. Thus $F(x) = F(y)$ and $f_x(r) = f_y(s)$. Since $\mathbb{F}/\rho = (F/\rho, (f/\rho)_{[(x,r)]_\rho}) : X/\rho \to Y$ is the fuzzy quotient, $(F/\rho)([(x,r)]_\rho) = F(x) = F(y) = (F/\rho)([(y,s)]_\rho)$ and $(f/\rho)_{[(x,r)]_\rho}(u) = f_x(r) = f_y(s) = (f/\rho)_{[(y,s)]_\rho}(v)$. Thus there exists $C \in \rho_{\mathbb{F}/\rho}$ such that $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in C$. So $\rho_{\mathbb{F}}/\rho \subset \rho_{\mathbb{F}/\rho}$. Now let $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in A \in \rho_{\mathbb{F}/\rho}$. Then $F(x) = (F/\rho)([(x,r)]_\rho) = (F/\rho)([(y,s)]_\rho) = F(y)$ and $f_x(r) = (f/\rho)_{[(x,r)]_\rho}(u) = (f/\rho)_{[(y,s)]_\rho}(v) = f_y(s)$. Thus there exists $C \in \rho_{\mathbb{F}}$ such that $((x,y),(r,s)) \in B$. Moreover, $u = \bigwedge_{t \in r/\rho_I(x)} t$ and $v = \bigwedge_{t \in s/\rho_I(y)} t$. Thus there exists $C \in \rho_{\mathbb{F}}/\rho$ such that $(([(x,r)]_\rho, [(y,s)]_\rho), (u,v)) \in C$. So $\rho_{\mathbb{F}/\rho} \subset \rho_{\mathbb{F}}/\rho$. Hence $\rho_{\mathbb{F}}/\rho = \rho_{\mathbb{F}/\rho}$. □

The following is an example of the use of Theorem 4.11.

Proposition 4.12. Let $\rho, \sigma \in \mathrm{F}Rel_E(X)$ such that $\rho \subset \sigma$ and let $\mathbb{F} = (F, f_x) : X \to X/\sigma$ be the canonical fuzzy function. Then $(X/\rho)/(\sigma/\rho) \approx X/\sigma$.

Proof. Since $\mathbb{F} = (F, f_x) : X \to X/\sigma$ is the canonical fuzzy function, by Theorem 4.5, $\sigma = \rho_{\mathbb{F}}$. Thus, by Proposition 4.9, $\mathbb{F}/\rho : X/\rho \to X/\sigma$ is a fuzzy function. So, by Theorem 4.11, $\sigma/\rho = \rho_{\mathbb{F}/\rho}$. Moreover, by Proposition 4.10, $\mathbb{F}/\rho$ is surjective. Hence, by Corollary 4.6, $(X/\rho)/(\sigma/\rho) \approx X/\sigma$. □

Problem 4.A. Let $\rho, \sigma, \eta \in \mathrm{F}Rel_E(X)$ such that $\rho \subset \sigma \subset \eta$. If $\mathbb{F} = (F, f_x) : X \to X/\rho$, $\mathbb{G} = (G, g_x) : X \to X/\sigma$ and $\mathbb{H} = (H, h_x) : X \to X/\eta$ are the canonical fuzzy functions, then $\mathbb{H}/\rho = \mathbb{H}/\sigma \circ \mathbb{G}/\rho$?

Problem 4.B. Let ρ and σ be any fuzzy equivalence relations in X. Then do the following hold?
(1) $X/(\rho \circ \sigma) \approx (X/\rho)/(\rho \circ \sigma/\rho)$.
(2) $X/\rho \approx (X/\rho \cap \sigma)/(\rho/\rho \cap \sigma)$.

References

[1] J. C. Bezdek and J. D. Harris, *Fuzzy partitions and relations; an axiomatic basis for clustering*, Fuzzy Sets and Systems **1** pp.111-127,1978.

[2] K. A. Dib and Nabil L. Youssef, *Fuzzy cartesian product, fuzzy relations and fuzzy functions*, *Fuzzy Sets and Systems* **41** pp.299-315,1991.

[3] M. A. Erceg, *Functions, equivalence relations, quotient spaces and subsets in fuzzy set theory*, Fuzzy Sets and Systems **3** pp.75-92,1980.

[4] J. A. Goguen, *L-fuzzy sets, J. Math. Anal. Appl.* **8** pp.145-174,1967.

[5] K. Hur, H. W. Kang and K.C.Lee, *Fuzzy equivalence relations and fuzzy partitions, Honam Math. J.* **28(3)** pp.291-315,2006.

[6] V. Murali, *Fuzzy equivalence relations, Fuzzy Sets and Systems,* **30** pp.155-163,1989.

[7] W. C. Nemitz, *Fuzzy relations and fuzzy functions, Fuzzy Sets and Systems* **19** pp.177-191,1986.

[8] L. A. Zadeh, *Fuzzy sets, Inform. and Control* **8** pp.338-353,1965.

Keon Chang Lee
Department of Computer Science,
Dongshin University,
Naju, Cheonnam, Korea 520-714
E-mail: kclee@dsu.ac.kr