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THE EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO p–LAPLACE EQUATION WITH PERIODIC BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we consider p–Laplace equation which models the turbulent flow in a porous medium. Using a continuation principle (cf. [R. Manásevich and J. Mawhin, *Periodic solutions for nonlinear systems with p-Lplacian-like operators*, J. Diff. Equa. **145**(1998), 367-393]), we prove the existence of solutions for p–Laplace equation subject to periodic boundary conditions, under some sign and growth conditions for f. With the help of Leray-Schauder degree theory, the multiplicity of periodic solutions for p–Laplace equation is obtained under the similar conditions above and some known results are improved.

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1. Introduction and main results

The turbulent flow in a porous medium is a fundamental mechanics problem. To study this type problem, Leibenson [6] introduced the following model

$$u_t = \frac{\partial}{\partial x}(\frac{\partial(u^m)}{\partial x}|\frac{\partial(u^m)}{\partial x}|^{P-1}), \tag{1}$$

where $m \geq 2, \frac{1}{2} \leq p \leq 1$. Generally, when $m > 1$, Eq.(1) is called Porous Medium Equation [1]; when $0 < m < 1$, it is called Diffusion Equation; when $m = 1$, it is called Heat Equation, which often appears in non-Newtonian liquid [4]. For the study of Eq.(1), someone reduced Eq.(1) into the following p–Laplace equation

$$(\phi_p(u'))' = f(t, u, u'), \tag{2}$$

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where $\phi_p(s) = |s|^{p-2}s$. Obviously, when $p = 2$, Eq.(2) becomes a general second order differential equation.

In recent years, some important results relative to Eq.(2) with periodic boundary conditions have been obtained [2,5,7-9]. Cranas and Lee [3] have discussed the boundary value problems for second order differential equation

$$u'' = f(t, u, u'), \quad t \in [0, T]$$

with boundary conditions

$$u(0) = u(T), \quad u'(0) = u'(T)$$

using the main assumptions as follows:

(A_1) there exists a constant $M \geq 0$ such that

$$uf(t, u, 0) > 0, \quad \text{for } |u| > M, t \in [0, T];$$

(A_2) there is $\psi \in C([0, +\infty), R^+)$ such that

$$|f(t, u, v)| \leq \psi(|v|), \quad \text{for } (t, u, v) \in [0, T] \times [-M, M] \times R,$$

where

$$\int_0^{+\infty} \frac{s}{\psi(s)} ds > 2M.$$

In this paper, we generalize the result in [3] to $p-$Laplace equation

$$(\phi_p(u'))' = f(t, u, u'), \quad t \in [0, T], \tag{3}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{4}$$

and we obtain the following existence result.

Theorem 1. *Let $f : [0, T] \times R^2 \longrightarrow R$ be continuous. Assume that*

(H_1) *there exist r_1, r_2 with $r_1 < r_2$, such that*

$$f(t, r_1, 0) < 0, \quad f(t, r_2, 0) > 0, \quad \text{for } t \in [0, T];$$

(H_2) *there is $\psi \in C([0, +\infty), R^+)$ such that*

$$|f(t, u, v)| \leq \psi(|v|), \quad \text{for } (t, u, v) \in [0, T] \times [r_1, r_2] \times R,$$

where

$$\int_0^{+\infty} \frac{s^{\frac{1}{p-1}}}{\psi(s^{\frac{1}{p-1}})} ds > r_2 - r_1,$$

Then there exists a solution of BVP (3)(4).

It is easy to see that the condition (A_1) in [3] is stronger than the condition (H_1) of Theorem 1. Hence we improve the result of [3] to some extent.

If the condition (H_1) is replaced by

(H_3) *there exist $r_1 < R_1 < r_2 < R_2$, such that*

$$f(t, r_1, 0) < 0, \quad f(t, R_1, 0) > 0, \quad f(t, r_2, 0) < 0, \quad f(t, R_2, 0) > 0, \quad \text{for } t \in [0, T],$$

then we can also get a multiplicity result.

Theorem 2. *Assume that the assumption (H_2) in Theorem 1 and (H_3) hold. Then, there exist at least three different solutions $u_1, u_2, u_3 \in C^1[0,T]$ of BVP (3)(4) satisfying:*

$$r_1 < u_1(t) < R_1, \quad r_2 < u_2(t) < R_2, \quad R_1 < u_3(t) < r_2, \quad for\, t \in [0,T].$$

2. Existence results

Throughout the paper, we shall use the classical spaces $C[0,T], C^1[0,T], L^1[0,T]$ and set

$$C_T^1[0,T] = \{u \in C^1[0,T] : u(0) = u(T), u'(0) = u'(T)\}.$$

We denote the norm in $C[0,T]$ by $\|\cdot\|_\infty$. Suppose Ω is an open bounded set in $C_T^1[0,T]$, i.e. there exist two positive constants M^*, M^{**} such that

$$\Omega = \{u(t) \in C_T^1[0,T] : \|u\|_\infty < M^*, \|u'\|_\infty < M^{**}\}.$$

Set

$$\Omega \cap R = \{u(t) \in C_T^1[0,T] : u(t) \equiv u_0 = \text{constant}, |u_0| < M^*\}$$

and

$$\partial\Omega \cap R = \{u(t) \in C_T^1[0,T] : u(t) \equiv u_0 = \text{constant}, |u_0| = M^*\}.$$

Moreover, we will need the following lemma.

Lemma 1. [7] *Assume that $f : [0,T] \times R^2 \longrightarrow R$ is continuous and such that the following conditions hold.*

(B_1) The problem

$$(\phi_p(u'))' = \lambda f(t,u,u'), \quad t \in [0,T], \lambda \in (0,1), \tag{5}$$

$$u(0) = u(T), \quad u'(0) = u'(T) \tag{6}$$

has no solution on $\partial\Omega$;

(B_2) The equation

$$F(a) := \frac{1}{T}\int_0^T f(t,a,0)dt = 0$$

has no solution on $\partial\Omega \cap R$;

(B_3) The Brower degree $deg_B(F, \Omega \cap R, 0) \neq 0$.

Then BVP (3)(4) has a solution in $\bar{\Omega}$.

Now we give the proof of *Theorem 1.*

Proof. For $(t,u,v) \in [0,T] \times R^2$ define

$$\bar{f}(t,u,v) = \begin{cases} f(t,r_2,v), & \text{for } u > r_2, \\ f(t,u,v), & \text{for } r_1 \le u \le r_2, \\ f(t,r_1,v), & \text{for } u < r_1. \end{cases} \tag{7}$$

The modified problem corresponding to (3)(4) is

$$(\phi_p(u'))' = \bar{f}(t,u,u'), \quad t \in [0,T],$$
$$u(0) = u(T), \quad u'(0) = u'(T).$$

In order to use Lemma 1, we consider the homotopy problem

$$(\phi_p(u'))' = \lambda \bar{f}(t, u, u'), \quad t \in [0, T], \tag{8}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{9}$$

where $\lambda \in (0, 1)$.

First, we can claim that

$$r_1 < u(t) < r_2, \quad \text{for } t \in [0, T], \lambda \in (0, 1), \tag{10}$$

where $u(t)$ is a possible solution of BVP (8)(9). Otherwise, there exists a point $t_0 \in [0, T)$ such that $u(t_0) = \min_{t\in[0,T]} u(t) \leq r_1$ or $u(t_0) = \max_{t\in[0,T]} u(t) \geq r_2$. Without loss of generality, assume that $u(t_0) = \max_{t\in[0,T]} u(t) \geq r_2$.

If $t_0 \in (0, T)$, then $u'(t_0) = 0$ and

$$(\phi_p(u'(t_0)))' = \lambda \bar{f}(t_0, u(t_0), u'(t_0)) = \lambda f(t_0, r_2, 0) > 0.$$

So, there exists a constant $\delta > 0$ such that $(\phi_p(u'(t)))' > 0$, for $t \in (t_0, t_0 + \delta)$. This implies that $\phi_p(u'(t))$ is increasing on $(t_0, t_0 + \delta)$. Thus

$$\phi_p(u'(t)) > \phi_p(u'(t_0)) = \phi_p(0) = 0, \quad t \in (t_0, t_0 + \delta),$$

which shows $u'(t) > 0$, $t \in (t_0, t_0 + \delta)$ from the monotonicity of ϕ_p. Namely, $u(t)$ is increasing on $(t_0, t_0 + \delta)$, contradicting $u(t_0) = \max_{t\in[0,T]} u(t)$.

If $t_0 = 0$, by $u(0) = u(T), u'(0) = u'(T)$, then $u'(0) = 0$. Immediately, a contradiction is obtained by a similar argument.

Second, we shall prove that there exists $M_0 > 0$ such that $\|u'\|_\infty \leq M_0$.

For fixed $t_0 \in [0, T]$, if $u'(t_0) \neq 0$, then there exists an interval $[\mu, \nu] \subset [0, T]$, $t_0 \in [\mu, \nu]$ such that $u'(t)$ has the same sign for $t \in [\mu, \nu]$ and either $u'(\mu) = 0$ or $u'(\nu) = 0$. Here, we might as well assume that $u'(t) > 0$, for $t \in [\mu, \nu]$ and $u'(\mu) = 0$. Multiplying (8) with $u'(t)$, we have

$$u'(t)(\phi_p(u'))' = \lambda u'(t)\bar{f}(s, u, u').$$

Noting that $r_1 < u(t) < r_2$, for $t \in [0, T]$ and the definition of $\bar{f}$, then

$$u'(t)(\phi_p(u'))' = \lambda u'(t) f(s, u, u').$$

Combining with condition (H_2) and $r_1 < u(t) < r_2$, for $t \in [0, 1]$, we get

$$u'(t)(\phi_p(u'))' \leq u'(t)\psi(u'(t)), \quad \text{for } t \in [\mu, \nu].$$

Namely

$$\frac{(\phi_p(u'(t))^{\frac{1}{p-1}}(\phi_p(u'(t)))'}{\psi(u'(t))} \leq u'(t), \quad \text{for } t \in [\mu, \nu]. \tag{11}$$

Integrating (11) over $[\mu, t_0]$, we get

$$\int_\mu^{t_0} \frac{(\phi_p(u'(t))^{\frac{1}{p-1}}(\phi_p(u'(t)))'}{\psi(u'(t))} dt \leq \int_\mu^{t_0} u'(t) dt.$$

Let $s = \phi_p(u'(t))$, we have

$$\int_0^{\phi_p(u'(t_0))} \frac{s^{\frac{1}{p-1}}}{\psi(s^{\frac{1}{p-1}})} ds \le u(t_0) - u(\mu) \le r_2 - r_1.$$

By hypothesis (H_2), we can find some constant M_1 (independent of λ and t_0) such that $\phi_p(u'(t_0)) \le M_1$. Hence, there exists M_0 (independent of λ) such that

$$\|u'\|_\infty \le M_0.$$

Next, we shall prove that the BVP (8)(9) has at least one solution by using Lemma 1. Set

$$\Omega = \{u(t) \in C_T^1[0,T] : r_1 < u(t) < r_2, t \in [0,T]; \|u'\|_\infty < M_0 + 1\}.$$

Obviously, the hypothesis (B_1) of Lemma 1 is satisfied. By assumption (H_1), we know that

$$\bar{f}(t, r_1, 0) < 0, \quad \bar{f}(t, r_2, 0) > 0.$$

Applying the monotonicity of ϕ_p, we immediately get

$$F(r_1) < 0, \quad F(r_2) > 0.$$

Thus

$$F(a) = \frac{1}{T}\int_0^T f(t, a, 0)dt \neq 0, \quad \text{for } a \in \partial\Omega \cap R.$$

Therefore, the hypothesis (B_2) of lemma 1 is true. Noting that $F(r_1)F(r_2) < 0$ and the property of Brouwer degree, we see that

$$\deg_B(F, \Omega \cap R, 0) = 1.$$

Hence, the hypothesis (B_3) of Lemma 1 is also satisfied. By Lemma 1, it can be shown that the BVP (8)(9) has one solution $u(t)$ and satisfying $r_1 < u(t) < r_2$, for $t \in [0,T]$. From (7), we get that $u(t)$ is a solution of BVP (3)(4). □

3. Multiplicity results

In order to prove the multiplicity result, we shall use the following lemmas. For more details we refer the readers to [7].

Lemma 2. [7] *For fixed $l(t) \in C[0,T]$, let us define*

$$G_l(a) = \int_0^1 \phi_p^{-1}(a + l(t))dt.$$

Then the function $G_l(a) : R \longrightarrow R$ has the following properties:

(1) *for any fixed $l(t) \in C[0,T]$, the equation $G_l(a) = 0$ has a unique solution $\bar{a}(l)$;*

(2) *the function $\bar{a} : C[0,T] \longrightarrow R$ is continuous and sends bounded sets into bounded sets.*

Let now $a : L^1 \longrightarrow R$ be defined by

$$a(h) = \bar{a}(H(h)), \quad H(h) = \int_0^t h(s)ds.$$

Then, it is clear that a is a continuous function which sends bounded sets of L^1 into bounded sets of R, and hence it is a completely continuous mapping.

Let us define the projectors P, Q respectively by

$$P : C_T^1 \longrightarrow C_T^1, P(u) = u(0), \quad Q : L^1 \longrightarrow L^1, Q(h) = \frac{1}{T}\int_0^T h(s)ds,$$

and the operator $J : L^1 \longrightarrow C_T^1$ given by

$$J(h)(t) = H\{\phi_p^{-1}[a((I-Q)h) + H((I-Q)h)]\}(t), \quad \text{for } t \in [0,T].$$

Obviously, the operator J is continuous and sends equi-integrable sets in L^1 into relatively compact sets in C_T^1.

Next, let us consider the auxiliary problem (5)(6) corresponding to (3)(4). Then BVP (5)(6) is equivalent to the problem $u = G_f(u,\lambda), \quad \lambda \in (0,1)$, where

$$G_f(u,\lambda) := Pu + QN_f(u) + (J \circ [\lambda(I-Q)N_f])(u),$$
$$N_f(u) = f(t,u,u').$$

Lemma 3. [7] *Assume that $f : [0,T] \times R^2 \longrightarrow R$ is continuous and Ω is an open bounded set in $C_T^1[0,T]$ such that the following conditions hold.*

(C_1) *For each $\lambda \in (0,1]$, the problem*

$$(\phi_p(u'))' = \lambda f(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has no solution on $\partial\Omega$;

(C_2) *The equation $F(a) := \frac{1}{T}\int_0^T f(t,a,0)dt = 0$ has no solution on $\partial\Omega \cap R$. Then*

$$deg_{LS}[I - G_f(\cdot,1),\Omega,0] = -deg_B[F,\Omega\cap R,0].$$

Now we prove **Theorem 2.**

Proof. For $(t,u,v) \in [0,T] \times R^2$, let us define three auxiliary operator $f_i(i = 1,2,3)$ by

$$f_i(t,u,v) = \begin{cases} f(t,R_i,v), & \text{for } u > R_i, \\ f(t,u,v), & \text{for } r_i \le u \le R_i, \\ f(t,r_i,v), & \text{for } u < r_i, \end{cases} \tag{12}$$

where $i = 1,2$ and

$$f_3(t,u,v) = \begin{cases} f(t,R_2,v), & \text{for } u > R_2, \\ f(t,u,v), & \text{for } r_1 \le u \le R_2, \\ f(t,r_1,v), & \text{for } u < r_1. \end{cases} \tag{13}$$

In order to use Lemma 3, we consider the homotopy problem corresponding to (3)(4)

$$(\phi_p(u'))' = \lambda f_i(t,u,u'), \quad t \in [0,T], \tag{14}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{15}$$

where $\lambda \in (0,1], i = 1,2,3$. For fixed i, we assume $\bar{u}_i(t)$ is a possible solution of BVP (14)(15). When $i = 1,2$, a similar argument in Theorem 1 shows that there exist two constants K_i (independent of λ) such that

$$r_i < \bar{u}_i(t) < R_i, \quad |\bar{u}_i'(t)| < K_i, \quad \text{for } t \in [0,T].$$

When $i = 3$, we can also obtain that there exists K_3 (independent of λ) such that

$$r_1 < \bar{u}_3(t) < R_2, \quad |\bar{u}_3'(t)| < K_3, \quad \text{for } t \in [0,T].$$

Let $K = \max_{i=1,2,3}\{K_i\}$. For $i = 1,2$, set

$$\Omega_i = \{u(t) \in C_T^1[0,T] : r_i < u(t) < R_i, \ |u'(t)| < K, \ t \in [0,T]\}.$$

And for $i = 3$, set

$$\Omega_3 = \{u(t) \in C_T^1[0,T] : r_1 < u(t) < R_2, \ |u'(t)| < K, \ t \in [0,T]\}.$$

Obviously, for each i, the hypothesis (C_1) and (C_2) of Lemma 3 are satisfied. Thus, by Lemma 3, we can get

$$\deg_{LS}[I - G_{f_i}(\cdot,1), \Omega_i, 0] = -\deg_B[F_i, \Omega_i \cap R, 0],$$

where $F_i(a) := \frac{1}{T}\int_0^T f_i(t,a,0)dt = 0$. A similar argument in Theorem 1 shows that $\deg_B[F_i, \Omega_i \cap R, 0] = 1$. Hence

$$\deg_{LS}[I - G_{f_i}(\cdot,1), \Omega_i, 0] = -1,$$

where $i = 1,2,3$. Noting the definition of f_i, we know that for each i

$$f_i(t,u,u') = f(t,u,u') \quad \text{for } u \in \Omega_i.$$

Thus

$$\deg_{LS}[I - G_f(\cdot,1), \Omega_i, 0] = -1, \quad i = 1,2,3,$$

and

$$\begin{aligned} &\deg_{LS}[I - G_f(\cdot,1), \Omega_3 \backslash \overline{\Omega_1 \cap \Omega_2}, 0] \\ &= \deg_{LS}[I - G_f(\cdot,1), \Omega_3, 0] - \deg_{LS}[I - G_f(\cdot,1), \Omega_1, 0] \\ &\quad -\deg_{LS}[I - G_f(\cdot,1), \Omega_2, 0] = 1. \end{aligned}$$

Consequently, the operator $G_f(\cdot,1)$ has at least three fixed points in $\Omega_i (i = 1,2)$ and $\Omega_3 \backslash \overline{\Omega_1 \cap \Omega_2}$ which are three different solutions u_1, u_2, u_3 of BVP (3)(4) satisfying:

$$r_1 < u_1(t) < R_1, \quad r_2 < u_2(t) < R_2, \quad R_1 < u_3(t) < r_2, \quad \text{for } t \in [0,T].$$

□

4. Discussion

In Theorem 1, we provide a natural and easily verifiable condition under which Eq.(1) has a periodic solution. This result improves the work of [3] to some extent.

On the other hand, we present some sufficient conditions to guarantee the existence of multiple periodic solutions for $p-$Laplace equation in Theorem 2. As far as we know, the similar results are very few.

In Theorem 1 and Theorem 2, the assumption (H_2) plays an important role which is always called Nagumo condition. When f is a polynomial, the condition (H_2) requires that the order of f is less than p. It is interesting that whether Eq.(1) still has periodic solutions if the order of f is greater than p. We leave this for future work.

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