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Meet preserving maps on stsc-biquantales

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Abstract

We investigate the properties of meet preserving maps on strictly two-sided, commutative biquantales. Using the properties of meet preserving maps, we induce the Zadeh image and preimage operators.

Key words : strictly two-sided, commutative biquantales, meet preserving maps, (left) right adjointness

1. Introduction and preliminaries

Quantales were introduced by Mulvey [8] as the non-commutative generalization of the lattice of open sets in topological spaces. Recently, quantales have arisen in an analysis of the semantics of linear logic systems developed by Girard [1], which supports part of foundation of theoretic computer science. Höhle *et al.* [4,5] introduced the notion of L-fuzzy relation on a complete quasi-monoidal lattice (including GL-monoid [2]) L instead of a completely distributive lattice or the unit interval[7,10]. The notion of L-fuzzy relation facilitated to study fuzzy equivalence relations, fuzzy rough sets, L-fuzzy topological structures [7,10].

In this paper, we investigate L-fuzzy relation on a strictly two-sided, commutative biquantale lattice L. We study the properties of meet preserving maps $\phi : L^X \to L^Y$.

2. Preliminaries

Definition 2.1. [6,8-10] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative biquantale* (stsc-biquantale, for short) iff it satisfies the following properties:

(L1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(L, \odot)$ is a commutative semigroup;

(L3) $a = a \odot 1$, for each $a \in L$;

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b);$$

(L5) $\odot$ is distributive over arbitrary meets, i.e.

$$(\bigwedge_{i\in\Gamma} a_i) \odot b = \bigwedge_{i\in\Gamma} (a_i \odot b).$$

Remark 2.2. [4,5,7,10,11](1) A completely distributive lattice (ref. [10]) is a stsc-biquantale. In particular, the unit interval $([0,1], \leq, \vee, \wedge, 0, 1)$ is a stsc-biquantale.

(2) The unit interval with a continuous t-norm t, $([0,1], \leq, t)$, is a stsc-biquantale.

(3) Let $(L, \leq, \odot)$ be a stsc-biquantale. For each $x, y \in L$, we define

$$x \to y = \bigvee \{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,
$(x \odot y) \leq z$ iff $x \leq (y \to z)$.

In this paper, we always assume that $(L, \leq, \odot, ^*)$ is a stsc-biquantale with strong negation * where $a^* = a \to 0$ and $a \oplus b = (a^* \odot b^*)^*$.

Let X be a nonempty set. All algebraic operations on L can be extended pointwisely to the set L^X as follows: for all $x \in X$, $\lambda, \mu \in L^X$ and $\alpha \in L$,

(1) $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$;
(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$;
(3) $1_X(x) = 1$, $\quad \alpha \odot 1_X(x) = \alpha$ and $1_\emptyset(x) = 0$;
(4) $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$ and $(\lambda \to \alpha)(x) = \lambda(x) \to \alpha$;
(5) $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$.

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Lemma 2.3. [6,11] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $(x \oplus y) \leq (x \oplus z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$

(4) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(5) $x \to (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \to y_i)$

(6) $(\bigwedge_{i\in\Gamma} x_i) \to y = \bigvee_{i\in\Gamma}(x_i \to y)$.

(7) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.

(8) $x \oplus (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \oplus y_i)$.

(9) $x \oplus (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \oplus y_i)$,

(10) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(11) $x \odot y = (x \to y^*)^*$, $x \oplus y = x^* \to y$.

(12) $(x \oplus z) \odot y \leq x \oplus (y \odot z)$.

Definition 2.4. [6,7,10] Let $\phi : M \to N$ and $\psi : N \to M$ be order-preserving maps between partially ordered sets M, N. ϕ is *left adjoint* of ψ, $\phi \dashv \psi$, iff $\phi(a) \leq b \Leftrightarrow a \leq \psi(b)$. Equivalently, $\phi \dashv \psi$ iff $id_M \leq \psi\circ\phi$ and $\phi\circ\psi \leq id_N$.

3. Meet preserving maps

Definition 3.1. Let $L(X, Y)$ be a subset of $(L^Y)^{(L^X)}$ if it satisfies

(J) $\psi(\bigwedge_{i\in\Gamma} \lambda_i) = \bigwedge_{i\in\Gamma} \psi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.

We denote $\Delta(X, Y)$ as a subset of $L(X, Y)$ satisfying

(D) $\psi(1_x \to \alpha) = \psi(1_x) \to \alpha$ for each $\alpha \in L$ and $x \in X$.

Theorem 3.2. For $\psi, \psi_1, \psi_2 \in L(X, Y)$, we define, for all $\lambda \in L^X, \rho \in L^Y$,

$$\psi^{-1}(\rho) = \bigvee\{\lambda \in L^X \mid \psi(\lambda^*) \geq \rho^*\},$$

$$\psi_1 \circ \psi_2(\lambda) = \psi_1(\psi_2(\lambda)).$$

Then the following properties hold:

(1) $\psi^{\to}(\rho) = \bigwedge\{\lambda \in L^X \mid \psi(\lambda) \geq \rho\}$ such that $\psi^{\to}$ is a left adjoint of ψ with $\psi\circ\psi^{\to}(\rho) \geq \rho$ and $\lambda \geq \psi^{\to}\circ\psi(\lambda)$.

(2) $\psi^{-1}(\rho) = (\psi^{\to}(\rho^*))^*$ and $\psi^{-1} \in L(Y, X)$ such that

$$\psi(\lambda) \geq \rho \Leftrightarrow \lambda \geq \psi^{\to}(\rho) \Leftrightarrow \psi^{-1}(\rho^*) \geq \lambda^*$$

(3) $(\psi^{-1})^{-1} = \psi$.

(4) If $\psi_1 \leq \psi_2$, then $\psi_1^{-1} \leq \psi_2^{-1}$.

(5) If $\phi \in L(Y, Z)$, then $\phi \circ \psi \in L(X, Z)$ and $(\phi \circ \psi)^{-1} = \psi^{-1} \circ \phi^{-1} \in L(Z, X)$.

(6) If $\psi(1_{\{x\}} \to \lambda(x)) = \rho_x$ for all $x \in X$, then $\psi(\lambda) = \bigwedge_{z\in X} \rho_z$.

(7) If $\psi_1(1_{\{x\}} \to \alpha) = \psi_2(1_{\{x\}} \to \alpha)$ for all $x \in X$, then $\psi_1 = \psi_2$.

Proof. (1) Since ψ is a meet preserving map and $\psi^{\to}(\rho) = \bigwedge\{\lambda \in L^Y \mid \psi(\lambda) \geq \rho\}$, we have $\psi(\lambda) \geq \rho \Leftrightarrow \lambda \geq \psi^{\to}(\rho)$. By Definition 2.4, $\psi^{\to}$ is a left adjoint of ψ with $\psi \circ \psi^{\to}(\rho) \geq \rho$ and $\lambda \leq \psi^{\to} \circ \psi(\lambda)$.

(2) By Lemma 2.3(7), we have

$$\begin{aligned}\psi^{-1}(\rho) &= \bigvee\{\lambda \in L^X \mid \psi(\lambda^*) \geq \rho^*\} \\ &= \Big(\bigwedge\{\lambda^* \in L^X \mid \psi(\lambda^*) \geq \rho^*\}\Big)^* \\ &= (\psi^{\to}(\rho^*))^*.\end{aligned}$$

We have $\psi^{-1} \in L(Y, X)$ from:

$$\begin{aligned}\textstyle\bigwedge_{i\in\Gamma} \psi^{-1}(\rho_i) \geq \lambda &\Leftrightarrow \psi^{-1}(\rho_i) \geq \lambda, \ \forall i \in \Gamma \\ &\Leftrightarrow \psi(\lambda^*) \geq \rho_i^*, \ \forall i \in \Gamma \\ &\Leftrightarrow \psi(\lambda^*) \geq \textstyle\bigvee_{i\in\Gamma} \rho_i^* = (\bigwedge_{i\in\Gamma} \rho_i)^*, \\ &\Leftrightarrow \psi^{-1}(\textstyle\bigwedge_{i\in\Gamma} \rho_i) \geq \lambda.\end{aligned}$$

(3) By the definition of ψ^{-1}, we obtain:

$$\psi(\rho) \geq \lambda \Leftrightarrow \psi^{-1}(\lambda^*) \geq \rho^* \Leftrightarrow (\psi^{-1})^{-1}(\rho) \geq \lambda$$

(4) Since $\psi_2(\rho^*) \geq \psi_1(\rho^*) \geq \lambda^*$, it easily proved.

(5) It follows from

$$\begin{aligned}&(\phi \circ \psi)^{-1}(\rho) \geq \lambda \\ &\Leftrightarrow \phi(\psi(\lambda^*)) \geq \rho^* \Leftrightarrow \phi^{-1}(\rho) \geq \psi(\lambda^*)^* \\ &\Leftrightarrow \psi(\lambda^*) \geq \phi^{-1}(\rho)^* \Leftrightarrow \psi^{-1}(\phi^{-1}(\rho)) \geq \lambda.\end{aligned}$$

(6) For each $\lambda \in L^X$, we write $\lambda = \bigwedge_{z\in X}(1_{\{z\}} \to \lambda(z))$. Thus,

$$\begin{aligned}\psi(\lambda) &= \psi(\textstyle\bigwedge_{z\in X}(1_{\{z\}} \to \lambda(z))) \\ &= \textstyle\bigwedge_{z\in X} \psi(1_{\{z\}} \to \lambda(z)) \\ &= \textstyle\bigwedge_{z\in X} \rho_z.\end{aligned}$$

(7) For $\lambda = \bigwedge_{z\in X}(1_{\{z\}} \to \lambda(z))$, we have

$$\begin{aligned}\psi_1(\lambda) &= \psi_1(\textstyle\bigwedge_{z\in X}(1_{\{z\}} \to \lambda(z)) \\ &= \textstyle\bigwedge_{z\in X} \psi_1(1_{\{z\}} \to \lambda(z)) \\ &= \textstyle\bigwedge_{z\in X} \psi_2(1_{\{z\}} \to \lambda(z)) \\ &= \psi_2(\textstyle\bigwedge_{z\in X}(1_{\{z\}} \to \lambda(z)) \\ &= \psi_2(\lambda).\end{aligned}$$

□

Definition 3.3. By Theorem 3.2(6), a family $F = \{\psi(1_{\{x\}} \to \alpha) \mid \alpha \in L, x \in X\}$ generates $\psi \in L(X, Y)$.

Theorem 3.4. For $\psi, \psi_i \in L(X, Y)$ $i \in \{1, 2, 3\}$ and $\phi_j \in L(X, Y)$ $j \in \{1, 2\}$, we define two operations $\psi_1 \otimes \psi_2, \psi_1 \uplus \psi_2 \in L(X, Y)$ generating with

$$\psi_1 \otimes \psi_2(1_x \to \alpha) = \bigvee\{\psi_1(1_x \to \alpha_1) \odot \psi_2(1_x \to \alpha_2) \mid \alpha = \alpha_1 \odot \alpha_2\}$$

$$\psi_1 \uplus \psi_2(1_x \to \alpha) = \bigvee\{\psi_1(1_x \to \alpha_1) \oplus \psi_2(1_x \to \alpha_2) \mid \alpha = \alpha_1 \oplus \alpha_2\}.$$

Then the following properties hold:

(1) If $\phi_1 \leq \phi_2, \psi_1 \leq \psi_2$, then $\phi_1 \otimes \psi_1 \leq \phi_2 \otimes \psi_2$ and $\phi_1 \uplus \psi_1 \leq \phi_2 \uplus \psi_2$.

(2) If $\psi_2(1_X)=1_Y$, then $\psi_1\otimes\psi_2\geq\psi_1$.
(3)If $\psi_2(1_x^*)=0$ for all $x\in X$, $\psi_1\uplus\psi_2\geq\psi_1$.
(4) $(\psi_1\otimes\psi_2)\otimes\psi_3=\psi_1\otimes(\psi_2\otimes\psi_3)$ and $(\psi_1\uplus\psi_2)\uplus\psi_3=\psi_1\uplus(\psi_2\uplus\psi_3)$,
(5) $\psi_1\otimes\psi_2(\lambda_1\odot\lambda_2)\geq\psi_1(\lambda_1)\odot\psi_2(\lambda_2)$ and $\psi_1\uplus\psi_2(\lambda_1\oplus\lambda_2)\geq\psi_1(\lambda_1)\oplus\psi_2(\lambda_2)$. Furthermore,

$$\psi_1\otimes\psi_2(\lambda)\geq\bigvee\{\psi_1(\lambda_1)\odot\psi_2(\lambda_2)\mid\lambda=\lambda_1\odot\lambda_2\},$$

$$\psi_1\uplus\psi_2(\lambda)\geq\bigvee\{\psi_1(\lambda_1)\oplus\psi_2(\lambda_2)\mid\lambda=\lambda_1\oplus\lambda_2\}.$$

(6)

$$\psi_1\otimes\psi_2=\bigwedge\{\psi\in L(X,Y)\mid\psi(\lambda_1\odot\lambda_2)\geq\psi_1(\lambda_1)\odot\psi_2(\lambda_2)\},$$

$$\psi_1\uplus\psi_2=\bigwedge\{\psi\in L(X,Y)\mid\psi(\lambda_1\oplus\lambda_2)\geq\psi_1(\lambda_1)\oplus\psi_2(\lambda_2)\},$$

(7) $(\psi_1\otimes\psi_2)^{-1}=\psi_1^{-1}\uplus\psi_2^{-1}\in L(Y,X)$
(8) $(\psi_1\uplus\psi_2)^{-1}=\psi_1^{-1}\otimes\psi_2^{-1}\in L(Y,X)$.
(9) If $\varphi_1,\varphi_2\in L(Y,Z)$, then $(\varphi_1\otimes\varphi_2)\circ(\phi_1\otimes\phi_2)\geq(\varphi_1\circ\phi_1)\otimes(\varphi_2\circ\phi_2)$. and $(\varphi_1\uplus\varphi_2)\circ(\phi_1\uplus\phi_2)\geq(\varphi_1\circ\phi_1)\uplus(\varphi_2\circ\phi_2)$.

Proof. (1) Straightforward.
(2) Since $\bigwedge_{x\in X}\psi_2(1_x\to 1)=\psi_2(1_X)=1_Y$ implies $\psi_2(1_x\to 1)=1_Y$, for $\lambda=\bigwedge_{x\in X}(1_x\to\lambda_1(x)\odot\lambda_2(x))$, we have

$$\begin{aligned}&\psi_1\otimes\psi_2(\lambda)\\&=\bigwedge_{x\in X}\Big\{\bigvee\Big(\psi_1(1_x\to\lambda_1(x))\odot\psi_2(1_x\to\lambda_2(x))\Big)\mid\\&\quad\lambda(x)=\lambda_1(x)\odot\lambda_2(x)\Big\}\\&\geq\bigwedge_{x\in X}\Big\{\Big(\psi_1(1_x\to\lambda(x))\odot\psi_2(1_x\to 1)\Big)\mid\\&\quad\lambda(x)=\lambda(x)\odot 1(x)\Big\}\\&=\bigwedge_{x\in X}\psi_1(1_x\to\lambda(x))=\psi_1(\lambda).\end{aligned}$$

(3) Since $\bigwedge_{x\in X}\psi_2(1_x\to 0)=\psi_2(\bar{0})=\bar{0}$, for $\lambda=\bigwedge_{x\in X}(1_x\to\lambda(x)\oplus 0)$, we have

$$\begin{aligned}&\psi_1\uplus\psi_2(\lambda)\\&\geq\bigwedge_{x\in X}\Big\{\Big(\psi_1(1_x\to\lambda(x))\oplus\psi_2(1_x\to 0)\Big)\mid\\&\quad\lambda(x)=\lambda(x)\oplus 0\Big\}\\&=\bigwedge_{x\in X}\psi_1(1_x\to\lambda(x))=\psi_1(\lambda).\end{aligned}$$

(4) Suppose there exists $(1_x\to\alpha)\in L^X$ with $(\psi_1\otimes(\psi_2\otimes\psi_3))(1_x\to\alpha)\not\geq((\psi_1\otimes\psi_2)\otimes\psi_3)(1_x\to\alpha)$. Then there exist $\alpha_i\in L$ with $\alpha=\alpha_1\odot\alpha_2$ such that

$$\begin{aligned}&(\psi_1\otimes(\psi_2\otimes\psi_3))(1_x\to\alpha)\\&\not\geq(\psi_1\otimes\psi_2)(1_x\to\alpha_1)\odot\psi_3(1_x\to\alpha_2).\end{aligned}$$

By (L5), there exist β_1 and β_2 with $\alpha_1=\beta_1\odot\beta_2$ such that

$$\begin{aligned}&(\psi_1\otimes(\psi_2\otimes\psi_3))(1_x\to\alpha)\\&\not\geq(\psi_1(1_x\to\beta_1)\odot\psi_2(1_x\to\beta_1))\odot\phi_3(1_x\to\alpha_2).\end{aligned}$$

On the other hand, since $(\beta_1\odot\beta_2)\odot\alpha_2=\beta_1\odot(\beta_2\odot\alpha_2)$,

$$\begin{aligned}&(\psi_1\otimes(\psi_2\otimes\psi_3))(1_x\to\alpha)\\&\geq\psi_1(1_x\to\beta_1)\odot(\psi_2(1_x\to\beta_2)\odot\psi_3(1_x\to\alpha_2)).\end{aligned}$$

It is a contradiction. Thus, $\psi_1\otimes(\psi_2\otimes\psi_3)\geq(\psi_1\otimes\psi_2)\otimes\psi_3$.
Similarly, $\psi_1\otimes(\psi_2\otimes\psi_3)\leq(\psi_1\otimes\psi_2)\otimes\psi_3$ and $(\psi_1\uplus\psi_2)\uplus\psi_3=\psi_1\uplus(\psi_2\uplus\psi_3)$.
(5)

$$\begin{aligned}&\psi_1\otimes\psi_2(\lambda)\\&=\bigvee\Big\{\Big(\bigwedge_{x\in X}(\psi_1(1_x\to\lambda_1(x))\odot\psi_2(1_x\to\lambda_2(x))\Big)\mid\\&\lambda=\bigwedge_{x\in X}(1_x\to\lambda_1(x)\odot\lambda_2(x))\Big\}\\&\geq\bigvee\Big\{\Big(\bigwedge_{x\in X}\psi_1(1_x\to\lambda_1(x))\Big)\odot\Big(\bigwedge_{y\in X}\psi_2(1_y\to\lambda_2(y))\Big)\mid\\&\lambda=\lambda_1\odot\lambda_2\Big\}\\&=\bigvee\{\psi_1(\lambda_1)\odot\psi_2(\lambda_2)\mid\lambda=\lambda_1\odot\lambda_2\}.\end{aligned}$$

Other cases are similarly proved.
(6) By (5), $\psi_1\uplus\psi_2(\lambda_1\oplus\lambda_2)\geq\psi_1(\lambda_1)\oplus\psi_2(\lambda_2)$. Since $\lambda_1=\bigwedge_{x\in X}(1_x\to\lambda_1(x))$ and $\lambda_2=\bigwedge_{y\in X}(1_y\to\lambda_2(y))$, we have

$$\begin{aligned}\lambda_1\oplus\lambda_2&=\bigwedge_{x\in X}(1_x\to\lambda_1(x))\oplus\bigwedge_{y\in X}(1_y\to\lambda_2(y))\\&=\bigwedge_{x\in X}\bigwedge_{y\in X}\Big((1_x\to\lambda_1(x))\oplus(1_y\to\lambda_2(y))\Big)\\&=\bigwedge_{x\in X}\Big((1_x\to\lambda_1(x))\oplus(1_x\to\lambda_2(x))\Big)\\&=\bigwedge_{x\in X}((1_x\to(\lambda_1(x)\oplus\lambda_2(x))).\end{aligned}$$

If $\psi(\lambda_1\oplus\lambda_2)\geq\psi_1(\lambda_1)\oplus\psi_2(\lambda_2)$, we have

$$\begin{aligned}&\psi(\lambda_1\oplus\lambda_2)\\&=\bigwedge_{x\in X}\psi(1_x\to(\lambda_1(x)\oplus\lambda_2(x))\\&=\bigwedge_{x\in X}\psi\Big((1_x\to\lambda_1(x))\oplus(1_x\to\lambda_2(x))\Big)\\&\geq\bigwedge_{x\in X}\Big(\psi_1((1_x\to\lambda_1(x))\oplus\psi_2((1_x\to\lambda_2(x))\Big)\\&=\psi_1\uplus\psi_2(\lambda_1\oplus\lambda_2).\end{aligned}$$

(7) Suppose $(\psi_1\otimes\psi_2)^{-1}(1_x\to\alpha)\not\geq(\psi_1^{-1}\uplus\psi_2^{-1})(1_x\to\alpha)$. Then there exist $\alpha_1,\alpha_2\in L$ with $\alpha=\alpha_1\oplus\alpha_2$ such that

$$(\psi_1\otimes\psi_2)^{-1}(1_x\to\alpha)\not\geq\psi_1^{-1}(1_x\to\alpha_1)\oplus\psi_2^{-1}(1_x\to\alpha_2).$$

By the definitions of $\psi_1^{-1}(1_x\to\alpha_1)$ and $\psi_2^{-1}(1_x\to\alpha_2)$, there exist ρ_i with $\psi_1(\rho_1^*)\geq(1_x\to\alpha_1)^*$ and $\psi_2(\rho_2^*)\geq(1_x\to\alpha_2)^*$ such that

$$(\psi_1\otimes\psi_2)^{-1}(1_x\to\alpha)\not\geq\rho_1\oplus\rho_2.$$

On the other hand, since $\psi_1(\rho_1^*)\geq(1_x\to\alpha_1)^*$ and $\psi_2(\rho_2^*)\geq(1_x\to\alpha_2)^*$,

$$\begin{aligned}(\psi_1\otimes\psi_2)(\rho_1^*\odot\rho_2^*)&\geq\psi_1(\rho_1^*)\odot\psi_2(\rho_2^*)\\&\geq(1_x\to\alpha_1)^*\odot(1_x\to\alpha_2)^*.\end{aligned}$$

Since

$$(1_x \to \alpha_1) \oplus (1_x \to \alpha_2) = 1_x \to (\alpha_1 \oplus \alpha_2)$$

we have, $(\psi_1 \otimes \psi_2)^{-1}(1_x \to \alpha) \geq \rho_1 \oplus \rho_2$. It is a contradiction. Thus, $(\psi_1 \otimes \psi_2)^{-1} \geq \psi_1^{-1} \uplus \psi_2^{-1}$.

Suppose $(\psi_1 \otimes \psi_2)^{-1}(1_x \to \alpha) \not\leq (\psi_1^{-1} \uplus \psi_2^{-1})(1_x \to \alpha)$. Then there exists ρ with $(\psi_1 \otimes \psi_2)(\rho^*) \geq (1_x \to \alpha)^*$ such that

$$\rho \not\leq (\psi_1^{-1} \uplus \psi_2^{-1})(1_x \to \alpha).$$

Since $(1_x \to \alpha) \geq ((\psi_1 \otimes \psi_2)(\rho^*))^*$, we have

$$\begin{aligned}
&(1_x \to \alpha)\\
&\geq \Big(\bigwedge_{x\in X}\{\bigvee \psi_1(1_x \to \rho_1^*(x)) \odot \psi_2(1_x \to \rho_2^*(x)) \mid \\
&\rho^*(x) = \rho_1^*(x) \odot \rho_2^*(x)\}\Big)^*\\
&= \bigvee_{x\in X}\{\bigwedge \psi_1(1_x \to \rho_1^*(x))^* \oplus \psi_2(1_x \to \rho_2^*(x))^* \mid \\
&\rho^*(x) = \rho_1^*(x) \odot \rho^*(x)\}
\end{aligned}$$

Since $(\psi_1^{-1} \uplus \psi_2^{-1}) \in L(X,Y)$ and $\psi_i(1_x \to \rho_i^*(x)) \leq \psi_i(1_x \to \rho_i^*(x))$, we have

$$\psi_i^{-1}(\psi_i(1_x \to \rho_i^*(x))^*) \geq (1_x \to \rho_i^*(x))^*, \quad i = 1, 2.$$

It implies

$$\begin{aligned}
&(\psi_1^{-1} \uplus \psi_2^{-1})(1_x \to \alpha)\\
&\geq (\psi_1^{-1} \uplus \psi_2^{-1})\Big(\bigvee_{x\in X}\{\bigwedge \psi_1(1_x \to \rho_1^*(x))^*\\
&\oplus \psi_2(1_x \to \rho_2^*(x))^* \mid \rho^*(x) = \rho_1^*(x) \odot \rho^*(x)\}\Big)\\
&\geq \bigvee_{x\in X}\Big(\{\bigwedge \psi_1^{-1}(\psi_1(1_x \to \rho_1^*(x))^*)\\
&\oplus \psi_2^{-1}(\psi_2(1_x \to \rho_2^*(x))^*) \mid \rho^*(x) = \rho_1^*(x) \odot \rho^*(x)\}\Big)\\
&\geq \bigvee_{x\in X}\Big((1_x \to \rho_1^*(x))^*) \oplus (1_x \to \rho_2^*(x))^*\Big)\\
&= \bigvee_{x\in X}\Big((1_x \to \rho_1^*(x)) \odot (1_x \to \rho_2^*(x))\Big)^*\\
&= \bigvee_{x\in X}\Big(1_x \to (\rho_1^*(x) \odot \rho_2^*(x)\Big)^*\\
&= \bigvee_{x\in X}\Big(1_x \to \rho^*(x)\Big)^*\\
&= \Big(\bigwedge_{x\in X}(1_x \to \rho^*(x))\Big)^*\\
&= (\rho^*)^* = \rho.
\end{aligned}$$

(8) Since $(\psi_1^{-1} \otimes \psi_2^{-1})^{-1} = (\psi_1^{-1})^{-1} \uplus (\psi_2^{-1})^{-1} = \psi_1 \uplus \psi_2$, we have $\psi_1^{-1} \otimes \psi_2^{-1} = (\psi_1 \uplus \psi_2)^{-1}$.

(9) Suppose there exists $1_x \to \alpha \in L^X$ with $(\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)(1_x \to \alpha) \not\geq (\varphi_1 \circ \phi_1) \otimes (\varphi_2 \circ \phi_2)(1_x \to \alpha)$. Then there exist α_i with $\alpha = \alpha_1 \odot \alpha_2$ such that

$$\begin{aligned}
&(\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)(1_x \to \alpha)\\
&\not\geq (\varphi_1 \circ \phi_1)(1_x \to \alpha_1) \odot (\varphi_2 \circ \phi_2)(1_x \to \alpha_2).
\end{aligned}$$

But

$$\begin{aligned}
&(\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)(1_x \to \alpha)\\
&\geq (\varphi_1 \otimes \varphi_2)(\phi_1(1_x \to \alpha_1) \odot \phi_2(1_x \to \alpha_2)) \quad \text{(by (5))}\\
&\geq \varphi_1(\phi_1(1_x \to \alpha_1)) \odot \varphi_2(\phi_2(1_x \to \alpha_2)).
\end{aligned}$$

It is a contradiction. Thus, $(\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2) \geq (\varphi_1 \circ \varphi_1) \otimes (\varphi_2 \circ \phi_2)$.

By (7) and Theorem 3.2, we have

$$\begin{aligned}
&(\phi_1^{-1} \otimes \phi_2^{-1}) \circ (\varphi_1^{-1} \otimes \varphi_2^{-1}) \geq (\phi_1^{-1} \circ \varphi_1^{-1}) \otimes (\phi_2^{-1} \circ \varphi_2^{-1})\\
&\Rightarrow (\phi_1 \uplus \phi_2)^{-1} \circ (\varphi_1 \uplus \varphi_2)^{-1} \geq (\varphi_1 \circ \phi_1)^{-1} \otimes (\varphi_2 \circ \phi_2)^{-1}\\
&\Rightarrow (\varphi_1 \uplus \varphi_2) \circ (\phi_1 \uplus \phi_2) \geq (\varphi_1 \circ \phi_1) \uplus (\varphi_2 \circ \phi_2).
\end{aligned}$$

□

Theorem 3.5. Let $b : P(X) \to P(Y)$ be non-constant function with $b(\bigcap_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} b(A_i)$. Let $\psi_b : L^X \to L^Y$ be a function satisfying the following conditions:

(A) $\phi_b(1_A) = 1_{b(A)}$ for all $A \in P(X)$,

(B) $\phi_b(\bigwedge_{i\in\Gamma} \lambda_i) = \bigwedge_{i\in\Gamma} \phi_b(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$,

(C) $\phi_b(1_A \to \alpha) = 1_{b(A)} \to \alpha$, for all $A \in P(X)$.

Then we have the following properties:

(1) For each $\lambda \in L^X$, we have $\lambda = \bigwedge_{a\in L}(1_{\lambda_a} \to a)$ where $\lambda_a = \{x \in X \mid \lambda(x) \leq a\}$.

(2) $\phi_b(\lambda) = \bigwedge_{a\in L}(1_{b(\lambda_a)} \to a)$.

(3) $b^{\to}(B) = \bigcap\{A \in P^X \mid B \subset b(A)\}$ such that $b^{\to}$ is a left adjoint of b with $b^{\to}(\bigcap_{i\in\Gamma} B_i) = \bigcap_{i\in\Gamma} b^{\to}(B_i)$.

(4) If $b(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} b(A_i)$, $b^{-1}(B) = \bigcup\{A \in P^X \mid b(A) \subset B\}$ such that b^{-1} is a right adjoint of b.

(5) If $b(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} b(A_i)$, then $\phi_b^{\to}$ is a left adjoint of ϕ_b as follows:

$$\begin{aligned}
\phi_b^{\to}(\rho) &= \bigwedge\{\lambda \in L^X \mid \phi_b(\lambda) \geq \rho\}\\
&= \bigwedge_{a\in L}(1_{b^{-1}(\rho_a)} \to a) = \phi_{b^{-1}}(\rho).
\end{aligned}$$

(6) If $b(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} b(A_i)$, then

$$(\phi_b)^{-1}(\rho) = \bigvee_{a\in L}(a^* \odot 1_{b^{-1}(\rho_a^*)}).$$

Proof. (1) If $\lambda(x) = a$, then $x \in \lambda_a$. Thus

$$\bigwedge_{a\in L}(1_{\lambda_a} \to a)(x) \leq (1_{\lambda_a} \to a)(x) = 1 \to a = a = \lambda(x).$$

Hence $\lambda \geq \bigwedge_{a\in L}(1_{\lambda_a} \to a)$.

Suppose there exists $x \in X$ such that

$$\lambda(x) \not\leq \bigwedge_{a\in L}(1_{\lambda_a} \to a)(x).$$

Put $\lambda(x) = d$. Then there exists $e \in L$ such that

$$d = \lambda(x) \not\leq (1_{\lambda_e} \to e)(x).$$

It implies $x \in \lambda_e$, that is, $d = \lambda(x) \leq e$. It is a contradiction because

$$d = \lambda(x) \not\leq (1_{\lambda_e} \to e)(x) = e.$$

Hence $\lambda \leq \bigwedge_{a\in L}(1_{\lambda_a} \to a)$.

(2) Since $\lambda = \bigwedge_{a\in L}(1_{\lambda_a} \to a)$, we have

$$\begin{aligned}\phi_b(\lambda)(y) &= \phi_b(\bigwedge_{a\in L}(1_{\lambda_a} \to a))(y)\\ &= \bigwedge_{a\in L}\phi_b(1_{\lambda_a} \to a)(y)\\ &= \bigwedge_{a\in L}(\phi_b(1_{\lambda_a}) \to a)(y)\\ &= \bigwedge_{a\in L}(1_{b(\lambda_a)} \to a)(y)\end{aligned}$$

(3) Since $B \subset b(A)$ iff $b^{\to}(B) \subset A$, $b^{\to}$ is a left adjoint of b. Moreover, we have

$$\begin{aligned}b^{\to}(\bigcup_{i\in\Gamma} B_i) \subset A \quad &\Leftrightarrow \bigcup_{i\in\Gamma} B_i \subset b(A)\\ &\Leftrightarrow B_i \subset b(A), \forall i \in \Gamma\\ &\Leftrightarrow b^{\to}(B_i) \subset A, \forall i \in \Gamma\\ &\Leftrightarrow \bigcup_{i\in\Gamma} b^{\to}(B_i) \subset A.\end{aligned}$$

Hence $b^{\to}(\bigcup_{i\in\Gamma} B_i) = \bigcup_{i\in\Gamma} b^{\to}(B_i)$.

(4) Since $b(A) \subset B$ iff $A \subset b^{-1}(B)$, b^{-1} is a right adjoint of b. By a similar method as in (3), $b^{-1}(\bigcap_{i\in\Gamma} B_i) = \bigcap_{i\in\Gamma} b^{-1}(B_i)$.

(5) Since $\phi_b^{\to}(\rho)(x) = \bigwedge\{\lambda(x) \mid \phi_b(\lambda) \geq \rho\}$ and

$$\begin{aligned}&\phi_b(\lambda) = \bigwedge_{a\in L}(1_{b(\lambda_a)} \to a) \geq \bigwedge_{a\in L}(1_{\rho_a} \to a) = \rho\\ &\Leftrightarrow b(\lambda_a) \subset \rho_a, \forall a \in L\\ &\Leftrightarrow \lambda_a \subset b^{-1}(\rho_a), \forall a \in L\\ &\Leftrightarrow \lambda = \bigwedge_{a\in L}(1_{\lambda_a} \to a) \geq \bigwedge_{a\in L}(1_{b^{-1}(\rho_a)} \to a)\end{aligned}$$

we have $\phi_b^{\to}(\rho) = \bigwedge_{a\in L}(1_{b^{-1}(\rho_a)} \to a) = \phi_{b^{-1}}(\rho)$.

(6)

$$\begin{aligned}\phi_b^{-1}(\rho) &= (\phi_b^{\to}(\rho^*))^*\\ &= (\bigwedge_{a\in L}(1_{b^{-1}(\rho_a^*)} \to a))^*\\ &= \bigvee_{a\in L}(1^*_{b^{-1}(\rho_a^*)} \oplus a)^* \text{ (by Lemma 2.3(11))}\\ &= \bigvee_{a\in L}(1_{b^{-1}(\rho_a^*)} \odot a^*)\end{aligned}$$

□

Example 3.6. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be sets. Define $b : P(X) \to P(Y)$ as follows:

$$b(\emptyset) = \{y_1\},\ b(\{x_1\}) = \{y_1, y_2\},$$

$$b(\{x_2\}) = \{y_1\},\ b(X) = Y.$$

Since $b(\bigcap A_i) = \bigcap b(A_i)$, b has a left adjoint $b^{\to}$ defined as

$$b^{\to}(B) = \bigcap\{A \in P(X) \mid B \subset b(A)\}$$

We obtain

$$b^{\to}(\{y_1, y_2\}) = b^{\to}(\{y_2\}) = \{x_1\},$$

$$b^{\to}(\{y_1\}) = b^{\to}(\emptyset) = \emptyset.$$

Let $([0,1], \odot)$ be a continuous t-norm and $\lambda \in [0,1]^X$, $\rho \in [0,1]^Y$ as follows:

$$\lambda(x_1) = 0.5, \lambda(x_2) = 0.7, \quad \rho(y_1) = 0.4, \rho(y_2) = 0.5.$$

Since

$$\begin{aligned}\lambda &= (1_{\lambda_{0.5}} \to 0.5) \wedge (1_{\lambda_{0.7}} \to 0.7)\\ &= (1_{x_1} \to 0.5) \wedge (1_X \to 0.7)\\ \rho &= (1_{\rho_{0.4}} \to 0.4) \wedge (1_{\rho_{0.5}} \to 0.5)\\ &= (1_{y_1} \to 0.4) \wedge (1_Y \to 0.5)\end{aligned}$$

we obtain

$$\begin{aligned}\phi_b(\lambda)(y_1) &= \Big((1_{b(\{x_1\})} \to 0.5) \wedge (1_{b(X)} \to 0.7)\Big)(y_1)\\ &= \Big((1_Y \to 0.5) \wedge (1_Y \to 0.7)\Big)(y_1) = 0.5.\end{aligned}$$

$$\phi_b(\lambda)(y_2) = \Big((1_Y \to 0.5) \wedge (1_Y \to 0.7)\Big)(y_2) = 0.5.$$

Since $b(\bigcup A_i) = \bigcup b(A_i)$, b has a right adjoint b^{-1} defined as

$$b^{-1}(B) = \bigcup\{A \in P(X) \mid b(A) \subset B\}$$

We obtain

$$b^{-1}(\{y_1, y_2\}) = X, b^{-1}(\{y_1\}) = \{x_2\},$$

$$b^{-1}(\{y_2\}) = b^{-1}(\emptyset) = \emptyset.$$

$$\begin{aligned}\phi_b^{\to}(\rho)(x_1) &= \Big((1_{b^{-1}(\{y_1\})} \to 0.4) \wedge (1_{b^{-1}(Y)} \to 0.5)\Big)(x_1)\\ &= \Big((1_{\{x_2\}} \to 0.4) \wedge (1_X \to 0.5)\Big)(x_1) = 0.5.\end{aligned}$$

$$\phi_b^{\to}(\rho)(x_2) = \Big((1_{\{x_2\}} \to 0.4) \wedge (1_X \to 0.5)\Big)(x_2) = 0.4.$$

Furthermore, since

$$\rho^* = (1_{\rho^*_{0.6}} \to 0.6) \wedge (1_{\rho^*_{0.5}} \to 0.5) = (1_Y \to 0.6) \wedge (1_{\{y_1\}} \to 0.5)$$

we have

$$\begin{aligned}(\phi_b)^{-1}(\rho) &= \bigvee_{a\in L}(a^* \odot 1_{b^{-1}(\rho_a^*)})\\ &= (1_{b^{-1}(\{y_1\})} \odot 0.5) \wedge (1_{b^{-1}(Y)} \odot 0.4)\\ &= (1_{\{x_2\}} \odot 0.5) \wedge (1_X \odot 0.4)\end{aligned}$$

We obtain $(\phi_b)^{-1}(\rho)(x_1) = 0, (\phi_b)^{-1}(\rho)(x_2) = 0.4$.

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