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SPECTRAL METHODS AND HERMITE INTERPOLATION ON ARBITRARY GRIDS

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ABSTRACT. In this paper, spectral scheme based on Hermite interpolation for solving partial differential equations is presented. The idea of this Hermite spectral method comes from the spectral method on arbitrary grids of Carpenter and Gottlieb [J. Comput. Phys. 129(1996) 74-86] using the Lagrange interpolation.

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1. Introduction and preliminaries

For a function $f : [-1, 1] \to \mathbb{R}$ and a set $\chi_N := \{x_0, x_1, \cdots, x_N\}$, $N \geq 1$ with

$$-1 = x_N < x_{N-1} < \cdots < x_1 < x_0 = 1,$$

let $\mathcal{L}_N[\chi_N; f]$ and $\mathcal{H}_N[\chi_N; f]$ denote Lagrange and Hermite interpolation polynomials to f with respect to χ_N. For the case of Hermite interpolation, we will always assume that f is differentiable so that $\mathcal{H}_N[\chi_N; f]$ is well defined. In fact, $\mathcal{L}_N[\chi_N; f]$ and $\mathcal{H}_N[\chi_N; f]$ are the unique polynomials of degree $\leq N$ and $\leq 2N+1$ satisfying

$$\mathcal{L}_N[\chi_N; f](x_j) = f(x_j) \quad \text{and} \quad \begin{cases} \mathcal{H}_N[\chi_N; f](x_j) = f(x_j) \\ \mathcal{H}'_N[\chi_N; f](x_j) = f'(x_j) \end{cases} \tag{1.1}$$

for $j = 0, 1, 2, \cdots, N$, respectively. Let

$$Q(x) := (x - x_0)(x - x_1) \cdots (x - x_N).$$

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Then more precisely, the Hermite interpolation polynomial $\mathcal{H}_N[\chi_N; f]$ is given by

$$\mathcal{H}_N[f](x) := \mathcal{H}_N[\chi_N; f](x) := \sum_{j=0}^{2N+1} f_j H_j(x) \tag{1.2}$$

where

$$f_j := \begin{cases} f(x_j), & 0 \le j \le N \\ f'(x_{j-(N+1)}), & N+1 \le j \le 2N+1 \end{cases}$$

and the Hermite fundamental polynomials $H_j(x)$ are defined by

$$H_j(x) := \begin{cases} h_j(x), & 0 \le j \le N \\ k_{j-(N+1)}(x), & N+1 \le j \le 2N+1. \end{cases} \tag{1.3}$$

Here, $h_j(x)$ and $k_j(x)$ are polynomials of degree $\le 2N+1$ satisfying

$$\begin{cases} h_j(x_l) = \delta_{j,l}, & h_j'(x_l) = 0, \quad j, l = 0, 1, 2, \cdots N \\ k_j(x_l) = 0, & k_j'(x_l) = \delta_{j,l}, \quad j, l = 0, 1, 2, \cdots N \end{cases} \tag{1.4}$$

such that $h_j(x)$ and $k_j(x)$ are given by

$$\begin{aligned} h_j(x) &= \left(1 - \frac{Q''(x_j)}{Q'(x_j)}(x - x_j)\right)\left(\frac{Q(x)}{Q'(x_j)(x - x_j)}\right)^2 \\ k_j(x) &= (x - x_j)\left(\frac{Q(x)}{Q'(x_j)(x - x_j)}\right)^2 \end{aligned}$$

(see [7]). Then we know easily that for $\pi \in \mathcal{P}_{2N+1}$

$$\pi(x) = \sum_{j=0}^{N} \pi(x_j) H_j(x) + \sum_{j=N+1}^{2N+1} \pi'(x_{j-(N+1)}) H_j(x). \tag{1.5}$$

[1, 7] serve as good references for the Lagrange interpolation polynomial and we are interested in the spectral scheme based on Hermite interpolation in this paper.

Polynomial spectral methods is to approximate the unknown function by interpolation polynomials at some predescribed points and this methods have been extensively used in numerical solutions of partial differential equations(PDEs). Especially, [3, 4, 5, 6] treat the spectral methods with the node points related to orthogonal polynomials of ordinary differential equations such as Chebyshev polynomials and Legendre polynomials. Carpenter and Gottlieb [1] introduced the arbitrary-grid spectral method using Lagrange interpolation. In this paper, we consider the spectral method using Hermite interpolations instead of Lagrange interpolations. Since our new method uses the function values and its first derivative values, we only need half of the grids comparing with the Lagrange spectral method [1].

This paper is organized as follows. In Section 2, we introduce the differentiation matrix based on the Hermite interpolation and prove the properties of this matrix. In section 3, we consider the Legendre Galerkin method by describing how to apply our Hermite spectral method for the following hyperbolic system of conservation laws

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \tag{1.6}$$

subject to given initial and boundary conditions on bounded domains. In section 4, we investigate the relation of two differentiation matrices defined on different grids. Finally, we give numerical results in section 5.

2. The differentiation matrix using Hermite interpolation

In this subsection, we derive the differentiation matrix using the Hermite interpolation on $N+1$ distinct points

$$-1 = x_N < x_{N-1} < \cdots < x_1 < x_0 = 1$$

where $x_1, x_2, \ldots, x_{N-1}$ are arbitrary. First, we consider the derivative of $\mathcal{H}_N$ $[f](x)$.

$$d\mathcal{H}_N[f](x)/dx = \sum_{j=0}^{N} f(x_j)H_j'(x) + \sum_{j=N+1}^{2N+1} f'(x_{j-(N+1)})H_j'(x). \tag{2.1}$$

Then we have by (1.1) for $k = 0, 1, 2, \ldots, N$

$$\begin{aligned} f'(x_k) = \mathcal{H}_N[f]'(x_k) &= \sum_{j=0}^{2N+1} f_j H_j'(x_k) \\ &= \sum_{j=0}^{N} f(x_j)H_j'(x_k) + \sum_{j=N+1}^{2N+1} f'(x_{j-(N+1)})H_j'(x_k) \end{aligned}$$

and since

$$d^2\mathcal{H}_N[f](x)/dx^2 = \mathcal{H}_N[f]''(x) = \sum_{j=0}^{2N+1} f_j H_j''(x),$$

we have for $k = 0, 1, 2, \ldots, N$

$$\begin{aligned} \mathcal{H}_N[f]''(x_k) &= \sum_{j=0}^{2N+1} f_j H_j''(x_k) \\ &= \sum_{j=0}^{N} f(x_j)H_j''(x_k) + \sum_{j=N+1}^{2N+1} f'(x_{j-(N+1)})H_j''(x_k). \end{aligned} \tag{2.2}$$

Therefore, if we define $\mathbf{f}'$ and $\mathbf{f}$ as

$$\mathbf{f}' = \Big[f'(x_0), \cdots, f'(x_N), \mathcal{H}_N[f]''(x_0), \cdots, \mathcal{H}_N[f]''(x_N)\Big]^T$$

and

$$\mathbf{f} = \Big[f(x_0), \cdots, f(x_N), f'(x_0), \cdots, f'(x_N)\Big]^T$$

then we have the following equation from the equations (2.1) and (2.2):

$$\mathbf{f}' = D_H \mathbf{f} \tag{2.3}$$

where the differentiation matrix D_H is given by

$$D_H = \begin{bmatrix} H_0'(x_0) & \cdots & H_j'(x_0) & \cdots & H_{2N+1}'(x_0) \\ \vdots & & \vdots & & \vdots \\ H_0'(x_N) & \cdots & H_j'(x_N) & \cdots & H_{2N+1}'(x_N) \\ H_0''(x_0) & \cdots & H_j''(x_0) & \cdots & H_{2N+1}''(x_0) \\ \vdots & & \vdots & & \vdots \\ H_0''(x_N) & \cdots & H_j''(x_N) & \cdots & H_{2N+1}''(x_N) \end{bmatrix}. \tag{2.4}$$

On the other hand, there is an alternative representation of the derivative of $\mathcal{H}_N[f](x)$ which is denoted by

$$d\mathcal{H}_N[f](x)/dx = \sum_{j=0}^{N} f'(x_j)H_j(x) + \sum_{j=N+1}^{2N+1} \mathcal{H}_N[f]''(x_{j-(N+1)})H_j(x). \tag{2.5}$$

The equation (2.1) is obtained by the derivative of the equation (1.2) and we can obtain (2.5) using the fact that $d\mathcal{H}_N[f](x)/dx$ is a polynomial of degree $2N$. Here, since (2.1) is identical to (2.5), we can make a statement that (2.1) and (2.5) are also the same in the weak form. That is, the difference between these expressions is orthogonal to all polynomials of degree $\leq 2N+1$ so that we have for $0 \leq l \leq 2N+1$

$$\int_{-1}^{1} \Bigg(\sum_{j=0}^{N} [f(x_j)H_j'(x) - f'(x_j)H_j(x)] \tag{2.6}$$
$$+ \sum_{j=N+1}^{2N+1} [f'(x_{j-(N+1)})H_j'(x) - \mathcal{H}_N[f]''(x_{j-(N+1)})H_j(x)] \Bigg) H_l(x)dx = 0.$$

Here we introduce the matrices $\mathbf{M}(= (m_{l,j}))$ and $\mathbf{S}(= (s_{l,j}))$ with the entries

$$m_{l,j} = \int_{-1}^{1} H_j(x)H_l(x)dx \quad \text{and} \quad s_{l,j} = \int_{-1}^{1} H_j'(x)H_l(x)dx. \tag{2.7}$$

Then we obtain the following equation from the above equation (2.6):

$$\sum_{j=0}^{N} m_{l,j} f'(x_j) + \sum_{j=N+1}^{2N+1} m_{l,j}\mathcal{H}_N[f]''(x_{j-(N+1)}) \tag{2.8}$$
$$= \sum_{j=0}^{N} s_{l,j} f(x_j) + \sum_{j=N+1}^{2N+1} s_{l,j} f'(x_{j-(N+1)}), \quad 0 \leq l \leq 2N+1.$$

This equation (2.8) becomes

$$\mathbf{M}\mathbf{f}' = \mathbf{S}\mathbf{f}. \tag{2.9}$$

From the equation (2.3) and (2.9), with the assumption that $\mathbf{M}$ is invertible

$$D_H = \mathbf{M}^{-1}\mathbf{S} \tag{2.10}$$

must be satisfied. In fact,

$$\begin{aligned}(\mathbf{M}D_H)_{i,l} &= \sum_{j=0}^{2N+1} m_{i,j} d_{j,l} \\ &= \int_{-1}^{1} H_i(x) \left(\sum_{j=0}^{N} H_j(x) H_l'(x_j) + \sum_{j=N+1}^{2N+1} H_j(x) H_l''(x_{j-(N+1)}) \right) dx \\ &= \int_{-1}^{1} H_i(x) H_l'(x) dx \\ &= s_{i,l}.\end{aligned}$$

Here, we get the third equality of the above equation since $H_l'(x)$ is a polynomial of degree $\leq 2N+1$ and using (1.5). Assuming that $\mathbf{M}$ is invertible, this proves (2.10).

In order to approximate the derivative of the function we have defined a new Hermite spectral method using Hermite interpolations. We have rewritten the derivative matrix D_H defined in (2.4) as (2.10) using matrices $\mathbf{M}$ and $\mathbf{S}$. We will investigate the attractive properties of the matrices $\mathbf{M}$ and $\mathbf{S}$ in Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *The matrix* $\mathbf{M}$ *in (2.7) is a symmetric positive-matrix.*

Proof. It is easy to see that from the definition of $m_{l,j}$ in (2.7) the matrix $\mathbf{M}$ is symmetric. Now we prove that $\mathbf{M}$ is a positive definite. To show that the matrix $\mathbf{M}$ is a positive definite, we first introduce a vector $\mathbf{V}$ which is a $2N+2$ component vector:

$$\mathbf{V} = (v_0, \cdots, v_{2N+1}).$$

Then we have

$$\begin{aligned}\mathbf{V}^T\mathbf{M}\mathbf{V} &= \sum_{i=0}^{2N+1}\sum_{j=0}^{2N+1} m_{i,j} v_i v_j \\ &= \int_{-1}^{1} \sum_{i=0}^{2N+1}\sum_{j=0}^{2N+1} v_i H_i(x) v_j H_j(x) dx \\ &= \int_{-1}^{1} \left(\sum_{i=0}^{2N+1} v_i H_i(x) \right)^2 dx \geq 0.\end{aligned} \tag{2.11}$$

The equality sign holds only if $\mathbf{V}$ is the null vector. □

We can state the equation (2.11) is a vector space norm. To see the relationship between a vector space norm and a function space norm we consider a polynomial $v(x)$ that takes the values of its components at the grid points x_j. We define a polynomial $v(x)$ of degree $2N+1$ satisfying :

$$\begin{cases} v(x_j) = v_j, & 0 \leq j \leq N \\ v'(x_j) = v_{j+N+1}, & 0 \leq j \leq N \end{cases} \tag{2.12}$$

so that $v(x) = \sum_{j=0}^{2N+1} v_j H_j(x)$. Then

$$\begin{aligned} \mathbf{V}^T\mathbf{M}\mathbf{V} &= \int_{-1}^{1} \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} v_i H_i(x) v_j H_j(x) dx \qquad (2.13) \\ &= \int_{-1}^{1} v^2(x) dx \geq 0. \end{aligned}$$

Thus, the vector space norm $\mathbf{V}^T\mathbf{M}\mathbf{V}$ is equivalent to the function space norm $\int_{-1}^{1} v^2(x)dx$.

Lemma 2.2. *Let* $\mathbf{S}$ *be the defined in (2.7) and let* $\mathbf{V}$ *be defined as before. Then,*

$$\mathbf{V}^T\mathbf{S}\mathbf{V} = \frac{1}{2}(v_0^2 - v_N^2). \tag{2.14}$$

Proof. We first show that $\mathbf{S}$ is almost antisymmetric. We defined the matrix $s_{l,j}$ in (2.7). Using integration by parts, we obtain the following:

$$\begin{aligned} s_{l,j} &= \int_{-1}^{1} H_j'(x) H_l(x) dx \\ &= H_j(1)H_l(1) - H_j(-1)H_l(-1) - s_{j,l}. \end{aligned}$$

By the definition of the Hermite fundamental polynomials $H_j(x)$, since $H_j(1) = \delta_{j,0}$ and $H_j(-1) = \delta_{j,N}$,

$$s_{l,j} + s_{j,l} = H_j(1)H_l(1) - H_j(-1)H_l(-1) = \delta_{j,0}\delta_{l,0} - \delta_{j,N}\delta_{l,N}.$$

Thus,

$$\begin{aligned} \mathbf{V}^T\mathbf{S}\mathbf{V} &= \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} s_{i,j} v_i v_j = \frac{1}{2} \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} (s_{i,j} + s_{j,i}) v_i v_j \\ &= \frac{1}{2} \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} (\delta_{i,0}\delta_{j,0} - \delta_{i,N}\delta_{j,N}) v_i v_j = \frac{1}{2}(v_0^2 - v_N^2) \end{aligned}$$

This completes the proof of Lemma 2.2. □

As before, let $v(x)$ be the polynomial of degree $2N+1$ such that (2.12) is satisfied. Then,

$$\begin{aligned} \mathbf{V}^T\mathbf{S}\mathbf{V} &= \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} s_{i,j} v_i v_j = \int_{-1}^{1} \sum_{i=0}^{2N+1} \sum_{j=0}^{2N+1} v_i H_i(x) v_j H_j'(x) dx \\ &= \int_{-1}^{1} v(x) v'(x) dx = \frac{1}{2}(v^2(1) - v^2(-1)) = \frac{1}{2}(v_0^2 - v_N^2). \end{aligned}$$

3. The Legendre Galerkin method on arbitrary grids

In this section we describe how to apply our new Hermite spectral methods to the partial differential equation :

$$\begin{aligned} U_t(x,t) &= U_x(x,t), \quad -1 \le x \le 1 \\ U(x,0) &= f(x) \\ U(1,t) &= g(t). \end{aligned}$$

To use a new method, we consider another equation obtained from the spatial derivative of the above equation:

$$\begin{cases} U_t(x,t) = U_x(x,t), & U_{xt}(x,t) = U_{xx}(x,t) \quad -1 \le x \le 1 \\ U(x,0) = f(x), & U_x(x,0) = f'(x) \\ U(1,t) = g(t), & U_x(1,t) = g'(t). \end{cases} \tag{3.15}$$

Here, we use the fact $U_t(x,t) = U_x(x,t)$ to obtain $U_x(1,t) = g'(t)$. Now we consider the Hermite spectral method to apply the equation (3.15). We find a vector

$$\begin{aligned} \mathbf{u} &= \Big[u(x_0,t),\cdots,u(x_N,t),u'(x_0,t),\cdots,u'(x_N,t)\Big]^T \\ &=: \Big[u_0(t),\cdots,u_N(t),u_{x,0}(t),\cdots,u_{x,N}(t)\Big]^T \end{aligned}$$

that satisfies

$$\mathbf{M}\frac{d\mathbf{u}}{dt} = \mathbf{S}\mathbf{u} - \tau_1\mathbf{e}_0[u_0 - g(t)] - \tau_2\mathbf{e}_{N+1}[u_{x,0} - g'(t)] \tag{3.16}$$

where $\mathbf{e}_i$ is a unit vector which the $i+1$-th component is 1. The last two terms in equation (3.16) accounts for a weak imposition of the boundary condition. For this technique, see [2]. In the following, we estimate the appropriate τ_1 and τ_2 for stability of the numerical schemes.

Theorem 3.1. *The method described in (3.16) is stable for* $\tau_1 \ge 1/2$ *and* $\tau_2 \ge 0$.

Proof. We knew the matrix $\mathbf{M}$ is a symmetric positive-matrix. Multiplying $\mathbf{u}^T$ to (3.16) and using (2.14) we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mathbf{u}^T\mathbf{M}\mathbf{u} &= \frac{1}{2}(u_0^2 - u_N^2) - \tau_1\mathbf{u}^T\mathbf{e}_0[u_0 - g(t)] \\ &\quad -\tau_2\mathbf{u}^T\mathbf{e}_{N+1}[u_{x,0} - g'(t)]. \end{aligned} \tag{3.17}$$

For stability, we only consider the case $g(t) = g'(t) = 0$ because the error satisfies the homogeneous boundary condition. We can clearly determine in this case that if $\tau_1 \ge 1/2$ and $\tau_2 \ge 0$, then

$$\frac{1}{2}\frac{d}{dt}\mathbf{u}^T\mathbf{M}\mathbf{u} = \left(\frac{1}{2} - \tau_1\right)u_0^2 - \frac{1}{2}u_N^2 - \tau_2 u_{x,0}^2 \le 0 \tag{3.18}$$

and stability exists in the norm induced by the positive definite matrix $\mathbf{M}$. □

Considering the polynomial $u_N(x,t) \in \mathcal{P}_{2N+1}$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u_N(x,t)^2 dx &= \frac{1}{2}\frac{d}{dt}\mathbf{u}^T\mathbf{M}\mathbf{u} \\ &= \left(\frac{1}{2}-\tau_1\right)u_0^2 - \frac{1}{2}u_N^2 - \tau_2 u_{x,0}^2 \\ &= \left(\frac{1}{2}-\tau_1\right)u(1,t)^2 - \frac{1}{2}u(-1,t)^2 - \tau_2 u_x(1,t)^2. \end{aligned}$$

Thus, for the polynomial $u_N(x,t) \in \mathcal{P}_{2N+1}$ we have stability in the usual H_2 norm, provided $\tau_1 \geq 1/2$ and $\tau_2 \geq 0$.

Now, by multiplying equation (3.16) by $\mathbf{M}^{-1}$ and using equation (2.10) we obtain

$$\frac{d\mathbf{u}}{dt} = \mathbf{D}_H\mathbf{u} - \tau_1\mathbf{M}^{-1}\mathbf{e}_0[u_0 - g(t)] - \tau_2\mathbf{M}^{-1}\mathbf{e}_{N+1}[u_{x,0} - g'(t)]. \tag{3.19}$$

Theorem 3.2. *Let* $\mathbf{M}$ *be the mass defined in (2.7). Define the residual vector* $\mathbf{r}$ *by*

$$\mathbf{r} := \tau_1\mathbf{M}^{-1}\mathbf{e}_0[u_0 - g(t)] + \tau_2\mathbf{M}^{-1}\mathbf{e}_{N+1}[u_{x,0} - g'(t)].$$

Then

$$\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b} \tag{3.20}$$

where α, β, $\mathbf{a}$, *and* $\mathbf{b}$ *are defined as follows.*

$$\begin{aligned} \alpha &:= \tau_1[u_0 - g(t)] + c_N\tau_2[u_{x,0} - g'(t)], \\ \beta &:= -\tau_2[u_{x,0} - g'(t)], \end{aligned}$$

and $\mathbf{a} = (a_i)_{i=0}^{2N+1}$ *and* $\mathbf{b} = (b_i)_{i=0}^{2N+1}$ *with*

$$a_i = \begin{cases} R_a(x_i), & 0 \leq i \leq N \\ R_a'(x_{i-(N+1)}), & N+1 \leq i \leq 2N+1 \end{cases} \tag{3.21}$$

and

$$b_i = \begin{cases} R_b(x_i), & 0 \leq i \leq N \\ R_b'(x_{i-(N+1)}), & N+1 \leq i \leq 2N+1. \end{cases} \tag{3.22}$$

Here, $P_N(x)$ *is the Legendre polynomial of degree* N *and* $P_N(1) = 1$, *and*

$$\begin{aligned} c_N &:= 2N^2 + 4N + \frac{9}{4}, \\ R_a(x) &:= \frac{P_{2N+2}'(x) + P_{2N+1}'(x)}{2}, \\ R_b(x) &:= \frac{P_{2N+3}''(x) + 3P_{2N+2}''(x) + 3P_{2N+1}''(x) + P_{2N}''(x)}{8}. \end{aligned}$$

Proof. To prove (3.20), we will show that

$$\alpha\mathbf{M}\mathbf{a} + \beta\mathbf{M}\mathbf{b} = \tau_1\mathbf{e}_0[u_0 - g(t)] + \tau_2\mathbf{e}_{N+1}[u_{x,0} - g'(t)].$$

We first consider some properties. $\tilde{R}_a(x)$, $\tilde{R}_b(x)$ and $\tilde{\tilde{R}}_b(x)$ are primitive functions of $R_a(x)$, $R_b(x)$ and $\tilde{R}_b(x)$ such that they are defined as follows, respectively.

$$\begin{aligned}
\tilde{R}_a(x) &= \frac{P_{2N+2}(x) + P_{2N+1}(x)}{2}, \\
\tilde{R}_b(x) &= \frac{P'_{2N+3}(x) + 3P'_{2N+2}(x) + 3P'_{2N+1}(x) + P'_{2N}(x)}{8}, \\
\tilde{\tilde{R}}_b(x) &= \frac{P_{2N+3}(x) + 3P_{2N+2}(x) + 3P_{2N+1}(x) + P_{2N}(x)}{8}.
\end{aligned}$$

Then $\tilde{R}_a$ and $\tilde{\tilde{R}}_b$ are orthogonal to $\mathcal{P}_{2N}$ and $\mathcal{P}_{2N-1}$, respectively. Since

$$P_N(1) = 1, \quad P_N(-1) = (-1)^N P_N(1)$$

and

$$P'_N(1) = \frac{(N+1)N}{2}, \quad P'_N(-1) = (-1)^{N-1} P'_N(1)$$

(see [7]), we have that

$$\tilde{R}_a(1) = \tilde{\tilde{R}}_b(1) = 1, \quad \tilde{R}_a(-1) = \tilde{\tilde{R}}_b(-1) = 0 \tag{3.23}$$

and

$$\tilde{R}_b(1) = c_N, \quad \tilde{R}_b(-1) = 0. \tag{3.24}$$

From these properties we have for $0 \le l \le 2N+1$

$$\begin{aligned}
(\mathbf{Ma})_l &= \sum_{j=0}^{2N+1} m_{lj} a_j = \int_{-1}^{1} H_l(x) \sum_{j=0}^{2N+1} H_j(x) a_j dx \\
&= \int_{-1}^{1} H_l(x) R_a(x) dx \quad \text{by (3.21) and } R_a(x) \in \mathcal{P}_{2N+1} \\
&= H_l(x)\tilde{R}_a(x)\Big|_{-1}^{1} - \int_{-1}^{1} H'_l(x)\tilde{R}_a(x) dx \\
&= H_l(1)\tilde{R}_a(1) - H_l(-1)\tilde{R}_a(-1) = H_l(1) = \delta_{l,0}.
\end{aligned}$$

This induces that

$$\mathbf{Ma} = \mathbf{e}_0. \tag{3.25}$$

Similarly, we also have for $0 \le l \le 2N+1$,

$$\begin{aligned}
(\mathbf{Mb})_l &= \int_{-1}^{1} H_l(x) R_b(x) dx \quad \text{by (3.22) and } R_b(x) \in \mathcal{P}_{2N+1} \\
&= H_l(x)\tilde{R}_b(x)\Big|_{-1}^{1} - H'_l(x)\tilde{\tilde{R}}_b(x)\Big|_{-1}^{1} + \int_{-1}^{1} H''_l(x)\tilde{\tilde{R}}_b(x) dx \\
&= \delta_{l,0} \cdot c_N - \delta_{l,N} \cdot 0 - (\delta_{l,N+1} \cdot 1 - \delta_{l,2N+1} \cdot 0) \\
&= c_N \delta_{l,0} - \delta_{l,N+1}.
\end{aligned}$$

This means that

$$\mathbf{Mb} = c_N \mathbf{e}_0 - \mathbf{e}_{N+1}. \tag{3.26}$$

Thus, we have from (3.25), and (3.26) that

$$\begin{aligned} & \alpha\mathbf{Ma} + \beta\mathbf{Mb} \\ = \; & \alpha\mathbf{e}_0 + \beta(c_N\mathbf{e}_0 - \mathbf{e}_{N+1}) \\ = \; & (\tau_1[u_0 - g(t)] + c_N\tau_2[u_{x,0} - g'(t)])\mathbf{e}_0 - \tau_2[u_{x,0} - g'(t)](c_N\mathbf{e}_0 - \mathbf{e}_{N+1}) \\ = \; & \tau_1[u_0 - g(t)]\mathbf{e}_0 + \tau_2[u_{x,0} - g'(t)]\mathbf{e}_{N+1}. \end{aligned}$$

This completes the proof. □

Theorem 3.2 sheds light on the connection between Legendre Galerkin method and the method defined in (3.16) that uses the arbitrary grid points x_j.

Theorem 3.3. *The method defined in (3.16) is equivalent to the Legendre Galerkin method.*

Proof. We can derive the error equation using polynomial $u_N(x,t)$ of degree $2N+1$:

$$\begin{aligned} & \frac{u_N(x,t)}{\partial t} - \frac{u_N(x,t)}{\partial x} \\ & = -\alpha\frac{P'_{2N+2}(x) + P'_{2N+1}(x)}{2} - \beta\frac{P''_{2N+3}(x) + 3P''_{2N+2}(x) + P''_{2N+1}(x) + P''_{2N}(x)}{8} \\ & = -\alpha R_a(x) - \beta R_b(x) \in \mathcal{P}_{2N+1}. \end{aligned}$$

For any polynomial $\pi_{2N+1} \in \mathcal{P}_{2N+1}$ with $\pi_{2N+1}(1) = \pi'_{2N+1}(1) = 0$ by (3.23) and (3.24)

$$\begin{aligned} (R_a, \pi_{2N+1}) &= \int_{-1}^{1} R_a(x)\pi_{2N+1}(x)dx \\ &= \tilde{R}_a(x)\pi_{2N+1}(x)\Big|_{-1}^{1} - \int_{-1}^{1} \tilde{R}_a(x)\pi'_{2N+1}(x)dx \\ &= \tilde{R}_a(1)\pi_{2N+1}(1) - \tilde{R}_a(-1)\pi_{2N+1}(-1) = 0 \end{aligned}$$

and

$$\begin{aligned} (R_b, \pi_{2N+1}) &= \int_{-1}^{1} R_b(x)\pi_{2N+1}(x)dx \\ &= \tilde{R}_b(x)\pi_{2N+1}(x)\Big|_{-1}^{1} - \tilde{\tilde{R}}_b(x)\pi'_{2N+1}(x)\Big|_{-1}^{1} + \int_{-1}^{1} \tilde{\tilde{R}}_b\pi''_{2N+1}(x)dx \\ &= \tilde{R}_b(1)\pi_{2N+1}(1) - \tilde{R}_b(-1)\pi_{2N+1}(-1) \\ & \quad -(\tilde{\tilde{R}}_b(1)\pi'_{2N+1}(1) - \tilde{\tilde{R}}_b(-1)\pi'_{2N+1}(-1)) \\ &= 0. \end{aligned}$$

This means that for any polynomial $\pi_{2N+1} \in \mathcal{P}_{2N+1}$ with $\pi_{2N+1}(1) = \pi'_{2N+1}(1) = 0$,

$$\left(\frac{u_N(x,t)}{\partial t} - \frac{u_N(x,t)}{\partial x}, \pi_{2N+1}\right) = 0.$$

Thus, the method defined in (3.16) satisfies the definition of the Legendre Galerkin method. □

4. Two Hermite differentiation operators

In this section, we will show that two differentiation operators defined on different grids are similar and the modified differentiation matrices produced by (3.19) are similar.

In order for showing the relationship between differentiation matrices based on different grid-point distributions we consider two grids x_j and y_j ($j = 0, 1, \cdots, N$). We define $H_j^x(x)$ and $H_j^y(x)$ using Hermite interpolation polynomials as in (1.3) and (1.4) of degree $2N+1$ based on the set of points x_j and y_j, respectively. So we obtain two differentiation matrices D_H^x and D_H^y:

$$D_H^x = \begin{bmatrix} H_0^{x\prime}(x_0) & \cdots & H_j^{x\prime}(x_0) & \cdots & H_{2N+1}^{x}{}'(x_0) \\ \vdots & & \vdots & & \vdots \\ H_0^{x\prime}(x_N) & \cdots & H_j^{x\prime}(x_N) & \cdots & H_{2N+1}^{x}{}'(x_N) \\ H_0^{x\prime\prime}(x_0) & \cdots & H_j^{x\prime\prime}(x_0) & \cdots & H_{2N+1}^{x}{}''(x_0) \\ \vdots & & \vdots & & \vdots \\ H_0^{x\prime\prime}(x_N) & \cdots & H_j^{x\prime\prime}(x_N) & \cdots & H_{2N+1}^{x}{}''(x_N) \end{bmatrix}$$

and

$$D_H^y = \begin{bmatrix} H_0^{y\prime}(y_0) & \cdots & H_j^{y\prime}(y_0) & \cdots & H_{2N+1}^{y}{}'(y_0) \\ \vdots & & \vdots & & \vdots \\ H_0^{y\prime}(y_N) & \cdots & H_j^{y\prime}(y_N) & \cdots & H_{2N+1}^{y}{}'(y_N) \\ H_0^{y\prime\prime}(y_0) & \cdots & H_j^{y\prime\prime}(y_0) & \cdots & H_{2N+1}^{y}{}''(y_0) \\ \vdots & & \vdots & & \vdots \\ H_0^{y\prime\prime}(y_N) & \cdots & H_j^{y\prime\prime}(y_N) & \cdots & H_{2N+1}^{y}{}''(y_N) \end{bmatrix}$$

The following theorem shows that the two matrices are similar.

Theorem 4.1. *Define the matrix T_H by*

$$T_H = \begin{bmatrix} H_0^{x\prime}(y_0) & \cdots & H_j^{x\prime}(y_0) & \cdots & H_{2N+1}^{x}{}'(y_0) \\ \vdots & & \vdots & & \vdots \\ H_0^{x\prime}(y_N) & \cdots & H_j^{x\prime}(y_N) & \cdots & H_{2N+1}^{x}{}'(y_N) \\ H_0^{x\prime\prime}(y_0) & \cdots & H_j^{x\prime\prime}(y_0) & \cdots & H_{2N+1}^{x}{}''(y_0) \\ \vdots & & \vdots & & \vdots \\ H_0^{x\prime\prime}(y_N) & \cdots & H_j^{x\prime\prime}(y_N) & \cdots & H_{2N+1}^{x}{}''(y_N) \end{bmatrix} \tag{4.27}$$

Then

$$T_H^{-1} = \begin{bmatrix} H_0^{y\prime}(x_0) & \cdots & H_j^{y\prime}(x_0) & \cdots & {H_{2N+1}^{y}}'(x_0) \\ \vdots & & \vdots & & \vdots \\ H_0^{y\prime}(x_N) & \cdots & H_j^{y\prime}(x_N) & \cdots & {H_{2N+1}^{y}}'(x_N) \\ H_0^{y\prime\prime}(x_0) & \cdots & H_j^{y\prime\prime}(x_0) & \cdots & {H_{2N+1}^{y}}''(x_0) \\ \vdots & & \vdots & & \vdots \\ H_0^{y\prime\prime}(x_N) & \cdots & H_j^{y\prime\prime}(x_N) & \cdots & {H_{2N+1}^{y}}''(x_N) \end{bmatrix} \tag{4.28}$$

and

$$D_H^y = T_H D_H^x T_H^{-1}. \tag{4.29}$$

Proof. From (1.4) and (1.5) we have for $0 \le i \le N$ and $0 \le j \le 2N+1$

$$\sum_{l=0}^{N} H_l^x(y_i) H_j^y(x_l) + \sum_{l=N+1}^{2N+1} H_l^x(y_i) H_j^{y\prime}(x_{l-(N+1)}) = H_j^y(y_i) = \delta_{i,j}$$

and for $N+1 \le i \le 2N+1$ and $0 \le j \le 2N+1$

$$\begin{aligned} & \sum_{l=0}^{N} H_l^{x\prime}(y_{i-(N+1)}) H_j^y(x_l) + \sum_{l=N+1}^{2N+1} H_l^{x\prime}(y_{i-(N+1)}) H_j^{y\prime}(x_{l-(N+1)}) \\ = \;& H_j^{y\prime}(y_{i-(N+1)}) = \delta_{i,j}. \end{aligned}$$

This means that the product T_H and the right hand of (4.28) is the identity matrix I and by the same reason, we also have the product the right hand of (4.28) and T_H is I. This implies (4.28). Now, we will prove (4.29). The (i,j) element of $T_H D_H^x T_H^{-1}$ is

$$\sum_{l=0}^{2N+1} (T_H D_H^x)_{i,l} (T_H^{-1})_{l,j}.$$

From (1.5) we have for $0 \le i \le N$

$$\begin{aligned} (T_H D_H^x)_{i,l} &= \sum_{k=0}^{N} H_k^x(y_i) H_l^{x\prime}(x_k) + \sum_{k=N+1}^{2N+1} H_k^x(y_i) H_l^{x\prime}(x_{k-(N+1)}) \\ &= H_l^{x\prime}(y_i) \end{aligned}$$

and for $N+1 \le i \le 2N+1$

$$\begin{aligned} (T_H D_H^x)_{i,l} &= \sum_{k=0}^{N} H_k^{x\prime}(y_{i-(N+1)}) H_l^{x\prime}(x_k) + \sum_{k=N+1}^{2N+1} H_k^{x\prime}((y_{i-(N+1)})) H_l^{x\prime}(x_{k-(N+1)}) \\ &= H_l^{x\prime\prime}(y_{i-(N+1)}) \end{aligned}$$

so that the (i,j) element of $T_H D_H^x T_H^{-1}$ is for $0 \le i \le N$,

$$\begin{aligned}\sum_{l=0}^{2N+1} H_l^{x\prime}(y_i)(T_H^{-1})_{l,j} &= \sum_{l=0}^{N} H_l^{x\prime}(y_i) H_j^y(x_l) + \sum_{l=N}^{2N+1} H_l^{x\prime}(y_i) H_j^{y\prime}(x_{l-(N+1)}) \\ &= H_j^{y\prime}(y_i) = (i,j) \text{ element of } D_H^y\end{aligned}$$

and for $N \le i \le 2N+1$,

$$\begin{aligned}&\sum_{l=0}^{2N+1} H_l^{x\prime\prime}(y_{i-(N+1)})(T_H^{-1})_{l,j} \\ =\ &\sum_{l=0}^{N} H_l^{x\prime\prime}(y_{i-(N+1)}) H_j^y(x_l) + \sum_{l=N}^{2N+1} H_l^{x\prime\prime}(y_{i-(N+1)}) H_j^{y\prime}(x_{l-(N+1)}) \\ =\ &H_j^{y\prime\prime}(y_{i-(N+1)}) = (i,j) \text{ element of } D_H^y.\end{aligned}$$

Thus, (4.29) is proved. □

Let us consider a modified differentiation matrix. The differentiation matrix produced in equations (3.19) and (3.20) takes account of the boundary conditions:

$$D_H - \tau_1 A_1 - c_N \tau_2 A_2 + \tau_2 B$$

where the boundary matrices A_1, A_2 and B are defined as

$$(A_1)_{i,j} = a_i \delta_{j,0}, \ (A_2)_{i,j} = a_i \delta_{j,N+1} \quad \text{and} \quad (B)_{i,j} = b_i \delta_{j,N+1}.$$

Suppose now that we have two different grid-point distributions x_j and y_j. We have proved in Theorem 4.1 that D_H^x and D_H^y are similar,

$$D_H^y = T_H D_H^x T_H^{-1}$$

where the matrices T_H and T_H^{-1} are defined in (4.27) and (4.28). We will show that the same similarity transformation exists for the modified Hermite differentiation matrices. That is,

$$D_H^y - \tau_1 A_{1y} - c_N \tau_2 A_{2y} + \tau_2 B_y = T_H (D_H^x - \tau_1 A_{1x} - c_N \tau_2 A_{2x} + \tau_2 B_x) T_H^{-1}$$

or

$$A_{1y} = T_H A_{1x} T_H^{-1}, \ A_{2y} = T_H A_{2x} T_H^{-1} \quad \text{and} \quad B_y = T_H B_x T_H^{-1}. \tag{4.30}$$

Here, when a_i^x and a_i^y are defined in (3.21) based on the set of points x_j and y_j, respectively, $(A_{1x})_{ij} := a_i^x \delta_{j,0}$ and $(A_{1y})_{ij} := a_i^y \delta_{j,0}$. The others are defined similarly.

Consider element (i,j) of $T_H A_{1x} T_H^{-1}$. Then since $H_j^y(x_0) = \delta_{j,0}$ by (1.4), we have

$$\begin{aligned}
(T_H A_{1x} T_H^{-1})_{i,j} &= \sum_{l=0}^{2N+1} \sum_{m=0}^{2N+1} (T_H)_{i,m} (A_{1x})_{m,l} (T_H^{-1})_{l,j} \\
&= \sum_{l=0}^{2N+1} \sum_{m=0}^{2N+1} a_m^x \delta_{l,0} (T_H)_{i,m} (T_H^{-1})_{l,j} \\
&= (T_H^{-1})_{0,j} \sum_{m=0}^{2N+1} a_m^x (T_H)_{i,m} \\
&= H_j^y(x_0) \sum_{m=0}^{2N+1} a_m^x \begin{cases} H_m^x(y_i), & 0 \le i \le N \\ H_m^{x\prime}(y_{i-(N+1)}), & N+1 \le i \le 2N+1 \end{cases} \\
&= \delta_{j,0} \begin{cases} R_a(y_i), & 0 \le i \le N \\ R_a'(y_{i-(N+1)}), & N+1 \le i \le 2N+1 \end{cases} = (A_{1y})_{i,j}
\end{aligned}$$

which proves the first term of (4.30). Similarly,

$$\begin{aligned}
(T_H A_{2x} T_H^{-1})_{i,j} &= (T_H^{-1})_{N+1,j} \sum_{m=0}^{2N+1} a_m^x (T_H)_{i,m} \\
&= H_j^{y\prime}(x_0) \sum_{m=0}^{2N+1} a_m^x \begin{cases} H_m^x(y_i), & 0 \le i \le N \\ H_m^{x\prime}(y_{i-(N+1)}), & N+1 \le i \le 2N+1 \end{cases} \\
&= \delta_{j,N+1} \begin{cases} R_a(y_i), & 0 \le i \le N \\ R_a'(y_{i-(N+1)}), & N+1 \le i \le 2N+1 \end{cases} = (A_{2y})_{i,j}
\end{aligned}$$

and

$$\begin{aligned}
(T_H B_x T_H^{-1})_{i,j} &= (T_H^{-1})_{N+1,j} \sum_{m=0}^{2N+1} b_m^x (T_H)_{i,m} \\
&= H_j^{y\prime}(x_0) \sum_{m=0}^{2N+1} b_m^x \begin{cases} H_m^x(y_i), & 0 \le i \le N \\ H_m^{x\prime}(y_{i-(N+1)}), & N+1 \le i \le 2N+1 \end{cases} \\
&= \delta_{j,N+1} \begin{cases} R_b(y_i), & 0 \le i \le N \\ R_b'(y_{i-(N+1)}), & N+1 \le i \le 2N+1 \end{cases} = (B_y)_{i,j}
\end{aligned}$$

which prove the rest of (4.30). Therefore, we can say that the similarity transformation is valid even for the modified Hermite derivative matrix.

5. Numerical results

In this section, we compare the spectral method using the Hermite interpolations(HSM) with the spectral method using the Lagrange interpolation(LSM)

	t=0			t=2		
mesh	L_1	L_2	L_∞	L_1	L_2	L_∞
5	3.149E-5	6.034E-6	1.634E-6	7.617E-5	8.657E-6	1.472E-6
7	6.766E-9	1.581E-9	5.166E-10	3.784E-9	4.678E-10	1.723E-10
10	5.700E-15	1.648E-15	6.578E-16	5.511E-15	6.902E-16	3.339E-16

TABLE 1. Hermite spectral method on the uniform grids

	t=0			t=2		
mesh	L_1	L_2	L_∞	L_1	L_2	L_∞
11	2.471E-5	6.099E-6	2.167E-6	3.337E-5	5.536E-6	3.874E-6
15	6.984E-9	2.124E-9	9.051E-10	2.970E-9	5.059E-10	2.515E-10
21	7.669E-15	2.951E-15	1.557E-15	2.924E-15	7.439E-16	3.863E-16

TABLE 2. Lagrange spectral method on the uniform grids

	t=0			t=2		
mesh	L_1	L_2	L_∞	L_1	L_2	L_∞
5	1.807E-5	2.574E-6	5.834E-7	1.244E-5	1.449E-6	2.568E-7
7	2.062E-9	2.854E-10	5.999E-11	1.546E-9	1.884E-10	7.044E-11
10	4.563E-16	6.110E-17	1.308E-17	3.296E-16	4.231E-17	2.120E-17

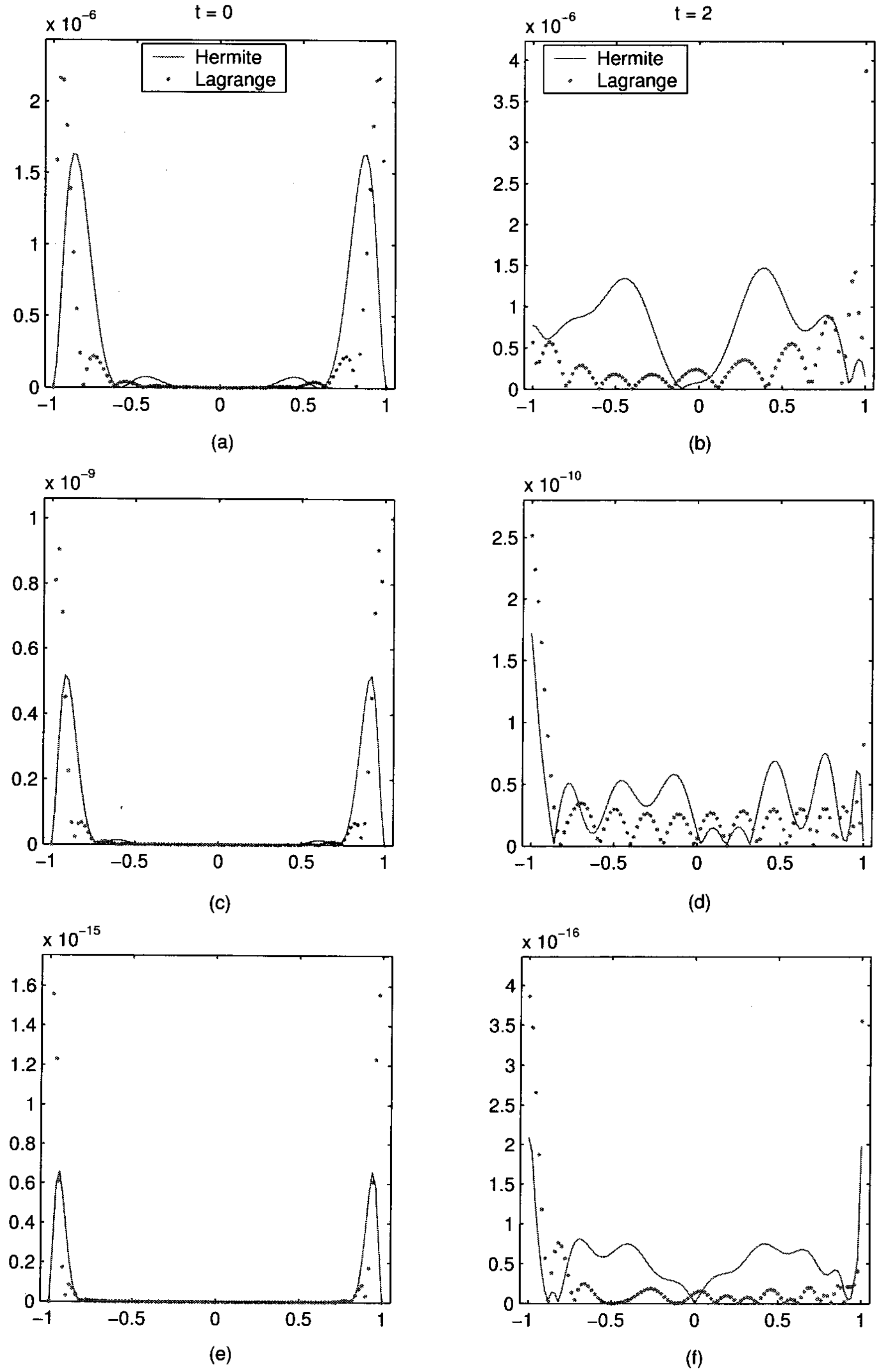
TABLE 3. Hermite spectral method on the Chebyshev GL grids

	t=0			t=2		
mesh	L_1	L_2	L_∞	L_1	L_2	L_∞
11	8.035E-6	9.878E-7	1.872E-7	1.258E-5	1.472E-6	2.689E-7
15	8.950E-10	1.093E-10	2.097E-11	1.532E-9	1.902E-10	8.100E-11
21	1.910E-16	2.338E-17	4.515E-18	8.054E-16	2.038E-16	1.891E-16

TABLE 4. Lagrange spectral method on the Chebyshev GL grids

at the Chebyshev GL grid points and the uniform grid points. More precisely, the linear partial differential equations (3.15) are solved with initial condition, $f(x) = \sin(\pi x)$, boundary condition, $g(t) = \sin(\pi(1+t))$, and the exact solution $u(x) = \sin(\pi(x+t))$. For the ODE solver fourth-order Runge-Kutta method is used with fixed time step. Here, we present the L_1,L_2, and L_∞ absolute errors at $t = 0$ and $t = 2$ for HSM(5,7,and 10 points) and LSM(11,15, and 21 points, resp.) at the uniform grids (Table 1 and 2) and the Chebyshev GL grids (Table 3 and 4).

Figure 1. The error function on the uniform grids : for (a),(b) ; using 5 points with Hermite Spectral Method (HSM) and 11 points Lagrange Spectral Method (LSM) at $t = 0$ and $t = 2$ resp., for (c),(d) ; 7(HSM),15(LSM) and for (e),(f) ; 10(HSM),21(LSM) at $t = 0$ ((a),(c),(e)) and at $t = 2$ ((b),(d),(f))


t = 0
t = 2
Hermite
Lagrange
x 10^-6
x 10^-9
x 10^-10
x 10^-15
x 10^-16
(a)
(b)
(c)
(d)
(e)
(f)

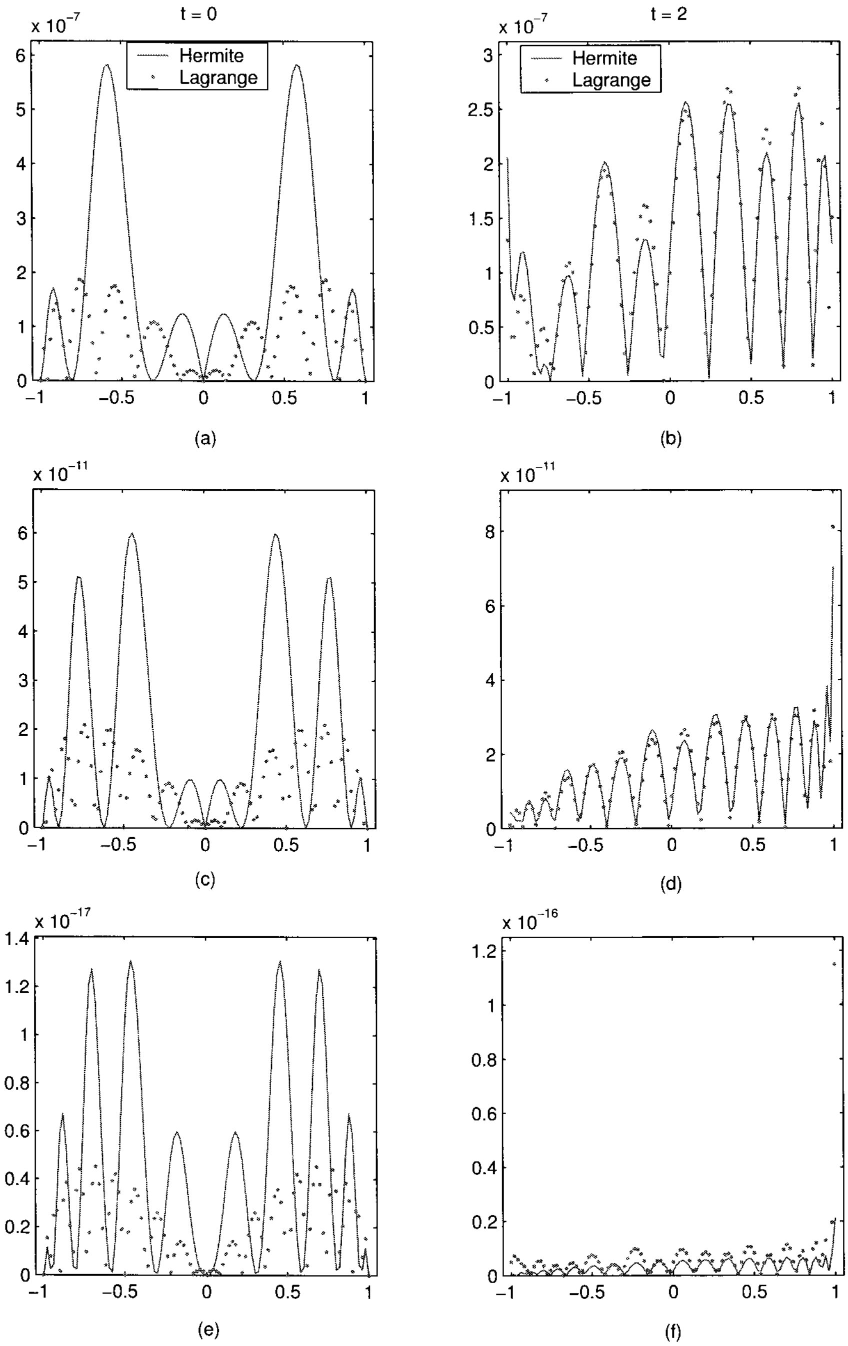

t = 0
x 10^-7
Hermite
Lagrange
(a)
t = 2
x 10^-7
Hermite
Lagrange
(b)
x 10^-11
(c)
x 10^-11
(d)
x 10^-17
(e)
x 10^-16
(f)

Figure 2. The error function on the Chebyshev GL grids : (a),(b); using 5 points with Hermite Spectral Method (HSM) and 11 points Lagrange Spectral Method (LSM) at $t = 0$ and $t = 2$ resp., for (c),(d) ; 7(HSM),15(LSM) and for (e),(f) ; 10(HSM),21(LSM) at $t = 0$ ((a),(c),(e)) and at $t = 2$ ((b),(d),(f))

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