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A Note on Possibilistic Correlation

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Abstract

Recently, Carlsson, Fullér and Majlender [1] presented the concept of possibilitic correlation representing an average degree of interaction between marginal distribution of a joint possibility distribution as compared to their respective dispersions. They also formulated the weak and strong forms of the possibilistic Cauchy-Schwarz inequality. In this paper, we define a new probability measure. Then the weak and strong forms of the Cauchy-Schwarz inequality are immediate consequence of probabilistic Cauchy-Schwarz inequality with respect to the new probability measure.

Key words : Possibilistic distribution, density function, Variance, Covariance, Cauchy-Schwarz inequality

1. Introduction

The notion of mean value of the function of random variables in probability theory plays a fundamental role in defining the basic measure of a probability distribution. Fuller and Majlender [4] presented the idea of interaction between a marginal distribution of a joint possibility distribution and introduced the notion of covariance between fuzzy numbers by their joint possibility distribution to measure the degree to which they interact. Recently, Carlsson, Fullér and Majlender [1] presented the concept of possibilitic correlation representing an average degree of interaction between the marginal distribution of a joint possibility distribution as compared to their respective dispersions. They also formulated the weak and strong forms of the possibilistic Cauchy-Schwarz inequality. In this note, we define a new probability measure. Then we show that the weak and strong forms of the possibilistic Cauchy-Schwarz inequality are immediate consequence of classical probability theory.

2. Preliminaries

A fuzzy number A is a fuzzy set in $\mathbb{R}$ that has a normal, fuzzy convex and continuous membership function of bounded support. The family of all fuzzy numbers will be denoted $\mathcal{F}$. Fuzzy numbers can be considered as possibility distributions [6-8]. If C is a fuzzy set in $\mathbb{R}^n$ then its γ-level set is defined by $[C]^\gamma = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid C(x_1, \dots, x_n) \geq \gamma\}$ for $0 < \gamma \leq 1$, and $[C]^\gamma = \mathrm{cl}\{(x_1, \dots, x_n) \in \mathbb{R}^n | C(x_1, \dots, x_n) > \gamma\}$(the closure of the support of C) for $\gamma = 0$. It is clear that if $A \in \mathcal{F}$ is a fuzzy number then $[A]^\gamma$ is a compact interval for all $\gamma \in [0, 1]$.

Let $A_i \in \mathcal{F}, i = 1, \dots, n$, be fuzzy numbers, and let C be a fuzzy set in $\mathbb{R}$. Then, C is said to be a *joint possibility distribution* of $A_i, i = 1, \dots, n$, if the following relationships hold [4]

$$A_i(x_i) = \sup_{x_j \in \mathbb{R}, j \neq i} C(x_1, \dots, x_n) \quad \forall x_i \in \mathbb{R}, i = 1, \dots, n.$$

Furthermore, in this case we will call A_i the ith *marginal possibility distribution* of C and use the notation $A_i = \pi_i(C)$, where π_i denotes the projection operator in $\mathbb{R}^n$ onto the ith axis, $i = 1, \dots, n$.

Let C be a joint possibility distribution in $\mathbb{R}^n$, let $g : \mathbb{R}^n \to \mathbb{R}$ be an integrable function, and let $\gamma \in [0, 1]$. Then, the *central value* of g on $[C]^\gamma$ is defined by [4]

$$\begin{aligned} & \mathcal{C}_{[C]^\gamma}(g) \\ = & \frac{1}{\int_{[C]^\gamma} dx} \int_{[C]^\gamma} g(x) dx \\ = & \frac{1}{\int_{[C]^\gamma} dx_1 \dots dx_n} \int_{[C]^\gamma} g(x_1, \dots, x_n) dx_1 \dots dx_n. \end{aligned}$$

Furthermore, if $[C]^\gamma$ is a degenerated set then we compute $\mathcal{C}_{[C]^\gamma}(g)$ as the limit case of a uniform approximation of $[C]^\gamma$ with non-degenerated sets [5].

Let C be a joint possibility distribution in $\mathbb{R}^n$, let $g : \mathbb{R}^n \to \mathbb{R}$ be an integrable function, and let f be a weighting function. The *expected value* of g on C with respect to

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f is defined by [4]

$$E_f(g;C) = \int_0^1 \mathcal{C}_{[C]^\gamma}(g)f(\gamma)d\gamma.$$

That is, $E_f(g;C)$ computes the f-weighted average of the central values of function g on the level sets of C.

Let us denote the projection functions on $\mathbb{R}^2$ by π_x and π_y, i.e. $\pi_x(u,v) = u$ and $\pi_y(u,v) = v$ for all $u, v \in \mathbb{R}$.

Let C be a joint possibility distribution in $\mathbb{R}^2$ with marginal possibility distributions $A = \pi_x(C)$ and $B = \pi_y(C)$, and let $\gamma \in [0,1]$. Then, the *measure of interactivity* between the γ-level sets of A and B (with respect to $[C]^\gamma$) is defined by [4]

$$\mathcal{R}_{[C]^\gamma}(\pi_x,\pi_y) = \mathcal{C}_{[C]^\gamma}((\pi_x - \mathcal{C}_{[C]^\gamma}(\pi_x))(\pi_y - \mathcal{C}_{[C]^\gamma}(\pi_y))).$$

In a possibilistic sense $\mathcal{R}_{[C]^\gamma}(\pi_x,\pi_y)$ computes the central value of the *interactivity function*

$$\mathfrak{g} = (\pi_x - \mathcal{C}_{[C]^\gamma}(\pi_x))(\pi_y - \mathcal{C}_{[C]^\gamma}(\pi_y))$$

on $[C]^\gamma$.

Now let A be a possibility distribution in $\mathbb{R}$, and let $\gamma \in [0,1]$. Then, the *measure of dispersion* of $[A]^\gamma$ is defined by

$$\mathcal{R}_{[A]^\gamma}(\text{id},\text{id}) = \mathcal{C}_{[A]^\gamma}\left(\left(\text{id} - \mathcal{C}_{[A]^\gamma}(\text{id})\right)^2\right).$$

Let C be a joint possibility distribution with marginal possibility distribution $A = \pi_x(C)$ and $B = \pi_y(C)$, and let f be a weighting function. Then, the *measure of covariance* between A and B (with respect to their joint distribution C and weighting function f) is defined by [4]

$$\text{Cov}_f(A,B) = E_f(\mathfrak{g};C) = \int_0^1 \mathcal{R}_{[C]^\gamma}(\pi_x,\pi_y)f(\gamma)d\gamma,$$

where $\mathfrak{g} \equiv \mathfrak{g}_{[C]^\gamma}$ stands for the interactivity function associated with $[C]^\gamma$, $\gamma \in [0,1]$. That is, the covariance of A and B is computed as the expected value of the interactivity function on the joint distribution C.

Now let $A \in \mathcal{F}$ be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)], \gamma \in [0,1]$, and let f be a weighting function. The *measure of variance* of A with respect to f is defined as [4]

$$\text{Var}_f(A) = E_f(\mathfrak{h};A) = \int_0^1 \mathcal{R}_{[A]^\gamma}(\text{id,id})f(\gamma)d\gamma,$$

where $\mathfrak{h} \equiv \mathfrak{h}_{[A]^\gamma} = \left(\text{id} - \mathcal{C}_{[A]^\gamma}(\text{id})\right)^2$ denotes the dispersion function of the level set $[A]^\gamma, \gamma \in [0,1]$.

Carlsson, Fullér and Majlender [1] formulated the following weak and strong forms of the possibilistic Cauchy-Schwarz inequality.

Theorem 2.1. ([1]) Let C be a joint possibility distribution in $\mathbb{R}^2$, and let f be a weighting function. Then

$$(E_f(\mathfrak{g};C))^2 \le E_f(\mathfrak{h}_x;C)E_f(\mathfrak{h}_y;C).$$

where $\mathfrak{h}_x = (\pi_x - \mathcal{C}_{[C]^\gamma}(\pi_x))^2$ and $\mathfrak{h}_y = (\pi_y - \mathcal{C}_{[C]^\gamma}(\pi_y))^2$.

Theorem 2.2. ([1]) Let C be a joint possibility distribution with marginal possibility distributions $A = \pi_x(C) \in \mathcal{F}, B = \pi_y(C) \in \mathcal{F}$, and let f be a weighting function. If $[C]^\gamma$ is convex for all $\gamma \in [0,1]$ then the following inequality holds:

$$(\text{Cov}_f(A,B))^2 \le \text{Var}_f(A)\text{Var}_f(B).$$

3. Main results

In this section, we define a new probability measure. Then we show that the weak and strong forms of the possibilistic Cauchy-Schwarz inequality are immediate consequence of probabilistic Cauchy-Schwarz inequality with respect to the new probability measure.

Define a new probability measure P_f on $\mathbb{R}^n$ such that for any Borel measurable set A of $\mathbb{R}^n$

$$P_f(A) = \int_0^1 \mathcal{C}_{[C]^\gamma}(I_A)f(\gamma)d\gamma,$$

where I_A is a indicator function on A.

Lemma 3.1. P_f is a probability measure on $\mathbb{R}^n$.

Proof. We first note that if we define $P_\gamma(A) = \mathcal{C}_{[C]^\gamma}(I_A)$, then P_γ is a probability measure with uniform distribution on $[C]^\gamma$. $P_f(\emptyset) = 0$ and $P_f(\mathbb{R}^n) = 1$ are clear. Let A_n be a sequence of disjoint Borel subsets of $\mathbb{R}^n$. Then we have that

$$\begin{aligned} P_f(\cup_{n=1}^\infty A_n) &= \int_0^1 P_\gamma(\cup_{n=1}^\infty A_n)f(\gamma)d\gamma \\ &= \int_0^1 \sum_{n=1}^\infty P_\gamma(A_n)f(\gamma)d\gamma \\ &= \sum_{n=1}^\infty \int_0^1 P_\gamma(A_n)f(\gamma)d\gamma \\ &= \sum_{n=1}^\infty P_f(A_n), \end{aligned}$$

which completes the proof. □

Lemma 3.2. Let C be a joint possibility distribution in $\mathbb{R}^n$, let $g : \mathbb{R}^n \to \mathbb{R}$ be an integrable function and let f be a weighting function. Then we have that $E_f(g;\ C) = \int g dP_f$.

Proof. Let g be an indicator I_B, where B is a Borel subsets of $\mathbb{R}^n$. Then

$$E_f(g;\ C) = \int_0^1 \mathcal{C}_{[C]^\gamma}(I_A)f(\gamma)d\gamma = P_f(B) = \int g dP_f$$

so that $E_f(g;\ C)$ and $\int g dP_f$ are equal.

Now let g be a nonnegative simple function, say $g(x) = \sum_{j=1}^n a_j I_{B_j}(x)$, the B_j disjoint Borel subsets of $\mathbb{R}^n$. Then

$$\begin{aligned} E_f(g;C) &= \int_0^1 \mathcal{C}_{|C|^\gamma}(\sum_{j=1}^n a_j I_{B_j})f(\gamma)d\gamma \\ &= \sum_{j=1}^n a_j \int_0^1 \mathcal{C}_{|C|^\gamma}(I_{B_j})f(\gamma)d\gamma \\ &= \sum_{j=1}^n a_j \int I_{B_j} dP_f \\ &= \int (\sum_{j=1}^n a_j I_{B_j})dP_f \\ &= \int g\ dP_f. \end{aligned}$$

Again, both integrals are equal.

If g is a nonnegative Borel measurable function, let $g_1, g_2, \cdots$, be nonnegative simple functions with $g_n \uparrow g$. We have just proved that

$$E_f(g_n;C) = \int g_n\ dP_f;$$

hence by the monotone convergence theorem,

$$E_f(g;C) = \int g\ dP_f,$$

and again both integrals are equal.

Finally, if $g = g^+ - g^-$ is an arbitrary Borel measurable function and we have

$$\begin{aligned} E_f(g;C) &= E_f(g^+;C) - E_f(g^-;C) \\ &= \int g^+\ dP_f - \int g^-\ dP_f \\ &= \int (g^+ - g^-)\ dP_f \\ &= \int g\ dP_f. \end{aligned}$$

where the second equality comes from what we have already proved. □

The following result is an immediate consequence of Theorem 5 [1].

Lemma 3.3. Let C be a joint possibility distribution with marginal possibility distribution $A = \pi_x(C) \in \mathcal{F}, B = \pi_y(C) \in \mathcal{F}$, and let $\gamma \in [0, 1]$. If $[C]^\gamma$ is convex then

$$E_f(\mathfrak{h}_x;C) \le Var_f(A).$$

We now consider the following Cauchy-Schwarz inequality

$$\begin{aligned} & Cov_f(A,\ B) = E_f(\mathfrak{g};C) \\ = &\ \int (\pi_x - \mathcal{C}_{|C|^\gamma}(\pi_x))(\pi_y - \mathcal{C}_{|C|^\gamma}(\pi_y))dP_f \\ \le &\ (\int (\pi_x - \mathcal{C}_{|C|^\gamma}(\pi_x))^2 dP_f)^{\frac{1}{2}}(\int (\pi_y - \mathcal{C}_{|C|^\gamma}(\pi_y)^2)dP_f)^{\frac{1}{2}} \\ = &\ (E_f(\mathfrak{h}_x;C))^{\frac{1}{2}}(E_f(\mathfrak{h}_y;C))^{\frac{1}{2}} \\ \le &\ (Var_f(A))^{\frac{1}{2}}(Var_f(B))^{\frac{1}{2}}, \end{aligned}$$

where the second inequality comes from Lemma 3 under the assumption that $[C]^\gamma$ is convex for all $\gamma \in [0,\ 1]$, which proves Theorem 1 and 2.

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