J. Appl. Math. & Informatics Vol. 27(2009), No. 3 - 4, pp. 955 - 961
Website: http://www.kcam.biz

CHRACTERIZATIONS OF THE PARETO DISTRIBUTION BY RECORD VALUES

SE-KYUNG CHANG

ABSTRACT. In this paper, we establish some characterizations which is satisfied by the independence of the upper record values from the Pareto distribution. We prove that $X \in PAR(1,\beta)$, $\beta > 0$, if and only if $\frac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$, $1 \leq m < n$ are independent. We show that $X \in PAR(1,\beta)$, $\beta > 0$, if and only if $\frac{X_{U(n)}+X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$, $n \geq 1$ are independent. And we characterize that $X \in PAR(1,\beta)$, $\beta > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n)}+X_{U(n+1)}}$ and $X_{U(n)}$, $n \geq 1$ are independent.

AMS Mathematics Subject Classification : 62E10, 60E10, 62H10
Key words and phrases : Pareto distribution, record value, characterization, independence.

1. Introduction

The record value model was introduced by Chandler [4]. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed (i.i.d.) random variables with a cumulative distribution function (cdf) $F(x)$ and a probability density function (pdf) $f(x)$. Suppose $Y_n = \max\left\{X_1, X_2, \cdots, X_n\right\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence, if $Y_j > Y_{j-1}$ for $j > 1$. And we suppose that X_1 is a first upper record value. The indices at which the upper record values occur are given by the record times $\left\{U(n),\ n \geq 1\right\}$, where $U(n) = \min\left\{j | j > U(n-1)\,,\, X_j > X_{U(n-1)}\,,\, n \geq 2\right\}$ with $U(1) = 1$.

Received November 27, 2008. Accepted February 26, 2009.

A continuous random variable X is said to have the Pareto distribution with two parameters $\alpha > 0$ and $\beta > 0$ if it has a cdf $F(x)$ of the form

$$F(x) = \begin{cases} 1 - \left(\dfrac{x}{\alpha}\right)^{-\beta}, & x > 1,\ \alpha > 0,\ \beta > 0 \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

A notation that designates that X has the cdf (1) is $X \in PAR(\alpha, \beta)$.

Some characterizations by the independence of the upper record values are known. In [2] and [3], Ahsanullah studied, if $X \in WEI(\theta, \alpha)$, $\theta > 0$, $\alpha > 0$, then $\dfrac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$, $0 < m < n$ are independent. And Ahsanullah proved, if $X \in PAR(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$, then $\dfrac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$, $0 < m < n$ are independent. Above results are characterized by the necessary condition. We can get characterizations of the the necessary and sufficient condition from the Pareto distribution.

In this paper, we will give characterizations of the Pareto distribution with the parameter $\alpha = 1$ by the independence of the upper record values.

2. Main results

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(1) = 0$ and $F(x) < 1$ for $x > 1$. Then $F(x) = 1 - x^{-\beta}$, $x > 1$, $\beta > 0$, if and only if $\dfrac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$, $1 \leq m < n$ are independent.*

Proof. If $F(x) = 1 - x^{-\beta}$, $x > 1$, $\beta > 0$, then the joint pdf $f_{m,n}(x, y)$ of $X_{U(m)}$ and $X_{U(n)}$ is

$$f_{m,n}(x,y) = \frac{\beta^n\, (ln\, x)^{m-1}\, (ln\, y - ln\, x)^{n-m-1}}{\Gamma(m)\Gamma(n-m)\ x\, y^{\beta+1}}$$

for $1 < x < y$, $\beta > 0$ and $1 \leq m < n$.

Consider the functions $V = \dfrac{X_{U(n)}}{X_{U(m)}}$ and $W = X_{U(m)}$. It follows that $x_{U(m)} = w$, $x_{U(n)} = vw$ and $|J| = w$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v,w) = \frac{\beta^n\, (ln\, v)^{n-m-1}\, (ln\, w)^{m-1}}{\Gamma(m)\Gamma(n-m)\, v^{\beta+1}\, w^{\beta+1}} \tag{2}$$

for $v > 1$, $w > 1$, $\beta > 0$ and $1 \leq m < n$.

The marginal pdf $f_V(v)$ of V is given by

$$f_V(v) = \int_1^\infty f_{V,W}(v,w)\, dw = \frac{\beta^{n-m}\, (ln\, v)^{n-m-1}}{\Gamma(n-m)\, v^{\beta+1}} \tag{3}$$

for $v > 1$, $\beta > 0$ and $1 \leq m < n$. Also, the pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{\beta^m (\ln w)^{m-1}}{\Gamma(m)\, w^{\beta+1}} \tag{4}$$

for $w > 1$, $\beta > 0$ and $m \geq 1$.

From (2), (3) and (4), we obtain $f_V(v) f_W(w) = f_{V,W}(v, w)$.

Hence $V = \frac{X_{U(n)}}{X_{U(m)}}$ and $W = X_{U(m)}$ are independent for $1 \leq m < n$.

Now we will prove the sufficiency condition. The joint pdf $f_{m,n}(x, y)$ of $X_{U(m)}$ and $X_{U(n)}$ is

$$f_{m,n}(x,y) = \frac{(R(x))^{m-1} r(x)(R(y) - R(x))^{n-m-1} f(y)}{\Gamma(m)\Gamma(n-m)}$$

for $1 < x < y$ and $1 \leq m < n$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}(R(x)) = \frac{f(x)}{1 - F(x)}$.

Let us use the transformations $V = \frac{X_{U(n)}}{X_{U(m)}}$ and $W = X_{U(m)}$. The Jacobian of the transformations is $|J| = w$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v,w) = \frac{f(vw)\,(R(vw) - R(w))^{n-m-1}\,(R(w))^{m-1}\, r(w)\, w}{\Gamma(m)\Gamma(n-m)} \tag{5}$$

for $v > 1$, $w > 1$ and $1 \leq m < n$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^{m-1} f(w)}{\Gamma(m)} \tag{6}$$

for $w > 1$ and $m \geq 1$.

From (5) and (6), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{f(vw)(R(vw) - R(w))^{n-m-1} r(w)\, w}{\Gamma(n-m) f(w)}$$

for $v > 1$, $w > 1$ and $1 \leq m < n$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}(R(x)) = \frac{f(x)}{1 - F(x)}$.

That is,

$$f_V(v) = \frac{1}{\Gamma(n-m+1)} \left(\frac{\partial}{\partial v} \left(ln\, \frac{1 - F(w)}{1 - F(vw)} \right)^{n-m} \right) \frac{1 - F(vw)}{1 - F(w)},$$

where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}(R(x)) = \frac{f(x)}{1 - F(x)}$.

Since V and $W$are independent, we must have

$$1 - F(vw) = (1 - F(v))(1 - F(w))$$

for $v > 1$ and $w > 1$.

Let $1 - F(x) = G(x)$. Then we have

$$G(vw) = G(v)G(w) \tag{7}$$

for $v > 1$ and $w > 1$.

By the functional equations (see, [1]), the only nonconstant continuous solution of (7) with the boundary conditions $G(0) = 0$ is

$$G(x) = x^{-\beta}$$

for $x > 1$ and $\beta > 0$. Thus we have

$$F(x) = 1 - x^{-\beta}$$

for $x > 1$ and $\beta > 0$. This completes the proof. □

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(1) = 0$ and $F(x) < 1$ for $x > 1$. Then $F(x) = 1 - x^{-\beta}$, $x > 1$, $\beta > 0$, if and only if $\frac{X_{U(n)} + X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$, $n \geq 1$ are independent.*

Proof. In the same manner as Theorem 2.1, $\frac{X_{U(n)} + X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $n \geq 1$.

Now we will prove the sufficiency condition. The joint pdf $f_{n,n+1}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x, y) = \frac{(R(x))^{n-1} r(x) f(y)}{\Gamma(n)}$$

for $1 < x < y$ and $n \geq 1$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}(R(x)) = \frac{f(x)}{1 - F(x)}$.

Let us use the transformations $V = \frac{X_{U(n)} + X_{U(n+1)}}{X_{U(n)}}$ and $W = X_{U(n)}$. The Jacobian of the transformations is $|J| = w$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{f((v-1)w)\,(R(w))^{n-1}\, r(w)\, w}{\Gamma(n)} \tag{8}$$

for $v > 2$, $w > 1$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^{n-1} f(w)}{\Gamma(n)} \tag{9}$$

for $w > 1$ and $n \geq 1$.

From (8) and (9), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{f((v-1)w)\, r(w)\, w}{f(w)}$$

for $v > 2$, $w > 1$ and $n \geq 1$, where $r(x) = \dfrac{f(x)}{1-F(x)}$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(-\frac{1-F((v-1)w)}{1-F(w)}\right).$$

Since V and $W$are independent, we must have

$$1 - F((v-1)w) = (1 - F(v-1))(1 - F(w))$$

for $v > 2$ and $w > 1$.

In the same manner as Theorem 2.1, let us use the substitutions $1 - F(x) = G(x)$ and $(v-1) = v_1$. Then we get

$$G(v_1\, w) = G(v_1)G(w) \tag{10}$$

for $v_1 > 1$ and $w > 1$.

By the functional equations (see, [1]), the only nonconstant continuous solution of (10) with the boundary conditions $G(0) = 0$ is

$$G(x) = x^{-\beta}$$

for $x > 1$ and $\beta > 0$. Thus we have

$$F(x) = 1 - x^{-\beta}$$

for $x > 1$ and $\beta > 0$. This completes the proof. □

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(1) = 0$ and $F(x) < 1$ for $x > 1$. Then $F(x) = 1 - x^{-\beta}$, $x > 1$, $\beta > 0$, if and only if $\dfrac{X_{U(n)}}{X_{U(n)} + X_{U(n+1)}}$ and $X_{U(n)}$, $n \geq 1$ are independent.*

Proof. In the same manner as Theorem 2.1, $\dfrac{X_{U(n)}}{X_{U(n)} + X_{U(n+1)}}$ and $X_{U(n)}$ are independent for $n \geq 1$.

Now we will prove the sufficiency condition. By the same manner as Theorem 2.2, let us use the transformations $V = \frac{X_{U(n)}}{X_{U(n)}+X_{U(n+1)}}$ and $W = X_{U(n)}$. The Jacobian of the transformations is $|J| = \dfrac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{f\left(\frac{(1-v)\,w}{v}\right)\, (R(w))^{n-1}\, r(w)\, w}{\Gamma(n)\, v^2} \tag{11}$$

for $0 < v < \frac{1}{2}$, $w > 1$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^{n-1} f(w)}{\Gamma(n)} \tag{12}$$

for $w > 1$ and $n \geq 1$.

From (11) and (12), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{f\left(\frac{(1-v)w}{v}\right) r(w)\, w}{v^2 f(w)}$$

for $0 < v < \frac{1}{2}$, $w > 1$ and $n \geq 1$, where $r(w) = \frac{f(w)}{1 - F(w)}$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(\frac{1 - F\left(\frac{(1-v)w}{v}\right)}{1 - F(w)}\right).$$

Since V and $W$are independent, we must have

$$1 - F\left(\frac{(1-v)\,w}{v}\right) = \left(1 - F\left(\frac{1-v}{v}\right)\right)(1 - F(w))$$

for $0 < v < \frac{1}{2}$ and $w > 1$.

In the same manner as Theorem 2.1, let us use the substitutions $1 - F(x) = G(x)$ and $\frac{(1-v)}{v} = v_2$. Then we obtain

$$G(v_2\, w) = G(v_2)G(w) \tag{13}$$

for $v_2 > 1$ and $w > 1$.

By the functional equations (see, [1]), the only nonconstant continuous solution of (13) with the boundary conditions $G(0) = 0$ is

$$G(x) = x^{-\beta}$$

for $x > 1$ and $\beta > 0$. Thus we have

$$F(x) = 1 - x^{-\beta}$$

for $x > 1$ and $\beta > 0$. This completes the proof. □

REFERENCES

1. J. Aczel, *Lectures on Functional Equations and Their Applications*, Academic Press, NY, 1966.
2. M. Ahsanuallah, *Record Statistics*, Nova Science Publishers, Inc., NY, 1995.
3. M. Ahsanuallah, *Record Values-Theory and Applications*, University Press of America, Inc., NY, 2004.
4. K. N. Chandler, The distribution and frequency of record values, J. R. Stat. Soc. B, **14** (1952), 220-228.

Se-Kyung Chang serves as a full-time lecturer in Department of Mathematics Education, Cheongju University. Her research interests characterizations of various distributions by record values or order statistics.

Department of Mathematics Education, Cheongju University, Korea
e-mail: skchang@cju.ac.kr