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MULTIPLE POSITIVE SOLUTIONS OF PERIODIC BOUNDARY VALUE PROBLEMS WITH IMPULSE

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ABSTRACT. At least two positive solutions of a first-order periodic boundary value problem with impulse are obtained by establishing a new cone and the theorem of fixed point index. And at the end of this paper we give an example to illustrate the application of our main results.

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1. Introduction

Let $0 =< t_1 < t_2 < \ldots < t_m = N$ be given. In this paper we examine the first order impulsive equation

$$\begin{cases} x'(t) + \lambda x(t) = f(t, x(t)), & t \in J,\ t \neq t_k,\ k = 1, 2, \ldots, m, \\ x(t_k^+) = x(t_k^-) + I(x(t_k)), & k = 1, 2, \cdots, m, \\ x(0) = x(N). \end{cases} \qquad (1)$$

Here $\lambda \in \mathcal{R} \setminus \{\theta\}, \mathcal{N} > 1, \mathcal{I} \in \mathcal{C}(\mathcal{R}, \mathcal{R})$, $f : [0, \mathcal{N}] \times \mathcal{R} \to \mathcal{R}$ is continuous on $(t, x) \in [0, N] \setminus \{t_1, \ldots, t_m\} \times \mathcal{R}$, $f(t_k^-, x)$, $f(t_k^+, x)$ exist and $f(t_k^-, x) = f(t_k, x)$. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$ where $x(t_k^+)$ (respectively $x(t_k^-)$) denote the right limit (respectively left limit) of $x(t)$ at $t = t_k$.

The processes of impulsive are naturally seen in such fields as: chemotherapy; population dynamics; optimal control; ecology; industrial robotics; biotechnology and physics. Recently Eq. (1) ($x \in R^+$, $R^+ = [0, +\infty)$) has been studied in a number of papers, e.g. [1-5]. As both f and I are bounded, Li in [1] obtain a solution by the Schaeffer's theorem. And the result is developed in [2]. Particularly, in [3], Chen and Sun obtain the existence results of the solution by the upper and lower solution method and the monotone iterative technique. In this

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paper, we obtained at least two positive solutions by the theorem of fixed point index and the hypotheses is improved.

The article is organized as follows. In Section 1 some hypotheses are given and in Section 2 we present some preliminary ideas associated with the impulsive BVP (1). Section 3 contains the main results of the paper and is devoted to the existence of solutions to (1). Our new results extend those of [1-5].

For convenience and simplicity in the following discussion, we always use the notations:

$$J = [0, N], \quad J' = J \setminus \{t_1, t_2, \dots, t_m\};$$
$$f_0 = \liminf_{x\to 0+} \min_{t\in J} \frac{f(t,x)}{\lambda x}, \quad I_0 = \liminf_{x\to 0+} \frac{I(x)}{\lambda x},$$
$$f^0 = \limsup_{x\to 0+} \max_{t\in J} \frac{f(t,x)}{\lambda x}, \quad I^0 = \limsup_{x\to 0+} \frac{I(x)}{\lambda x},$$
$$f_\infty = \liminf_{x\to +\infty} \min_{t\in J} \frac{f(t,x)}{\lambda x}, \quad I_\infty = \liminf_{x\to +\infty} \frac{I(x)}{\lambda x},$$
$$f^\infty = \limsup_{x\to +\infty} \max_{t\in J} \frac{f(t,x)}{\lambda x}, \quad I^\infty = \limsup_{x\to +\infty} \frac{I(x)}{\lambda x}.$$

In this paper, some of the following hypotheses are satisfied:

(H1) There exists $r > 0$ such that $0 \le x \le r$ and $t \in J$ implies

$$0 \le \frac{f(t,x)}{\lambda} \le \eta r, \quad 0 \le \frac{I(x)}{\lambda} \le \eta_k r,$$

where $\eta, \eta_k \ge 0$ and

$$\eta + \lambda \frac{e^{|\lambda| N}}{1 - e^{-\lambda N}} \sum_{k=1}^{m} \eta_k < 1.$$

(H2) $f_0 \ge 0, I_0 \ge 0, f_\infty \ge 0, I_\infty \ge 0$ and

$$f_0 + I_0 \frac{m}{N} e^{(-\lambda-|\lambda|)N} > 1, \quad f_\infty + I_\infty \frac{m}{N} e^{(-\lambda-|\lambda|)N} > 1.$$

(H3) There exists $r > 0$ such that $e^{(-\lambda-|\lambda|)N} r \le x \le r$ and $t \in J$ implies

$$\frac{f(t,x)}{\lambda} \ge \eta r, \quad \frac{I(x)}{\lambda} \ge \eta_k r,$$

where $\eta, \eta_k \ge 0$ and

$$\eta + \lambda \frac{e^{-|\lambda| N}}{1 - e^{-\lambda N}} \sum_{k=1}^{m} \eta_k > 1.$$

(H4) $f^0 \ge 0, I^0 \ge 0, f^\infty \ge 0, I^\infty \ge 0$ and

$$(f^0 + I^0 \frac{m}{N}) e^{(-\lambda-|\lambda|)N} > 1, \quad (f^\infty + I^\infty \frac{m}{N}) e^{(-\lambda-|\lambda|)N} > 1.$$

$x(t) \in PC(J) \cap C^1(J')$ is a positive solution of (1) if $x(t)$ satisfies (1) and $x(t) > 0, t \in J$.

Let $PC(J) = \{$x: x(t) is continuous at $t \neq t_1$, left continuous at $t = t_i$ and its right-hand limit at $t = t_i$ exists, $i = 1, 2, \ldots, m\}$. Obviously, $PC[J, R]$ is a Banach space with norm $\|x\| = \sup_{t \in J} |x(t)|$. $P = \{x \in PC(J) : x(t) \geq 0 \text{ for } t \in J\}$ is a cone in space $PC(J)$.

A map $x \in PC(J)$ is called a positive solution of the BVP (1) if it satisfies (1) and $x \in P$, $x(t) \neq 0$ for all $t \in J \setminus \{0, t_1, t_2, \ldots, t_m, 1\}$.

2. Several Lemmas

We denote $K = \{x \in PC(J) : x(t) \geq \mathrm{e}^{(-\lambda - |\lambda|)\mathrm{N}} \parallel \mathrm{x} \parallel, \forall \mathrm{t} \in \mathrm{J}\}$ and $K_r = \{x \in K : \parallel x \parallel < r\}$. For any $x \in K$, let

$$(Ax)(t) = \int_0^N g(t,s) f(s, x(s)) \mathrm{ds} + \sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{g}(\mathrm{t}, \mathrm{t_k}) \mathrm{I}(\mathrm{x}(\mathrm{t_k})), \quad \mathrm{t} \in \mathrm{J}. \tag{2}$$

where

$$g(t,s) = \frac{1}{1 - \mathrm{e}^{-\lambda \mathrm{N}}} \begin{cases} \mathrm{e}^{-\lambda(\mathrm{t}-\mathrm{s})}, & 0 \leq s \leq t \leq N, \\ \mathrm{e}^{-\lambda(\mathrm{N}+\mathrm{t}-\mathrm{s})}, & 0 \leq t \leq s \leq N. \end{cases}$$

Then $A\colon K \longrightarrow P$ is completely continuous. What's more, the positive fixed points of A are the positive solutions of the BVP (1).

Lemma 1.([1]) *Let $A\colon K \longrightarrow K$ be a completely continuous mapping. If*

(i) $\inf_{x \in \partial\Omega} \|Ax\| > 0$;

(ii) $Ax \neq tx$ *for every* $x \in \partial K_r, t \in (0, 1]$.

Then $i(A, K_r, K) = 0$.

Lemma 2. *Let $A\colon K \longrightarrow K$ be a completely continuous mapping and $Ax \neq x$ for $x \in \partial K_r$. Then*

(i) If $\|Ax\| \leq \|x\|$, for $x \in \partial K_r$, then $i(A, K_r, K) = 1$.

(ii) If $\|Ax\| \geq \|x\|$, for $x \in \partial K_r$, then $i(A, K_r, K) = 0$.

Lemma 3. *Let $A\colon K \to K$ be a completely continuous operator with $\mu Ax \neq x$ for every $x \in \partial K_r$ and $0 < \mu \leq 1$. Then $i(A, K_r, K) = 1$.*

Lemma 4. *Assume $(H1)$ is satisfied, Then $i(A, K_r, P) = 1$.*

Proof. $\forall\, x \in \bar{K}_r$, by $(H1)$, it is easy to see

$$\begin{aligned} (Ax)(t) &= \int_0^N g(t,s) f(s, x(s)) \mathrm{ds} + \sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{g}(\mathrm{t}, \mathrm{t_k}) \mathrm{I}(\mathrm{x}(\mathrm{t_k})) \\ &= \int_0^N \lambda g(t,s) \frac{f(s, x(s))}{\lambda} \mathrm{ds} + \sum_{\mathrm{k}=1}^{\mathrm{m}} \lambda \mathrm{g}(\mathrm{t}, \mathrm{t_k}) \frac{\mathrm{I}(\mathrm{x}(\mathrm{t_k}))}{\lambda} \\ &\geq 0. \end{aligned}$$

Thus $A(K_r) \subset P$. $\forall x \in \partial K_r$, we have

$$\begin{aligned}(Ax)(t) &= \int_0^N \lambda g(t,s)\frac{f(s,x(s))}{\lambda}\mathrm{ds} + \sum_{k=1}^{m} \lambda g(t,t_k)\frac{I(x(t_k))}{\lambda} \\ &\leq r(\eta + \lambda \frac{e^{|\lambda|N}}{1-e^{-\lambda N}} \sum_{k=1}^{m} \eta_k) < r = \|x\|.\end{aligned}$$

so $\|Ax\| < \|x\|$, $\forall x \in \partial K_r$. On the other way, it is easy to get $Ax \neq x$, $\forall x \in \partial K_r$. From Lemma 2, $i(A, K_r, P) = 1$. □

Lemma 5. *If* $(H3)$ *is satisfied, then* $i(A, K_r, P) = 0$.

Proof. $\forall x \in \bar{K}_r$, we have $x(t) \geq e^{(-\lambda-|\lambda|)N} \| x \|$. It follows from $(H3)$ that

$$\begin{aligned}(Ax)(t) &= \int_0^N g(t,s)f(s,x(s))\mathrm{ds} + \sum_{k=1}^{m} g(t,t_k)I(x(t_k)) \\ &= \int_0^N \lambda g(t,s)\frac{f(s,x(s))}{\lambda}\mathrm{ds} + \sum_{k=1}^{m} \lambda g(t,t_k)\frac{I(x(t_k))}{\lambda} \\ &\geq 0.\end{aligned}$$

So $A(K_r) \subset P$. $\forall x \in \partial K_r$, we get

$$\begin{aligned}(Ax)(\frac{T}{2}) &\geq \int_0^T g(\frac{T}{2},s)f(s,x(s))\mathrm{ds} + \sum_{k=1}^{m} g(\frac{T}{2},t_k)I(x(t_k)) \\ &\geq r(\eta + \lambda \sum_{k=1}^{m} g(t,t_k)\eta_k) > r = \|x\|,\end{aligned}$$

which shows that $\|Ax\| > \|x\|$, $\forall x \in \partial K_r$. It is obviously that $Ax \neq x$, $\forall x \in \partial K_r$. Therefore, from Lemma 2, $i(A, K_r, P) = 0$. □

3. Main results

Theorem 1. *Suppose that* $(H1)$ *and* $(H2)$ *hold. Then* (1.1) *at least has two positive solutions* x_1 *and* x_2*, witch satisfy* $0 < \| x_1 \| < r < \| x_2 \|$.

Proof. Suppose that $(H2)$ holds, then there exists $0 < \varepsilon < 1$ such that

$$(1-\varepsilon)f_0 + (1-\varepsilon)I_0\frac{m}{N}e^{(-\lambda-|\lambda|)N} > 1, \tag{3}$$

$$(1-\varepsilon)f_\infty + (1-\varepsilon)I_\infty\frac{m}{N}e^{(-\lambda-|\lambda|)N} > 1. \tag{4}$$

From the definition of f_0, I_0 and $(H2)$, we know there exists $0 < r_0 < r$ such that

$$\frac{f(t,x)}{\lambda} \geq (1-\varepsilon)f_0 x, \ \frac{I(x)}{\lambda} \geq (1-\varepsilon)I_0 x, \quad \forall t \in J,\ 0 < x < r_0.$$

Let $r_1 \in (0, r_0)$. Then $A(K_{r_1}) \subset P$ and $\forall\, x \in \partial K_{r_1}$, we have

$$\begin{aligned} (Ax)(\frac{T}{2}) &= \int_0^N g(\frac{T}{2}, s) f(s, x(s)) \mathrm{ds} + \sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{g}(\frac{\mathrm{T}}{2}, \mathrm{t_k}) \mathrm{I}(\mathrm{x}(\mathrm{t_k})) \\ &\geq (1-\varepsilon) \mathrm{e}^{(-\lambda-|\lambda|)\mathrm{N}} \mathrm{r_1} (\mathrm{f_0} + \mathrm{I_0} \frac{\mathrm{m}\lambda \mathrm{e}^{-\lambda \mathrm{N}}}{1 - \mathrm{e}^{-\lambda \mathrm{N}}}). \end{aligned}$$

Thus

$$\|Ax\| \geq (1-\varepsilon) \mathrm{e}^{(-\lambda-|\lambda|)\mathrm{N}} \mathrm{r_1} (\mathrm{f_0} + \mathrm{I_0} \frac{\mathrm{m}\lambda \mathrm{e}^{-\lambda \mathrm{N}}}{1 - \mathrm{e}^{-\lambda \mathrm{N}}}) > 0.$$

i.e. $\inf_{x \in \partial K_{r_1}} \|Ax\| > 0$. Next we will show $\mu Ax \neq x$ for $x \in \partial K_{r_1}$ and $\mu \geq 0$. In fact, assume there exist $x_0 \in \partial K_{r_1}$ and $\mu_0 \geq 1$ such that $\mu_0 A x_0 = x_0$. Then $x_0(t)$ satisfies

$$\begin{cases} x_0'(t) + \mu_0 \lambda x_0(t) = \mu_0 f(t, x_0(t)), & t \in J', \\ x_0(t_k^+) = x_0(t_k^-) + I(x_0(t_k)), & k = 1, 2, \cdots, m, \\ x(0) = x(N). \end{cases} \tag{5}$$

Integrating (5) from 0 to N, we have

$$\int_0^N x_0'(t) \mathrm{dt} + \mu_0 \lambda \int_0^{\mathrm{N}} \mathrm{x_0(t)dt} = \mu_0 \int_0^{\mathrm{N}} \mathrm{f(t, x_0(t))dt}.$$

Then

$$\begin{aligned} \int_0^N x_0(t) \mathrm{dt} &= \int_0^N \frac{f(t, x_0(t))}{\lambda} \mathrm{dt} + \sum_{\mathrm{k}=1}^{\mathrm{m}} \frac{\mathrm{I}(\mathrm{x_0}(\mathrm{t_k}))}{\lambda} \\ &\geq (1-\varepsilon) f_0 \int_0^N x_0(t)) \mathrm{dt} + (1-\varepsilon) \mathrm{I_0} \sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{x_0}(\mathrm{t_k}). \end{aligned}$$

Consequently $(1-\varepsilon) f_0 \leq 1$. What's more, we also get

$$(1-\varepsilon) f_0 + (1-\varepsilon) I_0 \frac{m}{N} \mathrm{e}^{(-\lambda-|\lambda|)\mathrm{N}} \leq 1,$$

which is contradict to (3). Thus, by $A(K_{r_1}) \subset P$ and Lemma 1, we obtain

$$i(A, K_{r_1}, P) = 0. \tag{6}$$

On the other way, by the definition of f_∞ and I_∞, we know there exists $R_1 > r$ such that

$$\frac{f(t,x)}{\lambda} \geq (1-\varepsilon) f_\infty x, \quad \frac{I(x)}{\lambda} \geq (1-\varepsilon) I_\infty x, \quad \forall\, t \in J, \ x \geq R_1. \tag{7}$$

Then there exists a constant C such that

$$\frac{f(t,x)}{\lambda} \geq (1-\varepsilon) f_\infty x - C, \quad \frac{I(x)}{\lambda} \geq (1-\varepsilon) I_\infty x - C, \quad \forall\, t \in J, \ x \geq 0.$$

Let $R = \max\{e^{(\lambda+|\lambda|)N}R_1, R_2, R_3\}$. Next we will show $\mu Ax \neq x$ for $x \in \partial K_R$ and $\mu \geq 1$. In fact, assume there exist $x_0 \in \partial K_R$ and $\mu \geq 1$ such that $\mu_0 Ax_0 = x_0$. Then $x_0(t)$ satisfies (5) and therefor

$$\int_0^N x_0(t))dt \geq (1-\varepsilon)f_\infty \int_0^N x_0(t))dt + (1-\varepsilon)I_\infty \sum_{k=1}^m x_0(t_k) - C(m+N).$$

If $f_\infty \leq 1$, by $(H2)$, we have

$$\|x_0\| \leq \frac{C(m+N)}{m(1-\varepsilon)I_\infty e^{(-\lambda-|\lambda|)N} - [1-(1-\varepsilon)f_\infty]N} \doteq R_2.$$

If $f_\infty > 1$, there exists $\varepsilon_1 > 0$ such that $(1-\varepsilon)f_\infty > 1$, we get

$$\|x_0\| \leq \frac{C(m+N)}{[(1-\varepsilon_1)f_\infty - 1]Ne^{(-\lambda-|\lambda|)N}} \doteq R_3.$$

Thus $\mu Ax \neq x$ for $x \in \partial K_R$ and $\mu \geq 1$. Nest we will show if $x \in \partial K_R$, $\inf_{x\in\partial K_R} \|Ax\| > 0$, $\forall t \in J$. Since $x(t) \geq e^{(-\lambda-|\lambda|)N}\|x\| = e^{(-\lambda-|\lambda|)N}R > R$, by (7), we know

$$\frac{f(t,x)}{\lambda} \geq e^{(-\lambda-|\lambda|)N}(1-\varepsilon)f_\infty R, \quad \frac{I(x)}{\lambda} \geq e^{(-\lambda-|\lambda|)N}(1-\varepsilon)I_\infty R, \quad \forall t \in J.$$

In a similar way, we can get $\inf_{x\in\partial K_R} \|Ax\| > 0$ and $A(K_R) \subset P$. consequently, from Lemma 1, we obtain

$$i(A, K_R, P) = 0. \tag{8}$$

In addition Eqs. (6), (8) and Lemma 3, we have

$$i(A, K_R \setminus \bar{K}_r, P) = -1, \quad i(A, K_r \setminus \bar{K}_{r_2}, P) = 1.$$

Thus A has fixed points $x_1 \in K_r \setminus \bar{K}_{r_2}$ and $x_2 \in K_R \setminus \bar{K}_r$, i.e. $x_1(t)$ and $x_2(t)$ are positive solutions of (1) with $0 < \|x_1\| < r < \|x_2\|$. □

Theorem 2. *Suppose that* $(H3)$ *and* $(H4)$ *hold. Then* (1) *at least has two positive solutions* x_1 *and* x_2, *witch satisfy* $0 < \| x_1 \| < r < \| x_2 \|$.

Proof. According to Lemma 5, we have that

$$i(A, K_r, P) = 0. \tag{9}$$

Suppose that (H_3) holds, there exists $0 < \varepsilon < \min\{e^{(\lambda+|\lambda|)N} - f^0, e^{(\lambda+|\lambda|)N} - f^\infty\}$ such that

$$e^{(\lambda+|\lambda|)N} - f^0 - \varepsilon < \frac{I^0+\varepsilon}{N}m, \tag{10}$$

$$e^{(\lambda+|\lambda|)N} - f^\infty - \varepsilon < \frac{I^\infty+\varepsilon}{N}m. \tag{11}$$

There exists a constant $0 < r_0 < r$ such that

$$\frac{f(t,x)}{\lambda} \leq (f^0+\varepsilon)x, \quad \frac{I(x)}{\lambda} \leq (\varepsilon + I_0)x, \quad \forall t \in J, \ 0 \leq x \leq r_0.$$

Let $r_1 \in (0, r_0)$.We now prove that $\mu Ax = x$ for any $x \in K_{r_1}$ and $0 < \mu \leq 1$. If this is not true, then there exist $\mu_0 \in \partial K_{r_1}$ and $0 < \mu_0 \leq 1$ such that $\mu_0 Ax_0 = x_0$. Then $x_0(t)$ satisfies Eq. (3.3). Integrate from 0 to T to obtain

$$\begin{aligned}\int_0^N x_0(t)\mathrm{dt} &= \int_0^N \frac{f(t, x_0(t))}{\lambda}\mathrm{dt} + \sum_{\mathrm{k}=1}^{\mathrm{m}} \frac{\mathrm{I}(\mathrm{x_0(t_k)})}{\lambda} \\ &\geq (\varepsilon + f^0)\int_0^N x_0(t)\mathrm{dt} + (\varepsilon + \mathrm{I}^0)\sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{x_0(t_k)}.\end{aligned}$$

Since $x_0(t) \in K_{r_1}$ and $f^0 + \varepsilon < \mathrm{e}^{(\lambda+|\lambda|)\mathrm{N}}$, we see that

$$\mathrm{e}^{(\lambda+|\lambda|)\mathrm{N}} - \mathrm{f}^0 - \varepsilon \geq \frac{\mathrm{I}^0 + \varepsilon}{\mathrm{N}}\mathrm{m},$$

which is a contradiction. By Lemma 3, we have

$$i(A, K_{r_1}, P) = 1. \tag{12}$$

On the other hand, there exists $H > r$ such that

$$\frac{f(t,x)}{\lambda} \leq (f^\infty + \varepsilon)x, \ \frac{I(x)}{\lambda} \leq (\varepsilon + I_\infty)x, \quad \forall\, t \in J,\ x \geq H.$$

And so there exist C such that

$$\frac{f(t,x)}{\lambda} \leq (f^\infty + \varepsilon)x + C, \ \frac{I(x)}{\lambda} \leq (\varepsilon + I_\infty)x + C, \quad \forall\, t \in J,\ x \geq 0. \tag{13}$$

Next we will show that if R is large enough, then $\mu Ax \neq x$ for any $x \in \partial K_R$ and $0 < \mu \leq 1$. In fact, if there exist $x_0 \in \partial K_R$ and $0 < \mu_0 \leq 1$ such that $\mu_0 Ax_0 = x_0$, then $x_0(t)$ satisfies Eq. (3.3). Integrate from 0 to 1 to obtain

$$\int_0^N x_0(t)\mathrm{dt} \geq (\varepsilon + f^\infty)\int_0^N x_0(t))\mathrm{dt} + (\varepsilon + \mathrm{I}^\infty)\sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{x_0(t_k)} + \mathrm{CN} + \mathrm{Cm}.$$

Since $x \in K_R$, from (13), and we have

$$\|x_0\| \leq \frac{C(N+m)}{(1 - f^\infty - \varepsilon)\mathrm{e}^{(-\lambda-|\lambda|)\mathrm{N}} - (\mathrm{I}^\infty + \varepsilon)\mathrm{m}} = R_1.$$

Let $R = \max\{r, R_1\}$. Then for any $x \in \partial K_R$ and $0 < \mu \leq 1$, we have $\mu Ax \neq x$. Hence hypothesis (ii) of Lemma 1 also holds. Consequently

$$i(A, K_R, P) = 1. \tag{14}$$

In addition Eq. (9), (12)and (14), we obtain

$$i(A, K_R \setminus \bar{K}_r, P) = 1, \quad i(A, K_r \setminus \bar{K}_{r_1}, P) = -1.$$

Thus, A has fixed points x_1 and x_2 in $K_r \setminus \bar{K}_{r_1}$ and $K_R \setminus \bar{K}_r$, which means $x_1(t)$ and $x_2(t)$ are positive solution of the problem (1.1) and $0 < \|x_1\| < r < \|x_2\|$. □

Theorem 3. *Suppose* $\frac{f(t,x)}{\lambda} \geq 0$, $\frac{I(x)}{\lambda} \geq 0$, *and*

$$\mathrm{e}^{(-\lambda-|\lambda|)\mathrm{N}}(\mathrm{f}_0 + \mathrm{m}\lambda\frac{\mathrm{e}^{-|\lambda|\mathrm{N}}}{1-\mathrm{e}^{-\lambda\mathrm{N}}}\mathrm{I}_0 > 1, \quad \mathrm{f}_0 + \mathrm{m}\lambda\frac{\mathrm{e}^{|\lambda|\mathrm{N}}}{1-\mathrm{e}^{-\lambda\mathrm{N}}}\mathrm{I}_0 < 1.$$

Then (1) has at least one positive solution.

Proof. The proof follows the ideas in the proof of Theorem 1. □

Theorem 4. *Suppose* $\frac{f(t,x)}{\lambda} \geq 0$, $\frac{I(x)}{\lambda} \geq 0$, *and*

$$f^0 + m\lambda\frac{\mathrm{e}^{|\lambda|\mathrm{N}}}{1-\mathrm{e}^{-\lambda\mathrm{N}}}I^0 < 1, \quad f^0 + m\lambda\frac{\mathrm{e}^{-|\lambda|\mathrm{N}}}{1-\mathrm{e}^{-\lambda\mathrm{N}}}I^0 > \frac{1}{\mathrm{e}^{(-\lambda-|\lambda|)\mathrm{N}}}.$$

Then (1) has at least one positive solution.

Proof. The proof follows the ideas in the proof of Theorem 2. □

3. Example

Consider the following impulsive boundary value problem:

$$\begin{cases} x'(t) = 10^{-3}x^3 + \frac{1}{10}\sin x - x + [5-t], & t \in [0,5],\ t \neq 1,2,3,4, \\ x(t_k^+) = x(t_k^-) + 30\mathrm{e}^{-10}, \\ x(0) = x(5). \end{cases} \tag{15}$$

where $f(t,x) = 10^{-3}x^3 + \frac{1}{10}\sin x + [5-t]$ ($[t]$ denotes the integer part of t), $I(x(t_k)) = 30\mathrm{e}^{-10}, \lambda = 1, \mathrm{N} = 5$. Let $r = 3$ and $\eta = \frac{1}{2}$, $\eta_k = 2 \times 10^{-3}$, it easy to see $(H1)$ and $(H2)$ hold. And so, by, we get (15) has two positive solutions x_1, x_2 with $0 < \|x_1\| < 3 < \|x_2\|$.

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