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COMPLETE CONVERGENCE FOR ARRAY OF ROWWISE DEPENDENT RANDOM VARIABLES

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ABSTRACT. Let $\{X_{ni}|1 \leq i \leq n,\ n \geq 1\}$ be an array of rowwise negatively associated random variables and let $\alpha > 1/2,\ 0 < p < 2$, $\alpha p \geq 1$. In this paper we discuss $n^{\alpha p-2}h(n)\max_{1\leq k\leq n}|\sum_{i=1}^{k} X_{ni}|/n^{\alpha} \to 0$ completely as $n \to \infty$ under not necessarily identically distributed with a suitable conditions and $h(x) > 0$ is a slowly varying function as $x \to \infty$. In addition, we obtained that $n^{\alpha p-2}h(n)\max_{1\leq k\leq n}|\sum_{i=1}^{k} X_{ni}|/n^{\alpha} \to 0$ completely as $n \to \infty$ if and only if $E|X_{11}|^p h(|X_{11}|^{1/\alpha}) < \infty$ and $EX_{11} = 0$ under identically distributed case and some corollaries are obtained.

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1. Introduction

Let $\{X_n|n \geq 1\}$ be a sequence of random variables. Hsu and Robbins (1947) introduced the concept of complete convergence of $\{X_n\}$. A sequence $\{X_n\}$ of random variables is said to converge completely to a constant c if

$$\sum_{n=1}^{\infty} P(|X_n - c| > \epsilon) < \infty \text{ for every } \epsilon > 0.$$

Moreover, it was proved that the sequence of arithmetic means of independent identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. This result has been

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generalized and extended in several directions and carefully studied by many authors (see, Pruitt(1966); Rohatgi(1971); Gut(1992); Wang et al.(1993); Kuczmaszewska and Szynal(1994); Ghosal and Chandra(1998); Hu et al.(1999,2001); Ahmed et al.(2002)).

Complete convergence for sequence of random variables play a central role in the area of limit theorems in probability theory and mathematical statistics. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel or Kolmogorov. Since then, serious attempts have been made to relax these strong conditions. For example, independence has been relaxed to pairwise independence or pairwise negative quadrant dependence or, even replaced by conditions of dependence such as mixing or martingale.

In particular, many authors showed that many results could be obtained by replacing i.i.d. condition by uniformly bounded condition. We recall that an array $\{X_{ni}|1 \leq i \leq n,\ n \geq 1\}$ of random variables is said to be uniformly bounded by a random variable X if for all n and $x \geq 0$,

$$\sup_{i\geq 1} P\Big(|X_{ni}| > x\Big) \leq P(|X| > x)$$

The main purpose of this paper, we discuss the complete convergence for sums of rowwise negatively associated random variables under suitable conditions, since independent and identically random variables are a spacial case of negatively associated random variables.

A finite sequence of random variables $\Big\{X_i|1 \leq i \leq n\Big\}$ is said to be negatively associated (NA) if for any two disjoint nonempty subsets A_1 and A_2 of $\{1, 2, \cdots, n\}$ and f_1 and f_2 are any two coordinatewise nondecreasing functions,

$$Cov\Big(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)\Big) \leq 0,$$

whenever the covariance is finite. If for every $n \geq 2$, $X_1, X_2, \cdots, X_n$ are NA, then the sequence $\{X_i|i \in N\}$ is said to be NA. This definition is introduced by Alam and Saxena (1981). Many authors derived several important properties about NA sequences and also discussed some applications in the area of statistics, probability, reliability and multivariate analysis. Compared to positively associated random variables, the study of NA random variables has received less attention in the literature. Readers may refer to Karlin and Rinott(1980 b), Joag-Dev and Proschan(1983), Matula(1992) and Roussas(1994) among others. Recently, some authors focussed on the problem of limiting behavior of partial sums of NA sequences. Su et al.(1997) derived some moment inequalities and weak convergence for NA sequences, Su and Qin(1997) studied some limiting results for NA sequences, Shao and Su(1999) discussed for law of the iterated logarithm, Liang(2000) and Baek et al(2003) considered some complete convergence for negatively dependent random variables.

2. Preliminaries

This section will contain a background materials which will be used in obtained the main results in the next section and C will represent positive constants whose value many change from one place to another.

Lemma 1. *(Hu et al.(1986)) For any $r \geq 1, E|X|^r < \infty$ if and only if*

$$\sum_{n=1}^{\infty} n^{r-1} P(|X| > \epsilon n) < \infty \text{ for any } \epsilon > 0 \text{ ;}$$

More precisely,

$$r2^{-r} \sum_{n=1}^{\infty} n^{r-1} P(|X| > n) \leq E|X|^r \leq 1 + r2^r \sum_{n=1}^{\infty} n^{r-1} P(|X| > n).$$

Lemma 2. *(Su et al.(1997), Shao (2000)) Let $p \geq 2$ and let $\{X_i | i \geq 1\}$ be a sequence of NA random variables with $EX_i = 0$ and $E|X_i|^p < \infty$. Then there exist constant $A_p > 0$ and $B_p > 0$ such that*

$$E|X_i|^p \leq A_p \left(\left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} + \sum_{i=1}^{n} E|X_i|^p \right),$$

$$E\left(\max_{1 \leq k \leq n} | \sum_{i=1}^{k} X_i|^p \right) \leq B_p \left(\left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} + \sum_{i=1}^{n} E|X_i|^p \right).$$

3. Main results

Theorem 1. *Suppose that $\alpha > \frac{1}{2}$, $0 < p < 2$ and $\alpha p \geq 1$. Suppose that $\left\{ X_{ni} | 1 \leq i \leq n, n \geq 1 \right\}$ be an array rowwise NA random variables with $EX_{ni} = 0$ and let $\sup_{i \geq 1} P(|X_{ni}| > x) \leq P(|X| > x)$ for all n and $x \geq 0$. Suppose that $h(x) > 0$ is a slowly varying function as $x \to \infty$. If $E|X|^p h(|X|^{1/\alpha}) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} h(n) P\left(\max_{1 \leq k \leq n} | \sum_{i=1}^{k} X_{ni}| \geq \epsilon n^{\alpha} \right) < \infty \quad \text{for every } \epsilon > 0. \tag{1}$$

Remark 1. On applying Theorem 1, we can extend the corresponding result of Baum and Katz(1965) from the i.i.d.case to the NA random variables. In particular, let $\left\{ X_{ni} | 1 \leq i \leq n, n \geq 1 \right\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$ and $\sup_{i \geq 1} P(|X_{ni}| > x) \leq P(|X| > x)$ for all n and $x \geq 0$.

(1) Let $r > 1, 1 \leq t < 2$. If $E|X|^{rt} < \infty$, then

$$\sum_{n=1}^{\infty} n^{r-2} P\left(|\sum_{i=1}^{n} X_{ni}| \geq \epsilon n^{1/t}\right) < \infty \quad \text{for every} \quad \epsilon > 0.$$

(2) Let $1 < t < 2$. If $E|X|^{t} < \infty$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(|\sum_{i=1}^{n} X_{ni}| > \epsilon n^{1/t}\right) < \infty \quad \text{for every} \quad \epsilon > 0.$$

(3) If $E(|X| log(1 + |X|)) < \infty$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(|\sum_{i=1}^{n} X_{ni}| > \epsilon n\right) < \infty \quad \text{for every} \quad \epsilon > 0.$$

Proof. Let $Y_{ni} = n^{\alpha} I(X_{ni} > n^{\alpha}) + X_{ni} I(|X_{ni}| \leq n^{\alpha}) - n^{\alpha} I(X_{ni} < -n^{\alpha})$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}| \geq \epsilon n^{\alpha}\right) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}| \geq \epsilon n^{\alpha}, |X_{ni}| < n^{\alpha}, 1 \leq i \leq n\right) \\
&+ \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\Big(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}| \geq \epsilon n^{\alpha}, \text{there exists } i \\
&\qquad\qquad \text{such that } |X_{ni}| \geq n^{\alpha}\Big) \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq i\leq n} |X_{ni}| \geq n^{\alpha}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n} |\sum_{i=1}^{k} Y_{ni} - EY_{ni}| \geq \epsilon n^{\alpha}\right) \\
&= I_1 + I_2 (\text{say}).
\end{aligned}$$

First, we show that $I_1 < \infty$. As to I_1,

$$\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq i\leq n} |X_{ni}| \geq n^{\alpha}\right) \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) \sum_{i=1}^{n} P(|X_{ni}| \geq n^{\alpha})
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{n=1}^{\infty} n^{\alpha p-1}h(n)P(|X| > n^{\alpha}) \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-1}h(n)\sum_{k=n}^{\infty} P(k \le |X|^{1/\alpha} < k+1) \\
&\le \sum_{n=1}^{\infty} P(k \le |X|^{1/\alpha} < k+1)\sum_{n=1}^{k} n^{\alpha p-1}h(n) \\
&\le C\sum_{k=1}^{\infty} h(|X|^{1/\alpha})k^{\alpha p}P(k \le |X|^{1/\alpha} < k+1) \\
&\le CE|X|^p h(|X|^{1/\alpha}) < \infty.
\end{aligned}$$

Secondly, in order to prove that $I_2 < \infty$, we prove that

$$n^{-\alpha} \max_{1\le k\le n}\left|\sum_{i=1}^{k} EY_{ni}\right| \to 0 \ as \ n \to \infty.$$

$$\begin{aligned}
&n^{-\alpha} \max_{1\le k\le n}\left|\sum_{i=1}^{k} EY_{ni}\right| \\
&\le n^{-\alpha} \max_{1\le k\le n}\left|\sum_{i=1}^{k} EX_{ni}I(|X_{ni}| \le n^{\alpha}) + \sum_{i=1}^{k} n^{\alpha}P(|X_{ni}| > n^{\alpha})\right| \\
&\le n^{1-\alpha}E|X|I(|X| \le n^{\alpha}) + nP(|X| > n^{\alpha}) \\
&= I_3 + I_4(\text{say}).
\end{aligned}$$

To show that $I_3 < \infty$ and $I_4 < \infty$, we first need to prove that

$$\sum_{k=1}^{\infty} kP(k \le |X|^{1/\alpha} < k+1) < \infty. \tag{2}$$

Without loss of generality, we assume that $h(x) \ge c > 0$ for $\alpha p = 1$, then we obtain that

$$\begin{aligned}
&\sum_{k=1}^{\infty} kP(k \le |X|^{1/\alpha} < k+1) \\
&\le C\sum_{k=1}^{\infty} kh(k)P(k \le |X|^{1/\alpha} < k+1) \\
&\le C\sum_{k=1}^{\infty} kP(k \le |X|^{1/\alpha} < k+1)h(|X|^{1/\alpha}) \\
&\le CE|X|^p h(|X|^{1/\alpha}) < \infty, \quad \text{by Lemma 2.1}
\end{aligned} \tag{3}$$

Also, noticing that $\alpha p > 1$ and $h(x)$ is a slowly varying function and for $k \geq n$, $k^{1-\alpha p}h^{-1}(k) < 1$, we have that

$$\begin{aligned}
&\sum_{k=1}^{\infty} kP(k \leq |X|^{1/\alpha} < k+1) \\
&\leq \sum_{k=1}^{N-1} kP(k \leq |X|^{1/\alpha} < k+1) + \sum_{k=N}^{\infty} kP(k \leq |X|^{1/\alpha} < k+1) \\
&\leq C + \sum_{k=N}^{\infty} k^{\alpha p}h(k)P(k \leq |X|^{1/\alpha} < k+1) \\
&\leq CE|X|^p h(|X|^{1/\alpha}) < \infty, \qquad (4)
\end{aligned}$$

which, together with (3) and (4), follows (2).

When $1/2 < \alpha \leq 1$, since $EX_{ni} = 0$ and $\sum_{k=1}^{\infty} kP(k \leq |X|^{1/\alpha} < k+1) < \infty$, we obtain that

$$\begin{aligned}
I_3 &= n^{1-\alpha}E|X|I(|X| \leq n^{\alpha}) \\
&= n^{1-\alpha}E|X|I(|X| > n^{\alpha}) \\
&\leq n^{1-\alpha}\sum_{k=n}^{\infty} k^{\alpha}P(k \leq |X|^{1/\alpha} < k+1) \\
&\leq C\sum_{k=n}^{\infty} kP(k \leq |X|^{1/\alpha} < k+1) < \infty.
\end{aligned}$$

When $\alpha > 1$, by $\sum_{k=1}^{\infty} kP(k \leq |X|^{1/\alpha} < k+1) < \infty$ and Kronecker Lemma, we obtain that

$$\begin{aligned}
I_3 &= n^{1-\alpha}E|X|I(|X| \leq n^{\alpha}) \\
&\leq Cn^{1-\alpha}\sum_{k=0}^{n} k^{\alpha}P(k \leq |X|^{1/\alpha} < k+1) \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Now, as to I_4, by $\sum_{k=1}^{\infty} kP(k \leq |X|^{1/\alpha} < k+1) < \infty$, we get

$$\begin{aligned}
I_4 &= nP(|X| > n^{\alpha}) \\
&\leq Cn\sum_{k=n}^{\infty} P(k \leq |X|^{1/\alpha} < k+1) \\
&\leq C\sum_{k=n}^{\infty} kP(k \leq |X|^{1/\alpha} < k+1) < \infty.
\end{aligned}$$

Hence, we have that

$$n^{-\alpha} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} EY_{ni} \right| \longrightarrow 0 \quad as\ n \to \infty.$$

Thus, it suffices to show that

$$I_2^* := \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1 \le k \le n} \mid \sum_{i=1}^{k} Y_{ni} - EY_{ni} \mid \ge \epsilon n^{\alpha} \right) < \infty \quad \text{for every} \quad \epsilon > 0.$$

In fact, we use Markov's inequality for a suitably large M which will be determined later, and note that $\{Y_{ni} \mid 1 \le i \le n, n \ge 1\}$ is still an array of rowwise NA random variables from the definition of NA.
Then, we obtain that for $M \ge 2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1 \le k \le n} \mid \sum_{i=1}^{k} Y_{ni} - EY_{ni} \mid \ge \epsilon n^{\alpha} \right) \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M} h(n) \sum_{i=1}^{n} E|Y_{ni}|^M + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M} h(n) \left(\sum_{i=1}^{n} E|Y_{ni}|^2 \right)^{M/2} \\
&= I_5 + I_6 (\text{say}).
\end{aligned}$$

First, as to I_5,

$$\begin{aligned}
I_5 &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M} h(n) \sum_{i=1}^{n} E|Y_{ni}|^M \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M} h(n) \sum_{i=1}^{n} (E|X_{ni}|^M I(|X_{ni}| \le n^{\alpha}) + n^{\alpha M} P(|X_{ni}| > n^{\alpha})) \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha M} h(n) E|X|^M I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-1} h(n) P(|X| > n^{\alpha}) \\
&= I_7 + I_8 (\text{say}).
\end{aligned}$$

When $\alpha(M-p) > 0$, we get that

$$\begin{aligned}
I_7 &= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha M} h(n) E|X|^M I(|X| \le n^{\alpha}) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha M} h(n) \sum_{k=1}^{n} n^{\alpha M} P(k \le |X|^{1/\alpha} < k+1) \\
&\le C \sum_{k=1}^{\infty} k^{\alpha M} P(k \le |X|^{1/\alpha} < k+1) \sum_{n=k}^{\infty} n^{\alpha p-1-\alpha M} h(n)
\end{aligned}$$

$$\le C\sum_{k=1}^{\infty} k^{\alpha p-1}h(|X|^{1/\alpha})P(k\le |X|^{1/\alpha}<k+1)$$
$$\le C\sum_{k=1}^{\infty} k^{\alpha p}h(|\ X\ |^{1/\alpha})P(k\le |X|^{1/\alpha}<k+1)$$
$$\le CE|X|^p h(|X|^{1/\alpha})<\infty. \tag{5}$$

Also, when $\alpha(M-p)>0$, we have that

$$I_8=\sum_{n=1}^{\infty} n^{\alpha p-1}h(n)P(|X|>n^{\alpha})$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha p-1}h(n)\sum_{k=n}^{\infty}P(k\le |X|^{1/\alpha}<k+1)$$
$$\le C\sum_{k=1}^{\infty}P(k\le |X|^{1/\alpha}<k+1)\sum_{n=1}^{k} n^{\alpha p-1}h(n)$$
$$\le C\sum_{k=1}^{\infty}h(|X|^{1/\alpha})k^{\alpha p}P(k\le |X|^{1/\alpha}<k+1)$$
$$\le CE|X|^p h(|\ X\ |^{1/\alpha})<\infty. \tag{6}$$

Thus, by (5) and (6), the proof of I_5 is completed.

If $\alpha p>2$, note that $E|X|^p<\infty$, taking that $M>2$, we get that

$$I_6\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M}h(n)\left(\sum_{i=1}^{n}EX^2I(|X|\le n^{\alpha})+\sum_{i=1}^{n}n^{2\alpha}P(|X|>n^{\alpha})\right)^{M/2}$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M}h(n)(E|X|^p n^{1+\alpha(2-p)})^{M/2}$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha p-2+(1-\alpha p)M/2}h(n)\ <\infty. \tag{7}$$

If $1<\alpha p\le 2$ and $0<r<p<2$, note that $E|X|^p h(|X|^{1/\alpha})<\infty$ implies $E|X|^{p-r}<\infty$, taking M large enough such that $M>\dfrac{2(\alpha p-1)}{(p-r)\alpha-1}$, we get that

$$I_6=\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M}h(n)\left(\sum_{i=1}^{n}E|Y_{ni}|^2\right)^{M/2}$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M}h(n)\left(\sum_{i=1}^{n}(E|X|^2 I(|X|\leq n^{\alpha})+n^{2\alpha}P(|X|^2>n^{\alpha})\right)^{M/2}$$

$$= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M+M/2}h(n)\left(\int_0^{n^{2\alpha}} P(|X|^2>x)dx\right)^{M/2}$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-M/2-(p-r)\alpha M/2}h(n) < \infty. \tag{8}$$

If $\alpha p = 1$ and $M=2$, similar to the above proof of I_5, we get $I_6<\infty$. (9)

Thus, by (7), (8) and (9), the proof of I_6 is completed. □

Theorem 2. *Let $\alpha > \frac{1}{2}$, $0<p<2$ and $\alpha p \geq 1$, and let $\left\{X_{ni}|1\leq i\leq n,\ n\geq 1\right\}$ be an array of rowwise NA random variables. Suppose that there is a random variable X such that $P(|X|>x)\leq \sup_{i\geq 1} P(|X_{ni}|\geq x)$ for all n and $x\geq 0$ and let $h(x)>0$ is a slowly varying function as $x\to\infty$. Assume that (1) holds, then*

$$E|X|^p h(|X|^{1/\alpha})<\infty \text{ and } \lim_{n\to\infty} n^{\alpha p-2}h(n)\left(\max_{1\leq k\leq n}\sum_{i=1}^{k} EX_{ni}/n^{\alpha}\right)=0.$$

Proof. Note that (1) implies that for every $\epsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\max_{1\leq k\leq n}|\sum_{i=1}^{k} X_{ni}|\geq \epsilon n^{\alpha}\right)<\infty \tag{10}$$

and

$$P\left(\max_{1\leq i\leq n}|X_{ni}|\geq \epsilon n^{\alpha}\right)\longrightarrow 0 \text{ as } n\to\infty. \tag{11}$$

Note that

$$\sum_{i=1}^{n} P(|X_{ni}|\geq \epsilon n^{\alpha})$$
$$=\sum_{i=1}^{n} P\left(|X_{ni}|\geq \epsilon n^{\alpha},\ \max_{1\leq k\leq i-1}|X_{nk}|<\epsilon n^{\alpha}\right)$$
$$+\sum_{i=1}^{n} P\left(|X_{ni}|\geq \epsilon n^{\alpha},\ \max_{1\leq k\leq i-1}|X_{nk}|\geq \epsilon n^{\alpha}\right)$$

$$= P\left(\max_{1\le i\le n}|X_{ni}|\ge \epsilon n^{\alpha}\right)+\sum_{i=1}^{n}P\left(|X_{ni}|\ge \epsilon n^{\alpha},\max_{1\le k\le i-1}|X_{nk}|\ge \epsilon n^{\alpha}\right)$$
$$= I_9 + I_{10}(\text{say}). \tag{12}$$

As to I_{10},

$$\begin{aligned}
I_{10} &= \sum_{i=1}^{n}P\left(|X_{ni}|\ge \epsilon n^{\alpha},\max_{1\le k\le i-1}|X_{nk}|\ge \epsilon n^{\alpha}\right)\\
&= \sum_{i=1}^{n}EI(|X_{ni}|\ge \epsilon n^{\alpha})I\left(\max_{1\le k\le i-1}|X_{nk}|\ge \epsilon n^{\alpha}\right)\\
&\le E\sum_{i=1}^{k}(I(|X_{ni}|\ge \epsilon n^{\alpha})-P(|X_{ni}|\ge \epsilon n^{\alpha}))I\left(\max_{1\le k\le n}|X_{nk}|\ge \epsilon n^{\alpha}\right)\\
&\quad+\sum_{i=1}^{n}P(|X_{ni}|\ge \epsilon n^{\alpha})P\left(\max_{1\le k\le n}|X_{nk}|\ge \epsilon n^{\alpha}\right)\\
&= I_{11}+I_{12}(\text{say}).
\end{aligned}\tag{13}$$

By applying NA and Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
I_{11} &\le \left\{Var\left(\sum_{i=1}^{n}I(|X_{ni}|\ge \epsilon n^{\alpha})\right)P\left(\max_{1\le k\le n}|X_{nk}|\ge \epsilon n^{\alpha}\right)\right\}^{1/2}\\
&\le \Big\{2(Var(\sum_{i=1}^{n}I(X_{ni}\ge \epsilon n^{\alpha}))+Var\left(\sum_{i=1}^{n}I(X_{ni}<-\epsilon n^{\alpha})\right)\\
&\qquad\qquad \times P\left(\max_{1\le k\le n}|X_{nk}|\ge \epsilon n^{\alpha}\right)\Big\}^{1/2}\\
&\le \Big\{8A_2\left\{\sum_{i=1}^{n}P(X_{ni}\ge \epsilon n^{\alpha})+\sum_{i=1}^{n}P(X_{ni}<-\epsilon n^{\alpha})\right\}\\
&\qquad\qquad P\left(\max_{1\le k\le n}|X_{nk}|\ge \epsilon n^{\alpha}\right)\Big\}^{1/2}\\
&= \left\{8A_2\sum_{i=1}^{n}P(|X_{ni}|\ge \epsilon n^{\alpha})P\left(\max_{1\le k\le n}|X_{nk}|\ge \epsilon n^{\alpha}\right)\right\}^{1/2}\\
&\le \frac{1}{2}\sum_{i=1}^{n}P(|X_{ni}|\ge \epsilon n^{\alpha})+4A_2P\left(\max_{1\le k\le n}|X_{nk}|\ge \epsilon n^{\alpha}\right),\\
&\qquad\qquad by\ (ab)^{1/2}\le (a+b)/2
\end{aligned}\tag{14}$$

Hence, from (12),(13) and (14), we obtain that

$$\frac{1}{2}\sum_{i=1}^{n} P(|X_{ni}| \geq \epsilon n^{\alpha}) \leq (1+C)P\left(\max_{1\leq i\leq n} |X_{ni}| \geq \epsilon n^{\alpha}\right)$$
$$+\sum_{i=1}^{n} P(|X_{ni}| \geq \epsilon n^{\alpha})P\left(\max_{1\leq k\leq n} |X_{nk}| \geq \epsilon n^{\alpha}\right)$$

and from (11),

$$\sum_{i=1}^{n} P(|X_{ni}| \geq \epsilon n^{\alpha}) \leq CP\left(\max_{1\leq k\leq n} |X_{nk}| \geq \epsilon n^{\alpha}\right) \text{ for sufficiently large } n. \quad (15)$$

Since (15) and $\max_{1\leq k\leq n} |X_{nk}| \leq 2 \max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}|$, we get that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)\sum_{i=1}^{n} P(|X_{ni}| \geq \epsilon n^{\alpha})$$
$$\leq \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}| \geq \epsilon n^{\alpha}\right),$$

which, together with (10) and assumptions, we have that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)\sum_{i=1}^{n} P(|X_{ni}| \geq \epsilon n^{\alpha}) < \infty,$$
$$i.e. \sum_{n=1}^{\infty} n^{\alpha p-1}h(n)P(|X| \geq \epsilon n^{\alpha}) < \infty,$$

by applying Lemma 1, we obtain $E|X|^p h(|X|^{1/\alpha}) < \infty$.

Now, under $E|X|^p(h|X|^{1/\alpha}) < \infty$ and Theorem 1, we obtain that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_{ni} - EX_{ni}\right| \geq \epsilon n^{\alpha}\right) < \infty \text{ for every } \epsilon > 0. \quad (16)$$

(10) and (16) yield

$$\lim_{n\to\infty} n^{\alpha p-2}h(n)\left(\max_{1\leq k\leq n} E\sum_{i=1}^{k} X_{ni}/n^{\alpha}\right) = 0$$

The proof is completed. □

As an application, we can obtain the following two corollaries.

Corollary 1. Let $\{X_{ni}|1 \leq i \leq n,\ n \geq 1\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$ and suppose that $\sup_{i\geq 1} P(|X_{ni}| > x) \leq P(|X| > x)$ and $P(|X| > x) \leq \sup_{i\geq 1} P(|X_{ni}| > x)$ for all n and $x \geq 0$. Suppose that

$\alpha > 1/2$, $0 < p < 2$, $\alpha p \geq 1$ and $h(x) > 0$ is a slowly varying function as $x \to \infty$. Then the following two statements are equivalent:

(1) $E|X|^p h(|X|^{1/\alpha}) < \infty$,

(2) $\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_{ni}\right| \geq \epsilon n^{\alpha}\right) < \infty$ for every $\epsilon > 0$.

Corollary 2. Let $\left\{X_{ni}|1 \leq i \leq n,\ n \geq 1\right\}$ be an array of identically distributed rowwise NA random variables. Suppose that $\alpha > 1/2$, $0 < p < 2$, $\alpha p \geq 1$ and $h(x) > 0$ is a slowly varying function as $x \to \infty$. Then the following two statements are equivalent:

(1) $E|X_{11}|^p h(|X_{11}|^{1/\alpha}) < \infty$ and $EX_{ni} = 0$,

(2) $\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_{ni}\right| \geq \epsilon n^{\alpha}\right) < \infty$ for every $\epsilon > 0$.

Theorem 3. *Suppose that $\alpha > 1/2$ and $p \geq 2$. Let $\left\{X_{ni}|1 \leq i \leq n,\ n \geq 1\right\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$ and $\sup_{i\geq 1} P(|X_{ni}| > x) \leq P(|X| > x)$ for all n and $x \geq 0$, and $h(x) > 0$ is a slowly varying function as $x \to \infty$. If $E|X|^p h(|X|^{1/\alpha}) < \infty$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} h(n) P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_{ni}\right| \geq \epsilon n^{\alpha}\right) < \infty \text{ for every } \epsilon > 0.$$

Proof. As for the proof of Theorem 1, by applying Lemma 2, we get that for $p \geq 2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} h(n) P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_{ni}\right| \geq \epsilon n^{\alpha}\right) \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n} h(n) \frac{1}{n^{\alpha p}} \left(\sum_{i=1}^{n} E|X_{ni}|^p + (\sum_{i=1}^{n} EX_{ni}^2)^{p/2}\right) \\
&= \sum_{n=1}^{\infty} n^{-(1+\alpha p)} h(n) \sum_{i=1}^{n} E|X_{ni}|^p + \sum_{n=1}^{\infty} n^{-(1+\alpha p)} h(n) \left(\sum_{i=1}^{n} EX_{ni}^2\right)^{p/2} \\
&= I_{13} + I_{14} \text{ (say)}.
\end{aligned}$$

Hence, by the fact that $\sup_{i\geq 1} E|X_{ni}|^p \leq E|X|^p < \infty$, we have that

$$I_{13} \leq C \sum_{n=1}^{\infty} n^{-\alpha p} h(n) E|X|^p < \infty,$$

and

$$\begin{aligned} I_{14} &\leq C \sum_{n=1}^{\infty} n^{-(1+\alpha p)} h(n)(nEX^2)^{p/2} \\ &\leq C \sum_{n=1}^{\infty} n^{-(1+\alpha p+p/2)} h(n) \\ &= C \sum_{n=1}^{\infty} n^{-1-(\alpha+1/2)p} h(n) < \infty. \end{aligned}$$

The proof is completed. □

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