Communications of the Korean Statistical Society
2009, Vol. 16, No. 3, 487–493

Estimation for the Time-t Discounted Price of Multiple Defaultable Zero Coupon Bond

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Abstract

We consider a multiple defaultable zero coupon bond. Assuming defaults occur according to a marked point process, we explain how to estimate the time-t discounted price of zero coupon bond by simulation. For the special case of a given specific random face value, we show that the real probability measure is the risk neutral probability measure. In this case the time-t discounted conditional price can be obtained by observing a single sample path upto the time t in the real world. Furthermore the time-t discounted price can be estimated by observing real situations or by simulation under the real probability measure.

Keywords: Multiple defaults, zero coupon bond, simulation, marked point process.

1. Introduction

In this paper, we consider multiple defaultable zero coupon bonds. The pricing of multiple defaultable bonds was first considered by Schönbucher (1998). He represented the term structure of defaultable bond prices in terms of forward rates. Duffie, Pederson and Singleton(DPS) (2003) constructed a model of the term structure of credit spreads on sovereign bonds. They showed that the cash flows promised by a sovereign bond can be discounted using a default-adjusted short-term discount rate. Following the framework of Lando (1998) and using the short-rate framework, Wong (2004) constructed a model of pricing zero coupon bond. His model was built on the assumption that the default generating process can be described by a Cox process and there exists an independent identically distributed process which marks each default. In this paper we explain how to estimate the time-t discounted price of multiple defaultable zero coupon bond by simulation. If the face value of bond is given as a special form of random variable, we show that the real probability is the risk neutral probability and hence, we can find it by simulation under the real probability measure.

Throughout this article we consider a marked point process $(T_n, Z_n; n \geq 1)$ defined on some measurable space $(\Omega, \mathcal{F})$, where T_n is a point process and Z_n is a sequence of random variables which have values (marks) on $(E, \mathcal{E}) = ((0,1), \mathcal{B}(0,1))$. If the n^{th} default occurs at time T_n, then the n^{th} loss rate Z_n, which has values in $(0,1)$, is generated by a given conditional distribution depending on the information upto the time. We associate to each $A \in \mathcal{E}$ the counting process $N_t(A)$ defined by

$$N_t(A) = \sum_{n \geq 1} 1(Z_n \in A) 1(T_n \leq t).$$

The internal history $\mathcal{F}_t^p$ of $(T_n, Z_n, n \geq 1)$ is defined by

$$\mathcal{F}_t^p = \sigma(N_s(A); 0 \leq s \leq t, A \in \mathcal{E}).$$

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Let

$$\mathcal{F}_t = \mathcal{F}_0 \vee \mathcal{F}_t^p, \quad t \geq 0.$$

We assume the existence of risk neutral probability measure P^* under which the discounted defaultable bond prices become a martingale. So all variables and processes are properly defined on the fixed probability space $(\Omega, \mathcal{F}, P^*)$.

This paper is organized as follows. In Section 2, we explain a method of estimating the time-t discounted price of zero coupon bond by simulation. In Section 3, under some restriction we show that the real probability measure is the risk neutral probability measure and hence, we can estimate the above time-t discounted price of zero coupon bond by simulation under real probability. Section 4 concludes this paper.

2. Estimation for the Time-t Discounted Price of Multiple Defaultable Bonds by Simulation

In this section we will explain a method that a time-t discounted multiple defaultable zero coupon bonds can be obtained by simulation. Let us consider a multiple defaultable zero-coupon bond with maturity date T, which promises to pay X at time T if no default occurs during the life of the bond. If the bond defaults before time T then the owner of the bond is given another bond of lower face value at each default event. In particular, if there are N_T defaults during the life of the bond, then the final pay-off would be

$$Z_T \equiv \prod_{n=1}^{N_T}(1 - Z_n)X,$$

where Z_n denotes the loss rate at the time of n^{th} default and N_T is the number of defaults up to the time T.

Since the discounted defaultable bond price is a martingale under the risk neutral probability,

$$E\left[\exp\left(-\int_0^T r(u)du\right)\prod_{n=1}^{N_T}(1 - Z_n)X \middle| \mathcal{F}_t\right] = \exp\left(-\int_0^t r(u)du\right)\prod_{n=1}^{N_t}(1 - Z_n)X, \tag{2.1}$$

where $r(u)$ is the short time interest rate. Multiplying both side of the Equation (2.1) by $\exp(\int_0^t r(u)du)$, the time-t discounted conditional price $v(\omega, t, T)$ of a multiple defautable zero-coupon bond which settles at time T is given as

$$v(\omega, t, T) = E\left[\exp\left(-\int_t^T r(u)du\right)\prod_{n=1}^{N_T}(1 - Z_n)X \middle| \mathcal{F}_t\right](\omega) = \prod_{n=1}^{N_t(\omega)}(1 - Z_n(\omega))X. \tag{2.2}$$

The righthand side of the above Equation (2.2) can be generated by a single simulation run using the following theorem whose proof is given in Bremaud (1981).

Theorem 1. *Let $(T_n, Z_n, n \geq 1)$ be an E-marked point process adapted to $\mathcal{F}_t$ Suppose that for each $n \geq 1$, there exists a regular conditional distribution of (S_{n+1}, Z_{n+1}) given $\mathcal{F}_{T_n}$ of the form*

$$P(S_{n+1} \in A, Z_{n+1} \in C | \mathcal{F}_{T_n}) = \int_A g^{(n+1)}(s, C)ds, \quad A \in \mathcal{B}_+,\ C \in \mathcal{E},$$

where $g^{(n+1)}(\omega, s, C)$ *is a finite kernel from* $(\Omega \times [0, \infty),\ \mathcal{F}_{T_n} \bigotimes \mathcal{B}_+)$ *into* $(E, \mathcal{E})$*, that is to say;*

1. *for fixed*$(\omega, s), C \to g^{(n+1)}(\omega, s, C)$ *is a finite measure on* $(E, \mathcal{E})$.
2. *for fixed* $C \in \mathcal{E}$*, the mapping* $(\omega, s) \to g^{(n+1)}(\omega, s, C)$ *is* $\mathcal{F}_{T_n} \bigotimes \mathcal{B}_+$ *- measurable.*

Then for all $n \geq 1$ *and all* $C \in \mathcal{E}$

$$P(Z_n \in C | \mathcal{F}_{T_n -}) = \frac{\lambda_{T_n}(C)}{\lambda_{T_n}(E)} P^* - a.s. \ \ on \ \ \{T_n < \infty\},$$

where

$$\lambda_t(C) = \frac{g^{(n+1)}(t - T_n, C)}{1 - \int_0^{t-T_n} g^{(n+1)}(x, E)dx} \ \ on \ \ (T_n, T_{n+1}].$$

Using the conditional densities $g^{n+1}(\omega, s, C)$ whose existence was assumed in the above theorem, for a given $\omega \in \Omega$ and $n \geq 0$, we can generate $S_{n+1} = T_{n+1} - T_n$ and Z_{n+1} as follows.

1. set $n = 0$, CP $= X$
2. by the inverse transform method (See Banks and Carson, 1984), generate S_{n+1} from the conditional distribution
$$P(S_{n+1} \leq t | \mathcal{F}_{T_n})(\omega) = \int_{-\infty}^{t} g^{n+1}(\omega, s, E)ds,$$
where $\mathcal{F}_{T_n} = \sigma(T_i, Z_i, 0 \leq i \leq n)$
3. set $T_{n+1} = T_n + S_{n+1}$
4. by the inverse transform method, generate Z_{n+1} from the conditional distribution
$$P(Z_{n+1} \leq t | \mathcal{F}_{T_{n+1}-})(\omega) = \frac{\lambda_{T_{n+1}}([0, t])(\omega)}{\lambda_{T_{n+1}}([0, 1])(\omega)} P^* - a.s. \ \ on \ \ \{T_{n+1} < \infty\},\ t < 1,$$
where $\mathcal{F}_{T_{n+1}-} = \sigma(T_0, Z_0, T_1, Z_1, \ldots, T_n, Z_n, T_{n+1})$
5. if $T_{n+1} < t$, CP = CP*$(1 - Z_{n+1})$, $n = n + 1$, GO TO (ii)
6. if $T_{n+1} \geq t$, STOP.

From the above formula we can find the time-t discounted conditional price of zero coupon bond by a single simulation run. If we denote the result of the i^{th} simulation run by CP(i), the time-t discounted price of multiple defaultable zero coupon bond

$$E\left[\exp\left(-\int_t^T r(u)du\right)\prod_{n=1}^{N_T}(1 - Z_n)X\right] = E\left[\prod_{n=1}^{N_t}(1 - Z_n)X\right]$$

can be estimated by $\sum_{i=1}^{N} \mathrm{CP}(i)/N$ for large N.

The difficulty of this method is that we have to prepare for infinite numbers of distributions to carry a single run of simulation. This difficulty can be avoided by restricting our attention to the special cases which satisfy all the three items.

- each interoccurence times S_n of point process can assume only finite number of values.
- each loss rates Z_n also can assume only finite number of values.
- the total number of defaults $N_T(E)$ is finite.

Under these assumption, the number of information in $\mathcal{F}_{T_n}$ is finite for all n. Hence we need only finite number of conditional distributions for a single run simulation. But the difficulty of finding a risk neutral probability still remaines unsolved. In the next section we will show that we can find the time-t discounted conditional price of zero coupon bond under the real probability measure if some restrictions are given.

3. Estimation under the Real Probability Measure

In the last section we explained how to simulate time-t discounted price of zero coupon bond by using conditional distributions under the risk neutral probability P^*. In this section we will show that the same thing can be obtained under the real probability measure. First, we need the following theorem.

Theorem 2. *Let $(T_n, Z_n, n \geq 1)$ be an E-marked point process adapted to $\mathcal{F}_t$. Let λ_t be the $(P, \mathcal{F}_t)$-predictable intensities of N_t with bound M, in other words $|\lambda_s| < M$ for all $s \geq 0$, $P-a.s.$ Assume the existence of $\mathcal{F}_t$- predictable loss rate process μ_t which has values in (0,1) and $\mu_{T_n} = Z_n$ for all $n \geq 1$. Let*

$$L_t = \left(\prod_{n \geq 1} (1 - \mu_{T_n}) 1(T_n \leq t) \right) \exp\left(\int_0^t \mu_s \lambda_s ds \right).$$

Then $(L_t\ 0 \leq t \leq T)$ is a $(P, \mathcal{F}_t)$ − martingale.

Proof: Define $f(t)$ and $g(t)$ by

$$f(t) = \prod_{0<s\leq t} (1 + (-\mu_s)\Delta N_s)$$
$$g(t) = \exp\left(\int_0^t \mu_s \lambda_s ds \right), \tag{3.1}$$

where $\prod_{0<s\leq t}(1 + (-\mu_s)\Delta N_s) = 1$ if $\Delta N_s = 0$ for all $s \in (0, t]$. By the product formula (Bremaud, 1981),

$$\begin{aligned} f(t)g(t) &= 1 + \int_0^t f(s-)dg(s) + \int_0^t g(s)df(s) \\ &= 1 + \int_0^t f(s-)g(s)\mu_s\lambda_s ds + \sum_{0<s\leq t} g(s)f(s-)(-\mu_s)\Delta N_s \\ &= 1 + \int_0^t f(s-)g(s-)(-\mu_s)dM_s, \end{aligned} \tag{3.2}$$

where $M_t = N_t - \int_0^t \lambda_s ds$.

Since $f(s-)g(s-)$ is left continuous, it is $\mathcal{F}_t$-predictable. By the assumption, μ_t is $\mathcal{F}_t$- predictable process. Hence $f(s-)g(s-)(-\mu_t)$ is $\mathcal{F}_t$- predictable. Using $\mu_s \in (0, 1)$ and $f(s-)g(s-) \leq \exp(\int_0^t \lambda_s ds)$, we know the following equation holds.

$$E\left[\int_0^t |f(s-)g(s-)(-\mu_s)|\lambda_s ds\right] \leq E\left[\exp\left(\int_0^t \lambda_s ds\right)\int_0^t \lambda_s ds\right] \leq \exp(Mt)Mt < \infty, \quad t \geq 0.$$

From this we have $\int_0^t f(s-)g(s-)(-\mu_s)dM_s$ is an $\mathcal{F}_t$-martingale (Bremaud, 1981). Hence, from the Equation (3.2),

$$f(t)g(t) = 1 + \int_0^t f(s-)g(s-)(-\mu_s)dM_s$$

is a $\mathcal{F}_t$-martingale. On the other hand, from the Equation (3.1)

$$\begin{aligned} f(t)g(t) &= \prod_{0<s\leq t}(1 + (-\mu_s)\Delta N_s)\exp\left(\int_0^t \mu_s\lambda_s ds\right) \\ &= \prod_{n\geq 1}(1 - \mu_{T_n})1(T_n \leq t)\exp\left(\int_0^t \mu_s\lambda_s ds\right) = L_t. \end{aligned}$$

Hence, L_t is an $\mathcal{F}_t$-martingale. □

We note this theorem holds for any probability measure P under which the assumptions of the theorem are satisfied. Since we can assume the assumptions of the theorem can be satisfied under the real probability measure, L_t is an $\mathcal{F}_t$-martingale under the real probability measure. In other words the real probability measure is the risk neutral probability measure for L_t.

Corollary 3. *On the assumptions of above theorem, let us add one more assumption that the face value X_t is given by $\exp(\int_0^t (r_s + \mu_s\lambda_s)ds)$, for all $0 \leq t \leq T$. Then the time-t discounted conditional price of multiple defautable zero coupon bond with the face value X_t is*

$$v(t,T) = \left(\prod_{n\geq 1}(1 - \mu_{T_n})1(T_n \leq t)\right)\exp\left(\int_0^t (r_s + \mu_s\lambda_s)ds\right).$$

Proof: Since L_t is a $(P, \mathcal{F}_t)$-martingale by the Theorem 2,

$$\begin{aligned} &E\left[\left(\prod_{n\geq 1}(1 - \mu_{T_n})1(T_n \leq T)\right)\exp\left(\int_0^T \mu_s\lambda_s ds\right)\Big|\mathcal{F}_t\right] \\ &= \left(\prod_{n\geq 1}(1 - \mu_{T_n})1(T_n \leq t)\right)\exp\left(\int_0^t \mu_s\lambda_s ds\right). \end{aligned} \tag{3.3}$$

This Equation (3.3) is equal to

$$\begin{aligned} &E\left[\exp\left(-\int_0^T r_s ds\right)\left(\prod_{n\geq 1}(1 - \mu_{T_n})1(T_n \leq T)\right)\exp\left(\int_0^T (r_s + \mu_s\lambda_s)ds\right)\Big|\mathcal{F}_t\right] \\ &= \exp\left(-\int_0^t r_s ds\right)\left(\prod_{n\geq 1}(1 - \mu_{T_n})1(T_n \leq t)\right)\exp\left(\int_0^t (r_s + \mu_s\lambda_s)ds\right). \end{aligned} \tag{3.4}$$

By multiplying $\exp(\int_0^t r_s ds)$ on both side of the above Equation (3.4), we have

$$
\begin{aligned}
v(t,T) &= E\left[\exp\left(-\int_t^T r_s ds\right)\left(\prod_{n\geq 1}(1-\mu_{T_n})1(T_n \leq T)\right)\exp\left(\int_0^T (r_s+\mu_s\lambda_s)ds\right)\Big|\mathcal{F}_t\right] \\
&= \left(\prod_{n\geq 1}(1-\mu_{T_n})1(T_n \leq t)\right)\exp\left(\int_0^t (r_s+\mu_s\lambda_s)ds\right). \qquad (3.5)
\end{aligned}
$$

□

From the note after Theorem 2, we know the real probability measure is the risk neutral probability measure for L_t which is equal to

$$\exp\left(-\int_0^t r_s ds\right)\left(\prod_{n\geq 1}(1-\mu_{T_n})1(T_n \leq T)\right)\exp\left(\int_0^t (r_s+\mu_s\lambda_s)ds\right).$$

In other words, the Equation (3.4) holds for the real probability measure. Consequently the Equation (3.5) shows the time-t discounted conditional price of zero coupon bond with the face value $\exp(\int_0^T (r_s + \mu_s\lambda_s)ds)$ can be obtained by observing a single sample path upto the time t in the real world. If N observations under similar condition are possible or N simulations under the real probability measure are possible, let us denote the results by CP(i) $i = 1, 2, \ldots, N$. As in the last section, the time-t discounted price of zero coupon bond with the face value $\exp(\int_0^t (r_s + \mu_s\lambda_s)ds)$ can be estimated by $\sum_{i=1}^N \text{CP}(i)/N$ for large N. In both cases, simulation or obserbation, we need to calculate the face value $\exp(\int_0^t (r_s + \mu_s\lambda_s)ds)$. This value can be approximated by using the definition of integration.

4. Concluding Remarks

In this paper we explained how to estimate the time-t discounted price of multiple defautable zero coupon bond by simulation. For this purpose we assumed the existence of conditional densities of interarrival times and loss rates under risk neutral probability. In the special case where the face value of bond is of the form $X_t = \exp(\int_0^t (r_s + \mu_s\lambda_s)ds$, we showed the time-$t$ discounted conditional price of zero coupon bond can be obtained simply by observing a single sample path upto the time t in the real world. We also suggested how to estimate the time-t discounted price by obserbations of real situations or by simulation under the real probability measure. Extension to the multitype multiple defautable zero coupon bond will appear soon in the next paper.

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Received February 2009; Accepted March 2009