# Analysis of Fuzzy Entropy and Similarity Measure for Non Convex Membership Functions

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## Abstract

Fuzzy entropy is designed for non convex fuzzy membership function using well known Hamming distance measure. Design procedure of convex fuzzy membership function is represented through distance measure, furthermore characteristic analysis for non convex function are also illustrated. Proof of proposed fuzzy entropy is discussed, and entropy computation is illustrated.

Key Words : Fuzzy entropy, non convex fuzzy membership function, distance measure

## 1. Introduction

Characterization and quantification of fuzziness are important issues about the data management. Especially the management of uncertainty affect in many system model and designing problem. The results about the fuzzy set entropy have been well known by the previous researchers [1-6]. Liu had proposed the axiomatic definitions of entropy, distance measure and similarity measure, and discussed the relations between these three concepts. Kosko viewed the relation between distance measure and fuzzy entropy. Bhandari and Pal gave a fuzzy information measure for discrimination of a fuzzy set relative to some other fuzzy set. Pal and Pal analyzed the classical Shannon information entropy. Also Ghosh used this entropy to neural network. However, all these results are based on the convex fuzzy membership functions.

For fuzzy set, uncertainty knowledge in fuzzy set can be obtained through analyzing fuzzy set itself. Thus most studies about fuzzy set are emphasized on considering membership function. At this point we have an interest for non convex fuzzy membership. Applying fuzzy entropy to non convex fuzzy membership function, first we analyze the characteristics for fuzzy sets. With previous result of fuzzy entropy, we have designed the fuzzy entropy for non convex membership function [7]. The fuzzy entropy was designed based on the distance measure. Entropy value is proportional to the difference area between fuzzy set membership function and crisp set. However, considered fuzzy membership function was restricted to convex-type fuzzy membership function.

In this paper, we extend the fuzzy entropy for convex membership function to the non convex membership function. To overcome sharpening and complementary properties of fuzzy entropy definition, it is required to add assumptions. To verify the usefulness of proposed fuzzy entropy for non convex membership function, we also utilize the definition of fuzzy entropy.

In the next chapter, the axiomatic definitions of entropy, previous fuzzy entropy for convex membership function are introduced. Preliminary study of non convex membership function is proposed in Chapter 3. Fuzzy entropy for non convex membership function is derived and proved in Chapter 4. Finally, conclusions are followed in Chapter 5. Notations of Liu's are used in this paper [4].

## 2. Fuzzy entropy

Study on the fuzzy entropy analysis has been studied through designing fuzzy entropy for fuzzy set. Furthermore most researches are emphasized on the design of explicit entropy measure with distance measure or fuzzy numbers. In this chapter, we first introduce definition of fuzzy entropy and our previous fuzzy entropy results.

### 2.1 Preliminary results

We introduce some preliminary results about axiomatic definitions of fuzzy entropy and related results. Definition 2.1 represents the axiomatic definition of fuzzy entropy.

**Definition 2.1** [4] A real function $e : F(X) \to R^+$ is called an entropy on $F(X)$, or $P(X)$ if $e$ has the following properties:

(E1)  $e(D) = 0, \forall D \in P(X)$

(E2)  $e([1/2]) = \max_{A \in F(X)} e(A)$

(E3)  $e(A^*) \le e(A)$, for any sharpening $A^*$ of $A$

(E4)  $e(A) = e(A^c), \forall A \in F(X)$.

where $[1/2]$ is the fuzzy set in which the value of the membership function is $1/2$, $R^+ = [0, \infty)$, $X$ is the universal set, $F(X)$ is the class of all fuzzy sets of $X$, $P(X)$ is the class of all crisp sets of $X$ and $D^c$ is the complement of $D$.

A lot of fuzzy entropy satisfying Definition 2.1 can be formulated. We have designed fuzzy entropy in our previous literature [7]. Proposed fuzzy entropies are designed based on the distance measure. We had also proved the usefulness in our previous literatures. Two fuzzy entropies have their own strong points in measure themselves. Now two fuzzy entropies are illustrated without proofs.

**Fuzzy entropy 1.** If distance $d$ satisfies $d(A,B) = d(A^c, B^c)$, $A,B \in F(X)$, then

$$e(A) = 2d((A \cap A_{near}), [1]) + 2d((A \cup A_{near}), [0]) - 2 \quad (1)$$

is fuzzy entropy.

**Fuzzy entropy 2.** If distance $d$ satisfies $d(A,B) = d(A^c, B^c)$, $A,B \in F(X)$, then

$$e(A) = 2d((A \cap A_{far}), [0]) + 2d((A \cup A_{far}), [1]) \quad (2)$$

is also fuzzy entropy.

Exact meaning of fuzzy entropy of fuzzy set $A$ is fuzziness of fuzzy set $A$ with respect to crisp set. We commonly consider crisp sets as $A_{near}$ or $A_{far}$. Here membership function of $A_{near}$ or $A_{far}$ are represented by

$$\mu_{A_{near}}(x) = 1 \text{ for } \mu_A(x) \geq 1/2,$$

$$\mu_{A_{near}}(x) = 0 \text{ for } \mu_A(x) < 1/2$$

and

$$\mu_{A_{far}}(x) = 0 \text{ for } \mu_A(x) \geq 1/2,$$

$$\mu_{A_{far}}(x) = 1 \text{ for } \mu_A(x) < 1/2.$$

In the above fuzzy entropies, one of well known Hamming distance is commonly used as distance measure between fuzzy sets $A$ and $B$,

$$d(A,B) = \frac{1}{2} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|$$

where $X = \{x_1, x_2, \cdots x_n\}$, $|k|$ is the absolute value of $k$. $\mu_A(x)$ is the membership function of $A \in F(X)$. Basically fuzzy entropy is proportional to the difference area between fuzzy membership function and its crisp sets.

Fuzzy entropy (1) and (2) satisfy Definition 2.1. However, Definition 2.1 does not restrict to convex fuzzy membership function. Next, we introduce non convex fuzzy membership function. Definition of non convex fuzzy membership function can be found in reference [8]. Non convex fuzzy sets are not common fuzzy membership function. Definition of non convexity derived from convexity definitely.

## 2.2 Non Convex Membership Function

By Jang et.al., it has been known that definition of convexity of a fuzzy set is not as strict as the common definition of

convexity of a function [8]. Definition 2.2 represents the definition of convexity for fuzzy set not general function.

**Definition 2.2** [8] A fuzzy set $A$ is convex if and only if for any $x_1$, $x_2 \in X$ and any $\lambda \in [0,1]$,

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\} \quad (3)$$

Non convex fuzzy set is said if it is not convex. Non convex fuzzy membership functions can be notified naturally 3 sub classes [9].

· Elementary non convex membership functions
· Time related non convex membership functions
· Consequent non convex membership functions

First, a discrete fuzzy set express elementary non convex fuzzy membership functions. However continuous domain non convex fuzzy set may be less common.

Next, time related non convex membership functions can be found in energy supply by time of day or year, mealtime by time of day. This fuzzy set is interesting as it is also sub-normal and never has a membership of zero.

Finally, Mamdani fuzzy inferencing is a typical example of consequent non convex sets. In a rule based fuzzy system the result of Mamdani fuzzy inferencing is a non convex fuzzy set where the antecedent and consequent fuzzy sets are triangular and/or trapezoidal.

Jang et.al insisted that the definition of convexity of a fuzzy set is not as strict as the common definition of convexity of a function [14]. Then the definition of convexity of a function is

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda f(x_1) + (1 - \lambda)f(x_2) \quad (4)$$

which is a tighter condition than (3). Following figures show two convex and non convex fuzzy membership functions.



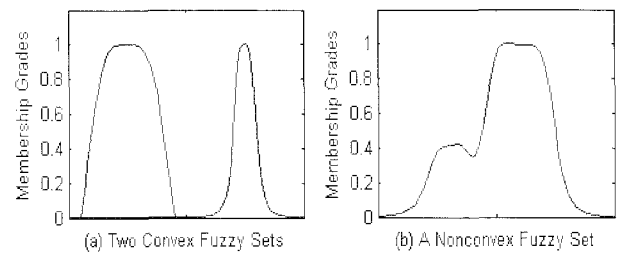(a) Two Convex Fuzzy Sets    (b) A Nonconvex Fuzzy Set

Fig.1 Convex MF and Non convex MF [8]

Fig. 1 (a) shows two convex fuzzy sets, the left fuzzy set satisfies both (3) and (4) while the right one satisfies (3) only. Hence the right one is not convex for a general function. Whereas, fuzzy set of Fig. 1(b) behaves a non convex fuzzy set. It does not satisfy even (3).

By the definition of Jang et.al., fuzzy entropies of two convex fuzzy membership functions of Fig. 1 (a) can be calculated. However if two fuzzy set are considered as one fuzzy set, then it has to be considered as non convex fuzzy set.
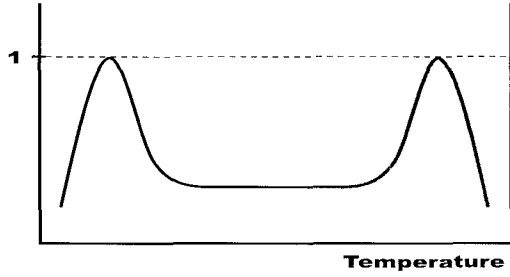
Fig. 2 Preference temperature of drinking milk

Fig. 2 represents the fuzzy membership function of preference temperature of drinking milk. Drinking milk temperature is proper as the non convex fuzzy membership function. Medium temperature is not popular to drink. For the non convex fuzzy membership function, how can we design fuzzy entropy ? Studies can be considered from the analysis of typical convex membership function fuzzy entropy. With the corresponding crisp set, we will formulate fuzzy entropy.

# 3. Fuzzy entropy for non convex Membership function

Now, we pay attention to the definition of fuzzy entropy for fuzzy set, particularly for non convex fuzzy set. Conditions of (E1) and (E2) are natural for convex or non convex membership function. However, conditions of (E3) and (E4) are important to decide structure of fuzzy entropy. Sharpening $A^*$ of non convex fuzzy set $A$ has to also satisfy nearer crisp set structure. Hence, $A^*$ approaches to crisp set $D$, it is natural to be satisfied (E1). For (E4), fuzzy entropy of complementary fuzzy membership is the same with original fuzzy membership function. To satisfy these characteristics, we have to consider corresponding crisp set properly.

### 3.1 Fuzzy entropy for non convex membership function

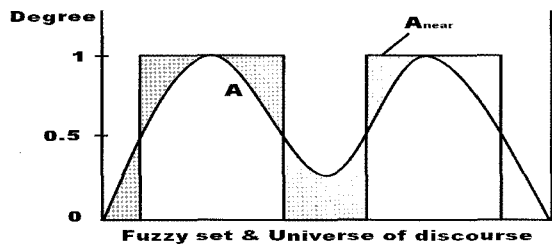In Fig. 3, we assign crisp set corresponding to non convex fuzzy membership function as follows



Fig. 3 Fuzzy set and crisp set

For every non convex fuzzy set $A$ we let crisp set $A_{near}$. Then two fuzzy entropy measure (1) and (2) are applicable to non convex fuzzy membership function. Now, we can show that two fuzzy entropy (1) and (2) are satisfied as fuzzy entropy as non convex fuzzy membership function [6,7]. It is essential to assign the crisp set $A_{near}$ of fuzzy set $A$. Crisp set $A_{near}$ of

non convex fuzzy set $A$ is also non convex. Next two fuzzy entropy measures are presented as fuzzy entropy of non convex membership function.

**Theorem 3.1** If distance $d$ satisfies $d(A,B)=d(A^c,B^c)$ for convex or non convex $A,B \in F(X)$, then

$$e(A) = 2d((A \cap A_{near}),[1]) + 2d((A \cup A_{near}),[0]) - 2$$

satisfies fuzzy entropy.

Proof is straight forward. Proposed fuzzy entropy is represented through the area between fuzzy membership function and corresponding crisp set. In Fig. 3, the fuzzy entropy is represented by shaded area. We have to check for non convex fuzzy set case. By analyzing the properties of Definition 2.1, It will be simple to prove (E1) and (E2). We summarize the proof of (E1) and (E2) briefly.

For (E1), $\forall D \in P(X)$,

$$e(D) = 2d((D \cap D_{near}),[1]) + 2d((D \cup D_{near}),[0]) - 2$$
$$= 2d(D,[1]) + 2d(D,[0]) - 2 = 0.$$

It is also satisfied from $D_{near} = D$. (E2) represents that crisp set $1/2$ has the maximum entropy 1. Therefore, the entropy measure $e([1/2])$ satisfies

$$e([1/2]) = 2d(([1/2] \cap [1/2]_{near}),[1]) + 2d(([1/2] \cup [1/2]_{near}),[0]) - 2$$
$$= 2d(([1/2] \cap [1]),[1]) + 2d(([1/2] \cup [1]),[0]) - 2$$
$$= 2 \cdot 1/2 + 2 \cdot 1 - 2 = 1.$$

In the above equation, $[1/2]_{near} = [1]$ is satisfied. Fuzzy entropy measure (1) was designed for normal fuzzy entropy. Hence, it has maximal value as one.

(E3) shows that the entropy of the sharpened version of fuzzy set $A$, $e(A^*)$, is less than or equal to $e(A)$. For the proof, $A^*_{near} = A_{near}$ is also used:

$$e(A^*) = 2d((A^* \cap A^*_{near}),[1]) + 2d((A^* \cup A^*_{near}),[0]) - 2$$
$$= 2d((A^* \cap A_{near}),[1]) + 2d((A^* \cup A_{near}),[0]) - 2$$
$$\leq 2d((A \cap A_{near}),[1]) + 2d((A \cup A_{near}),[0]) - 2$$
$$= e(A).$$

The inequality in the above equation is satisfied because the fuzzy entropy is proportional to the shaded area in Fig. 3 and the property, $d(A^*, A_{near}) \leq d(A, A_{near})$ in [10].

Finally, (E4) is proved using the assumption $d(A^c, B^c) = d(A, B)$; hence we have

$$e(A) = 2d((A \cap A_{near}),[1]) + 2d((A \cup A_{near}),[0]) - 2$$
$$= 2d((A^c \cup A^c_{near}),[0]) + 2d((A^c \cap A^c_{near}),[1]) - 2$$
$$= e(A^c).$$

Through our analysis to the non convex fuzzy membership function, we have found out that the fuzzy entropy (1) is applicable to non convex fuzzy membership function too. We can insist that our previous dual result can be also applicable to non convex fuzzy membership function.

**Theorem 3.2** If distance $d$ satisfies $d(A,B) = d(A^c, B^c)$ and for convex or non convex $A, B \in F(X)$,

$$e(A) = 2d\left((A \cap A_{far}), [0]\right) + 2d\left((A \cup A_{far}), [1]\right)$$

is also fuzzy entropy.

Proof is similar to those of Theorem 3.1. In Theorem 3.2, computation of $d\left((A \cap A_{far}), [0]\right)$ and $d\left((A \cup A_{far}), [1]\right)$ are preformed twice.

For non convex fuzzy set, it is also applicable with convex fuzzy entropy computation. However, proper assignment of crisp set is required to formulate fuzzy entropy measure.

### 3.2 Illustrative example

In this section, we present the example about deciding normal or fault condition from flight system coefficients. The flight control system is generally equipped with redundancy features in order to increase the safety of the aircraft, particularly in cases where the aircraft is damaged due to malfunctioning of the control surface region. If the extent of the damage is determined after the occurrence of the failure, the fault tolerant control system is capable of adapting to the various faults in real time. Therefore, the pilot or the flight control system accomplishes the mission or returns to the safety region. Here, $C_{m_{\delta e}}$, $C_{m_\alpha}$, and $C_{L_\alpha}$ are the pitching moment coefficient with changes of elevator deflection, change in pitching moment coefficient with angle of attack and change in lift coefficient with angle of attack, respectively. Scatter diagrams of $C_{m_{\delta e}}$ and $C_{m_\alpha}$ are easy to discriminate, which one is normal or fault. However data points of $C_{L_\alpha}$ are mixed, and it is not easy to discriminate. Scatter diagram of $C_{L_\alpha}$ is illustrated in Fig. 4.
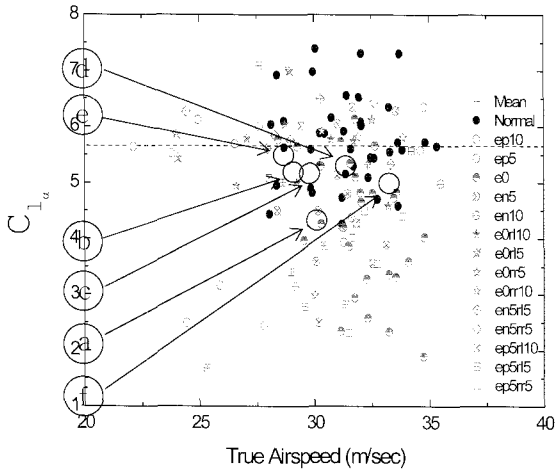


Fig. 4 Scatter diagram of $C_{L_\alpha}$ coefficients

For constructing the fuzzy membership function coefficient values of $C_{m_{\delta e}}$, $C_{m_\alpha}$ and $C_{L_\alpha}$ are partitioned into eight groups, and the number of data are normalized. The fault detection procedures comprise similarity computation and

detection. Similarity computations are performed by using the similarity measure, and the detection procedure requires further reference data and consideration. Mentioned similarity measure can be obtained through analysis of fuzzy entropy and distance measure [11]. Now we formulate fuzzy membership functions of $C_{m_{\delta e}}$ and $C_{L_\alpha}$ in Fig. 5. As shown in figures, fuzzy membership functions are shown as the non convex types. Here we present the similarity measure as follows.

The control surface stuck is determined by monitoring the value of $C_{m_{\delta e}}$; This facilitates the discrimination between normal and fault conditions. Hence, the two fuzzy membership functions are clearly separated. First, we consider the similarity measure using $C_{m_{\delta e}}$, as follows [11];

$$s_{m_{\delta e}}(F_N, F_F) = 2 - d((F_N \cap F_F), [1]_X) - d((F_N \cup F_F), [0]_X).$$

Here, $F_N$ and $F_F$ denote the normal and fault fuzzy membership functions from Fig. 5. Further, we propose another similarity measure using $C_{m_\alpha}$ and $C_{L_\alpha}$

$$s_{m_\alpha}(F_N, F_F) = 2 - d((F_N \cap F_F), [1]_X) - d((F_N \cup F_F), [0]_X)$$

and $s_{L_\alpha}(F_N, F_F) = 2 - d((F_N \cap F_F), [1]_X) - d((F_N \cup F_F), [0]_X).$

Six data points in Fig. 4 are selected to calculate and analyze the characteristics. Point **a** to **f** of $C_{L_\alpha}$ are also listed in Fig. 4, as follows:

**a** = 4.9974, **b** = 5.6780, **c** = 5.7549, **d** = 5.5991, **e** = 5.56238, **f** = 5.4462.

The locations of the six points are in the mixed area between the normal and fault conditions. It is necessary to calculate the similarity measure corresponding to the membership values; hence, the membership values of point **a** and **d** are illustrated in Fig. 5.

Now, we calculate the tendency of normal or fault conditions for six points. The value of $s_{m_{\delta e}}(F_N, F_F)$ is classified into normal and fault conditions by $s_{m_{\delta e}}(F_N, p)$ and $s_{m_{\delta e}}(p, F_F)$ for fault point **p**. Similarly, the other similarities $s_{m_\alpha}(F_N, F_F)$ and $s_{L_\alpha}(F_N, F_F)$ are also classified from **a** to **f**. In Table 1, Points **a**, **b** and **c** are fault data, whereas the others are all normal.

To calculate the degree of similarity for normal operation with data **p** can be obtained as;

$$s(F_N, p) = w_1 s_{m_{\delta e}}(F_N, p) + w_2 s_{m_\alpha}(F_N, p) + w_3 s_{L_\alpha}(F_N, p).$$

Calculating the degree of similarity for fault operation is obtained similar way. In Table 1, we notice that the similarity measure for three fault cases **a**, **b**, and **c** are greater than those for normal cases. Where the weighting factors $w_1$, $w_2$ and $w_3$ are all considered as one. This implies that the fault decisions are accurate by similarity computations and comparisons. Furthermore, three normal cases also represent correct decisions.
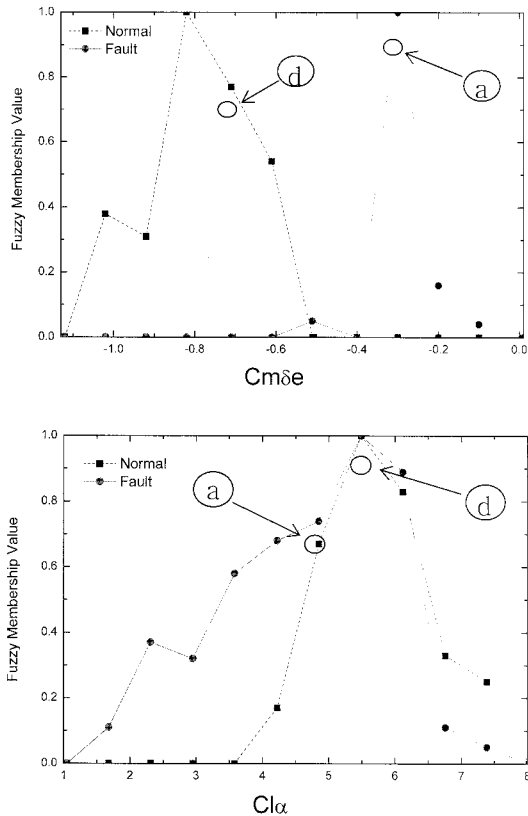
Fig. 5 Membership functions of the $C_{m_{\delta c}}$ and $C_{L_\alpha}$ coefficients

Table 1. Computation of Similarity Measure

|  | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| $S_{m_{\delta c}}(F_N,\mathrm{p})$ | 0 | 0 | 0 | 0.77 | 0.77 | 0.54 |
| $S_{m_{\delta c}}(\mathrm{p},F_F)$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $S_{m_\alpha}(F_N,\mathrm{p})$ | 0 | 0.42 | 0.83 | 0.83 | 1 | 0.42 |
| $S_{m_\alpha}(\mathrm{p},F_F)$ | 1 | 0.73 | 0.62 | 0.64 | 0.23 | 0.73 |
| $S_{L_\alpha}(F_N,\mathrm{p})$ | 0.67 | 1 | 1 | 1 | 1 | 1 |
| $S_{L_\alpha}(\mathrm{p},F_F)$ | 0.74 | 1 | 1 | 1 | 1 | 1 |
| $s(F_N,\mathrm{p})$ | 0.67 | 1.42 | 1.83 | **2.60** | **2.77** | **2.73** |
| $s(\mathrm{p},F_F)$ | **2.74** | **2.73** | **2.62** | 1.64 | 1.23 | 1.73 |

## 4. Conclusions

Fuzzy entropy of non convex fuzzy membership function is designed. Non convex fuzzy membership function is introduced and its property was discussed. Furthermore, characteristic analysis for non convex function is also illustrated. Our fuzzy entropy measure for fuzzy set is also applicable to non convex fuzzy membership function. We have discussed this fact, it is essential to assign corresponding crisp set. We have found out that the corresponding crisp set is also non convex set. We have

formulated similarity measure with fuzzy entropy and distance measure, fault decision problem is applied with this similarity measure.

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