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ROUGHNESS BASED ON INTUITIONISTIC FUZZY SUBGROUPS

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ABSTRACT. Using the notion of intuitionistic fuzzy subgroups, its roughness is discussed. With respect to a congruence relation on a group, several properties about the lower and upper approximations of a subset of a group are investigated.

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1. Introduction

The theory of fuzzy set was initiated by Zadeh[24] and so many researchers were conducted on the generalizations of the notion of fuzzy sets. The idea of *intuitionistic fuzzy set* was first initiated by Atanassov [2, 3, 4] as a generalization of the notion of fuzzy sets. Intuitionistic fuzzy sets are applied to groups and rings (see [1, 5, 9, 13, 14]). Various theories and methods have been proposed to deal with incomplete and insufficient information in classification, concept formation, and data analysis in data mining. For example, fuzzy set theory [24], rough sets [21], computing with words [23, 25, 26], linguistic dynamic systems [22, 23], and many others, have been developed and applied to real-world problems. The concept of a rough set was originally proposed by Pawlak [20, 21] as a formal tool for modelling and processing incomplete information in information systems. Rough set theory has been conceived as a tool to conceptualize, organize and analyze various types of data, in particular, to deal with inexact, uncertain or vague knowledge in applications related to artificial intelligence. The algebraic approach to rough sets was studied in [6, 7, 8, 12, 15, 16, 17, 18, 19]. The theory of rough set is an extension of set theory,

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in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in the rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a nonempty intersection with the set. In this paper, we consider roughness based on intuitionistic fuzzy subgroups. We investigate several properties about the lower and upper approximations of a subset of a group with respect to a congruence relation.

2. Preliminaries

As an important generalization of the notion of fuzzy sets in a non-empty set G, Atanassov [2, 3] introduced the concept of an *intuitionistic fuzzy set* (IFS for short) defined on a non-empty set G as objects having the form

$$A = \Big\{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in G\Big\},$$

where the functions $\mu_A : R \to [0,1]$ and $\gamma_A : R \to [0,1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\gamma_A(x)$) of each element $x \in G$ to A respectively, and $0 \le \mu_A(x) + \gamma_A(x) \le 1$ for all $x \in G$. Such defined objects are studied by many authors (see for example two journals: 1. *Fuzzy Sets and Systems* and 2. *Notes on Intuitionistic Fuzzy Sets*) and have many interesting applications not only in mathematics (see Chapter 5 in the book [4]). For the sake of simplicity, we shall use the symbol $A = (G; \mu_A, \gamma_A)$ for the IFS

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in G\}.$$

Let $A = (G; \mu_A, \gamma_A)$ and $B = (G; \mu_B, \gamma_B)$ be IFSs in a set G. We define

- $A \subseteq B \Leftrightarrow (\forall x \in G)\Big(\mu_A(x) \le \mu_B(x),\ \gamma_A(x) \ge \gamma_B(x)\Big).$
- $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.
- $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$.
- $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$.

An IFS $A = (G; \mu_A, \gamma_A)$ in a group G is called an *intuitionistic fuzzy subgroup* of a group G (see [10]) if it satisfies:

(i) $\mu_A(xy) \ge \min\Big\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(xy) \le \max\{\mu_A(x), \gamma_A(y)\Big\},$

(ii) $\mu_A(x^{-1}) \ge \mu_A(x)$ and $\gamma_A(x^{-1} \le \gamma_A(x)$.

for all $x, y \in G$. Let G be a group with identity e. Note that if $A = (G; \mu_A, \gamma_A)$ is an intuitionistic fuzzy subgroup of G, then

$$\mu_A(x^{-1}) = \mu_A(x),\ \gamma_A(x^{-1}) = \gamma_A(x),\ \mu_A(x) \le \mu_A(e),\ \text{ and }\ \gamma_A(x) \ge \gamma_A(e)$$

for all $x \in G$. An IFS $A = (G; \mu_A, \gamma_A)$ in a group G is called an *intuitionistic fuzzy normal subgroup* of G (see [?]) if it is an intuitionistic fuzzy subgroup of G and satisfies the conditions $\mu_A(xy) = \mu_A(yx)$ and $\gamma_A(xy) = \gamma_A(yx)$ for all $x, y \in G$.

3. Roughness based on intuitionistic fuzzy subgroups

In what follows let G denote a group with identity e unless otherwise specified.

Defitinition 3.1. Let $A = (G; \mu_A, \gamma_A)$ be an intuitionistic fuzzy normal subgroup of G. For each $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, the set

$$\mathscr{C}(A, \alpha, \beta) = U(\mu_A, \alpha) \cap L(\gamma_A, \beta)$$

is called a (α, β)*-level relation* of $A = (G; \mu_A, \gamma_A)$, where

$$U(\mu_A, \alpha) := \Big\{(a, b) \in G \times G \mid \mu_A(ab^{-1}) \geq \alpha\Big\}$$

and

$$L(\gamma_A, \beta) := \Big\{(a, b) \in G \times G \mid \gamma_A(ab^{-1}) \leq \beta\Big\}$$

which are called the α*-level relation* of μ_A and the β*-level relation* of γ_A, respectively.

An equivalence relation θ on G is called a *congruence relation* if it satisfies:

$$(\forall a, b, x \in G)\,((a, b) \in \theta \Rightarrow (ax, bx) \in \theta,\ (xa, xb) \in \theta).$$

Lemma 3.2. *Let $A = (G; \mu_A, \gamma_A)$ be an intuitionistic fuzzy normal subgroup of G and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, Then $\mathscr{C}(A, \alpha, \beta)$ is a congruence relation on G.*

Proof. Obviously $(a, a) \in \mathscr{C}(A, \alpha, \beta)$ for all $a \in G$. L et $a, b \in G$. If $(a, b) \in \mathscr{C}(A, \alpha, \beta)$, then $\mu_A(ab^{-1}) \geq \alpha$ and $\gamma_A(ab^{-1}) \leq \beta$. Since $A = (G; \mu_A, \gamma_A)$ is an intuitionistic fuzzy subgroup of G, it follows that

$$\mu_A(ba^{-1}) = \mu_A\Big((ab^{-1})^{-1}\Big) \geq \mu_A(ab^{-1}) \geq \alpha,$$

$$\gamma_A(ba^{-1}) = \gamma_A\Big((ab^{-1})^{-1}\Big) \leq \gamma_A(ab^{-1}) \leq \beta$$

so that $(b, a) \in \mathscr{C}(A, \alpha, \beta)$. Let $a, b, c \in G$ satisfy $(a, b) \in \mathscr{C}(A, \alpha, \beta)$ and $(b, c) \in \mathscr{C}(A, \alpha, \beta)$. Then

$$\begin{aligned} \mu_A(ac^{-1}) &= \mu_A((ae)c^{-1}) = \mu_A(a(b^{-1}b)c^{-1}) = \mu_A((ab^{-1})(bc^{-1})) \\ &\geq \min\Big\{\mu_A(ab^{-1}), \mu_A(bc^{-1})\Big\} \geq \min\{\alpha, \alpha\} = \alpha, \end{aligned}$$

$$\begin{aligned} \mu_A(ac^{-1}) &= \gamma_A((ae)c^{-1}) = \gamma_A(a(b^{-1}b)c^{-1}) = \gamma_A((ab^{-1})(bc^{-1})) \\ &\leq \max\Big\{\gamma_A(ab^{-1}), \gamma_A(bc^{-1})\Big\} \leq \max\{\beta, \beta\} = \beta, \end{aligned}$$

and so $(a,c) \in \mathscr{C}(A,\alpha,\beta)$. Therefore $\mathscr{C}(A,\alpha,\beta)$ is an equivalence relation on G. Now let $a,b,x \in G$ be such that $(a,b) \in \mathscr{C}(A,\alpha,\beta)$. Then

$$\begin{aligned}\mu_A((ax)(bx)^{-1}) &= \mu_A\Big((ax)(x^{-1}b^{-1})\Big) = \mu_A(a(xx^{-1})b^{-1}) \\ &= \mu_A(aeb^{-1}) = \mu_A(ab^{-1}) \geq \alpha,\end{aligned}$$

$$\begin{aligned}\gamma_A((ax)(bx)^{-1}) &= \gamma_A\Big((ax)(x^{-1}b^{-1})\Big) = \gamma_A(a(xx^{-1})b^{-1}) \\ &= \gamma_A(aeb^{-1}) = \gamma_A(ab^{-1}) \leq \beta,\end{aligned}$$

and thus $(ax,bx) \in \mathscr{C}(A,\alpha,\beta)$. Since $A = (G;\mu_A,\gamma_A)$ is intuitionistic fuzzy normal, we have

$$\begin{aligned}\mu_A\Big(((xa)(xb)^{-1}\Big) &= \mu_A\Big((xa)(b^{-1}x^{-1})\Big) = \mu_A\Big((b^{-1}x^{-1})(xa)\Big) \\ &= \mu_A\Big(b^{-1}(x^{-1}x)a\Big) = \mu_A(b^{-1}ea) \\ &= \mu_A(b^{-1}a) = \mu_A(ab^{-1}) \geq \alpha,\end{aligned}$$

$$\begin{aligned}\gamma_A\Big(((xa)(xb)^{-1}\Big) &= \gamma_A\Big((xa)(b^{-1}x^{-1})\Big) = \gamma_A\Big((b^{-1}x^{-1})(xa)\Big) \\ &= \gamma_A\Big(b^{-1}(x^{-1}x)a\Big) = \gamma_A(b^{-1}ea) \\ &= \gamma_A(b^{-1}a) = \gamma_A(ab^{-1}) \leq \beta\end{aligned}$$

which yields $(xa,xb) \in \mathscr{C}(A,\alpha,\beta)$. Therefore $\mathscr{C}(A,\alpha,\beta)$ is a congruence relation on G. □

Lemma 3.3. *Let $A = (G;\mu_A,\gamma_A)$ and $B = (G;\mu_B,\gamma_B)$ be intuitionistic fuzzy normal subgroups of G and let $\alpha,\beta \in [0,1]$ with $\alpha+\beta \leq 1$. Then*

$$\mathscr{C}\Big(A \cap B,\alpha,\beta\Big) = \mathscr{C}(A,\alpha,\beta) \cap \mathscr{C}(B,\alpha,\beta)).$$

Proof. Let $a,b \in G$ be such that $(a,b) \in \mathscr{C}(A \cap B,\alpha,\beta)$. Then $(a,b) \in U(\mu_A \wedge \mu_B,t) \cap L(\gamma_A \vee \gamma_B,s)$, and so $\min\Big\{\mu_A(ab^{-1}),\mu_B(ab^{-1})\Big\} = (\mu_A \wedge \mu_B)(ab^{-1}) \geq \alpha$ and $\max\Big\{\gamma_A(ab^{-1}),\gamma_B(ab^{-1})\Big\} = (\gamma_A \vee \gamma_B)(ab^{-1}) \leq \beta$. Hence $\mu_A(ab^{-1}) \geq \alpha$, $\mu_B(ab^{-1}) \geq \alpha$, $\gamma_A(ab^{-1}) \leq \beta$, and $\gamma_B(ab^{-1}) \leq \beta$. Therefore

$(a,b) \in \mathscr{C}(A,\alpha,\beta) \cap \mathscr{C}(B,\alpha,\beta)$ and so $\mathscr{C}(A \cap B,\alpha,\beta) \subseteq \mathscr{C}(A,\alpha,\beta) \cap \mathscr{C}(B,\alpha,\beta)$.

Conversely, if $(a,b) \in \mathscr{C}(A,\alpha,\beta) \cap \mathscr{C}(B,\alpha,\beta)$, then $(a,b) \in \mathscr{C}(A,\alpha,\beta) = U(\mu_A,\alpha) \cap L(\gamma_A,\beta)$ and $(a,b) \in \mathscr{C}(B,\alpha,\beta) = U(\mu_B,\alpha) \cap L(\gamma_B,\beta)$. Thus

$$(\mu_A \wedge \mu_B)(ab^{-1}) = \min\Big\{\mu_A(ab^{-1}),\mu_B(ab^{-1})\Big\} \geq \min\{\alpha,\alpha\} = \alpha,$$

$$(\gamma_A \vee \gamma_B)(ab^{-1}) = \max\Big\{\gamma_A(ab^{-1}),\gamma_B(ab^{-1})\Big\} \leq \min\{\beta,\beta\} = \beta,$$

which imply that $(a,b) \in U(\mu_A \wedge \mu_B,\alpha) \cap L(\gamma_A \vee \gamma_B,\beta) = \mathscr{C}(A \cap B,\alpha,\beta)$. This completes the proof. □

Let $A = (G; \mu_A, \gamma_A)$ be an intuitionistic fuzzy subgroup of G. For any $x \in G$, we denote by $[x]^A_{(\alpha,\beta)}$ the equivalence class of x with respect to $\mathscr{C}(A, \alpha, \beta)$, i.e.,

$$[x]^A_{(\alpha,\beta)} = \Big\{ y \in G \mid (x, y) \in \mathscr{C}(A, \alpha, \beta) \Big\}.$$

We will use the notation $[x]_{(\alpha,\beta)}$ instead of $[x]^A_{(\alpha,\beta)}$ if there is no confusion. For any nonempty subset X of G, the sets

$$G_*(\mathscr{C}(A, \alpha, \beta), X) := \Big\{ x \in G \mid [x]^A_{(\alpha,\beta)} \subseteq X \Big\},$$

$$G^*(\mathscr{C}(A, \alpha, \beta), X) := \Big\{ x \in G \mid [x]^A_{(\alpha,\beta)} \cap X \neq \emptyset \Big\}$$

are called, respectively, the *lower* and *upper approximations* of X with respect to $\mathscr{C}(A, \alpha, \beta)$.

Proposition 3.4. *Let $A = (G; \mu_A, \gamma_A)$ and $B = (G; \mu_B, \gamma_B)$ be intuitionistic fuzzy subgroups of G and let $\alpha, \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1$. Then*

(i) *$A \subseteq B$ implies $[a]^A_{(\alpha,\beta)} \subseteq [a]^B_{(\alpha,\beta)}$ for all $a \in G$.*

(ii) *If $\epsilon, \delta \in [01,]$ satisfies $\epsilon + \delta \leq 1$ and $(\epsilon, \delta) \leq (\alpha, \beta)$, i.e., $\epsilon \leq \alpha$ and $\delta \geq \beta$, then $[a]^A_{(\alpha,\beta)} \subseteq [a]^A_{(\epsilon,\delta)}$ for all $a \in G$.*

(iii) *If $\mathscr{C}(A, \alpha, \beta) \subseteq \mathscr{C}(B, \alpha, \beta)$, then $[x]^A_{(\alpha,\beta)} \subseteq [x]^B_{(\alpha,\beta)}$ for all $x \in G$.*

(iv) *For every $a \in G$ we have $[a]^{A\cap B}_{(\alpha,\beta)} = [a]^A_{(\alpha,\beta)} \cap [a]^B_{(\alpha,\beta)}$.*

Proof. (i) Let $x \in [a]^A_{(\alpha,\beta)}$. Then $(a, x) \in \mathscr{C}(A, \alpha, \beta) = U(\mu_A, \alpha) \cap L(\gamma_A, \beta)$, and so $\mu_A(ax^{-1}) \geq \alpha$ and $\gamma_A(ax^{-1}) \leq \beta$. Since $A \subseteq B$, it follows that

$$\mu_B(ax^{-1}) \geq \mu_A(ax^{-1}) \geq \alpha \quad \text{and} \quad \gamma_B(ax^{-1}) \leq \gamma_A(ax^{-1}) \leq \beta$$

so that $(a, x) \in U(\mu_B, \alpha) \cap L(\gamma_B, \beta) = \mathscr{C}(B, \alpha, \beta)$. Hence $x \in [a]^B_{(\alpha,\beta)}$.

(ii) If $x \in [a]^A_{(\alpha,\beta)}$, then $(a, x) \in \mathscr{C}(A, \alpha, \beta)$, and so $\mu_A(ax^{-1}) \geq \alpha \geq \epsilon$ and $\gamma_A(ax^{-1}) \leq \beta \leq \delta$. Hence $(a, x) \in U(\mu_B, \epsilon) \cap L(\gamma_B, \delta) = \mathscr{C}(A, \epsilon, \delta)$, and thus $x \in [a]^A_{(\epsilon,\delta)}$. This completes the proof.

(iii) Let $a \in [x]^A_{(\alpha,\beta)}$. Then $(x, a) \in \mathscr{C}(A, \alpha, \beta) \subseteq \mathscr{C}(B, \alpha, \beta)$, and so $a \in [x]^B_{(\alpha,\beta)}$. Hence $[x]^A_{(\alpha,\beta)} \subseteq [x]^B_{(\alpha,\beta)}$.

(iv) If $x \in [a]^{A\cap B}_{(\alpha,\beta)}$, then $(a, x) \in \mathscr{C}(A \cap B, \alpha, \beta) = U(\mu_A \wedge \mu_B, \alpha) \cap L(\gamma_A \vee \gamma_B, \beta)$ and so

$$\min \Big\{ \mu_A(ax^{-1}), \mu_B(ax^{-1}) \Big\} = (\mu_A \wedge \mu_B)(ax^{-1}) \geq \alpha,$$

$$\max \Big\{ \gamma_A(ax^{-1}), \gamma_B(ax^{-1}) \Big\} = (\gamma_A \vee \gamma_B)(ax^{-1}) \leq \beta.$$

It follows that $\mu_A(ax^{-1}) \geq \alpha$, $\gamma_A(ax^{-1}) \leq \beta$, $\mu_B(ax^{-1}) \geq \alpha$, and $\gamma_B(ax^{-1}) \leq \beta$ so that

$$(a, x) \in U(\mu_A, \alpha) \cap L(\gamma_A, \beta) = \mathscr{C}(A, \alpha, \beta)$$

and

$$(a, x) \in U(\mu_B, \alpha) \cap L(\gamma_B, \beta) = \mathscr{C}(B, \alpha, \beta),$$

that is, $x \in [a]^A_{(\alpha,\beta)} \cap [a]^B_{(\alpha,\beta)}$. Conversely, suppose that $x \in [a]^A_{(\alpha,\beta)} \cap [a]^B_{(\alpha,\beta)}$. Then

$$(a, x) \in \mathscr{C}(A, \alpha, \beta) = U(\mu_A, \alpha) \cap L(\gamma_A, \beta)$$

and

$$(a, x) \in \mathscr{C}(B, \alpha, \beta) = U(\mu_B, \alpha) \cap L(\gamma_B, \beta).$$

Hence $\mu_A(ax^{-1}) \geq \alpha$, $\gamma_A(ax^{-1}) \leq \beta$, $\mu_B(ax^{-1}) \geq \alpha$, and $\gamma_B(ax^{-1}) \leq \beta$. It follows that

$$(\mu_A \wedge \mu_B)(ax^{-1}) = \min\left\{\mu_A(ax^{-1}), \mu_B(ax^{-1})\right\} \geq \alpha,$$

$$(\gamma_A \vee \gamma_B)(ax^{-1}) = \max\left\{\gamma_A(ax^{-1}), \gamma_B(ax^{-1})\right\} \leq \beta$$

so that $(a, x) \in U(\mu_A \wedge \mu_B, \alpha) \cap L(\gamma_A \vee \gamma_B, \beta) = \mathscr{C}(A \cap B, \alpha, \beta)$. Therefore $x \in [a]^{A\cap B}_{(\alpha,\beta)}$. □

Proposition 3.5. *Let $A = (G; \mu_A, \gamma_A)$ and $B = (G; \mu_B, \gamma_B)$ be intuitionistic fuzzy subgroups of G and let $\alpha, \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1$. For any nonempty subsets X and Y of G, we have:*

(1) $G_*\Big(\mathscr{C}(A, \alpha, \beta), X\Big) \subseteq X \subseteq G^*(\mathscr{C}(A, \alpha, \beta), X)$.

(2) $G_*\Big(\mathscr{C}(A, \alpha, \beta), \emptyset\Big) = \emptyset = G^*(\mathscr{C}(A, \alpha, \beta), \emptyset)$.

(3) $G_*\Big(\mathscr{C}(A, \alpha, \beta), G\Big) = G = G^*(\mathscr{C}(A, \alpha, \beta), G)$.

(4) $G^*\Big(\mathscr{C}(A, \alpha, \beta), X \cup Y\Big) = G^*(\mathscr{C}(A, \alpha, \beta), X) \cup G^*(\mathscr{C}(A, \alpha, \beta), Y)$.

(5) $G_*\Big(\mathscr{C}(A, \alpha, \beta), X \cap Y\Big) = G_*(\mathscr{C}(A, \alpha, \beta), X) \cap G_*(\mathscr{C}(A, \alpha, \beta), Y)$.

(6) $X \subseteq Y$ *implies* $G_*\Big(\mathscr{C}(A, \alpha, \beta), X\Big) \subseteq G_*(\mathscr{C}(A, \alpha, \beta), Y)$.

(7) $X \subseteq Y$ *implies* $G^*\Big(\mathscr{C}(A, \alpha, \beta), X\Big) \subseteq G^*(\mathscr{C}(A, \alpha, \beta), Y)$.

(8) $G^*\Big(\mathscr{C}(A, \alpha, \beta), G^*(\mathscr{C}(A, \alpha, \beta), X)\Big) = G^*(\mathscr{C}(A, \alpha, \beta), X)$.

(9) $G_*\Big(\mathscr{C}(A, \alpha, \beta), G^*(\mathscr{C}(A, \alpha, \beta), X)\Big) = G^*(\mathscr{C}(A, \alpha, \beta), X)$.

(10) $G^*\Big(\mathscr{C}(A, \alpha, \beta), X \cap Y\Big) \subseteq G^*(\mathscr{C}(A, \alpha, \beta), X) \cap G^*(\mathscr{C}(A, \alpha, \beta), Y)$.

(11) $G_*\Big(\mathscr{C}(A, \alpha, \beta), X \cup Y\Big) \supseteq G_*(\mathscr{C}(A, \alpha, \beta), X) \cup G_*(\mathscr{C}(A, \alpha, \beta), Y)$.

(12) $\mathscr{C}(A, \alpha, \beta) \subseteq \mathscr{C}(B, \alpha, \beta)$ *implies* $G^*\Big(\mathscr{C}(A, \alpha, \beta), X\Big) \subseteq G^*(\mathscr{C}(B, \alpha, \beta), X)$.

Proof. (1) If $a \in G_*(\mathscr{C}(A, \alpha, \beta), X)$, then $a \in [a]^A_{(\alpha,\beta)} \subseteq X$. Hence the first inclusion is valid. If $a \in X$, then $a \in [a]^A_{(\alpha,\beta)}$, and so $a \in [a]^A_{(\alpha,\beta)} \cap X$. Therefore $a \in G^*(\mathscr{C}(A, \alpha, \beta), X)$.

(2) and (3) are straightforward.

(4) We have

$$
\begin{aligned}
a \in G^*\Big(\mathscr{C}(A,\alpha,\beta), X\cup Y\Big) &\Leftrightarrow [a]^A_{(\alpha,\beta)} \cap (X\cup Y) \neq \emptyset \\
&\Leftrightarrow ([a]^A_{(\alpha,\beta)} \cap X) \cup ([a]^A_{(\alpha,\beta)} \cap Y) \neq \emptyset \\
&\Leftrightarrow [a]^A_{(\alpha,\beta)} \cap X \neq \emptyset \text{ or } [a]^A_{(\alpha,\beta)} \cap Y \neq \emptyset \\
&\Leftrightarrow a \in G^*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) \text{ or } a \in G^*(\mathscr{C}(A,\alpha,\beta), Y) \\
&\Leftrightarrow a \in G^*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) \cup a \in G^*\Big(\mathscr{C}(A,\alpha,\beta), Y\Big).
\end{aligned}
$$

Hence $G^*\Big(\mathscr{C}(A,\alpha,\beta), X\cup Y\Big) = G^*(\mathscr{C}(A,\alpha,\beta), X) \cup G^*(\mathscr{C}(A,\alpha,\beta), Y)$.

(5) We have

$$
\begin{aligned}
a \in G_*\Big(\mathscr{C}(A,\alpha,\beta), X\cap Y\Big) &\Leftrightarrow [a]^A_{(\alpha,\beta)} \subseteq X\cap Y \\
&\Leftrightarrow [a]^A_{(\alpha,\beta)} \subseteq X \text{ and } [a]^A_{(\alpha,\beta)} \subseteq Y \\
&\Leftrightarrow a \in G_*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) \text{ and } a \in G_*(\mathscr{C}(A,\alpha,\beta), Y) \\
&\Leftrightarrow a \in G_*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) \cap a \in G_*(\mathscr{C}(A,\alpha,\beta), Y),
\end{aligned}
$$

and so (5) is valid.

(6) and (7). Since $X \subseteq Y$, $X\cap Y = X$ and $X \cup Y = Y$. By (4) and (5),

$$
\begin{aligned}
G^*\Big(\mathscr{C}(A,\alpha,\beta), Y\Big) &= G^*(\mathscr{C}(A,\alpha,\beta), X\cup Y) \\
&= G^*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) \cup G^*(\mathscr{C}(A,\alpha,\beta), Y)
\end{aligned}
$$

and

$$
\begin{aligned}
G_*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) &= G_*(\mathscr{C}(A,\alpha,\beta), X\cap Y) \\
&= G_*(\mathscr{C}(A,\alpha,\beta), X) \cap G_*(\mathscr{C}(A,\alpha,\beta), Y)
\end{aligned}
$$

which yield

$$G^*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) \subseteq G^*(\mathscr{C}(A,\alpha,\beta), Y)$$

and

$$G_*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) \subseteq G_*(\mathscr{C}(A,\alpha,\beta), Y),$$

respectively.

(8) By (1), $G_*\Big(\mathscr{C}(A,\alpha,\beta), X\Big) \subseteq G^*\Big(\mathscr{C}(A,\alpha,\beta), G_*(\mathscr{C}(A,\alpha,\beta), X)\Big)$. If

$$x \in G^*\Big(\mathscr{C}(A,\alpha,\beta), G_*(\mathscr{C}(A,\alpha,\beta), X)\Big),$$

then $[x]^A_{(\alpha,\beta)} \cap G_*(\mathscr{C}(A,\alpha,\beta), X) \neq \emptyset$ and so there exists $a \in G$ such that $a \in [x]^A_{(\alpha,\beta)}$ and $a \in G_*(\mathscr{C}(A,\alpha,\beta), X)$. Hence $(x,a) \in \mathscr{C}(A,\alpha,\beta)$ and $[a]^A_{(\alpha,\beta)} \subseteq X$. We claim that $[x]^A_{(\alpha,\beta)} \subseteq X$. To do this, let $b \in [x]^A_{(\alpha,\beta)}$. Then $(x,b) \in \mathscr{C}(A,\alpha,\beta)$. Since $\mathscr{C}(A,\alpha,\beta)$ is an equivalence relation, it follows that $(a,b) \in \mathscr{C}(A,\alpha,\beta)$ so that $b \in [a]^A_{(\alpha,\beta)}$.

(9) By (1), $G_*\Big(\mathscr{C}(A,\alpha,\beta), G^*(\mathscr{C}(A,\alpha,\beta),X)\Big) \subseteq G^*(\mathscr{C}(A,\alpha,\beta),X)$. Let $x \in G^*(\mathscr{C}(A,\alpha,\beta),X)$. Then $[x]^A_{(\alpha,\beta)} \cap X \neq \emptyset$, and so $a \in X$ and $a \in [x]^A_{(\alpha,\beta)}$ for some $a \in G$. We claim that $[x]^A_{(\alpha,\beta)} \subseteq G^*(\mathscr{C}(A,\alpha,\beta),X)$. If $b \in [x]^A_{(\alpha,\beta)}$, then $(x,b) \in \mathscr{C}(A,\alpha,\beta)$. Since $\mathscr{C}(A,\alpha,\beta)$ is an equivalence relation, it follows that $(b,a) \in \mathscr{C}(A,\alpha,\beta)$, i.e., $a \in [b]^A_{(\alpha,\beta)}$ so that $[b]^A_{(\alpha,\beta)} \cap X \neq \emptyset$, i.e., $b \in G^*(\mathscr{C}(A,\alpha,\beta),X)$. Hence $[x]^A_{(\alpha,\beta)} \subseteq G^*(\mathscr{C}(A,\alpha,\beta),X)$, and so

$$x \in G_*\Big(\mathscr{C}(A,\alpha,\beta), G^*(\mathscr{C}(A,\alpha,\beta),X)\Big).$$

(10) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, it follows from (7) that

$$G^*\Big(\mathscr{C}(A,\alpha,\beta), X\cap Y\Big) \subseteq G^*(\mathscr{C}(A,\alpha,\beta),X),$$

$$G^*\Big(\mathscr{C}(A,\alpha,\beta), X\cap Y\Big) \subseteq G^*(\mathscr{C}(A,\alpha,\beta),Y)$$

so that $G^*\Big(\mathscr{C}(A,\alpha,\beta), X\cap Y\Big) \subseteq G^*(\mathscr{C}(A,\alpha,\beta),X) \cap G^*(\mathscr{C}(A,\alpha,\beta),Y)$.

(11) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, the result follows from (6).

(12) Let $x \in G^*(\mathscr{C}(A,\alpha,\beta),X)$. Then there exists $a \in G$ such that $a \in X$ and $a \in [x]^A_{(\alpha,\beta)}$. It follows from Proposition 3.4(iii) that $a \in [x]^B_{(\alpha,\beta)} \cap X$ so that $x \in G^*(\mathscr{C}(B,\alpha,\beta),X)$. □

Proposition 3.6. *Let $A = (G;\mu_A,\gamma_A)$ be intuitionistic fuzzy subgroup of G and let $\alpha,\beta \in [0,1]$ be such that $\alpha+\beta \leq 1$. If X and Y are nonempty subsets of G, then*

(i) $G^*\Big(\mathscr{C}(A,\alpha,\beta), XY\Big) = G^*(\mathscr{C}(A,\alpha,\beta),X)G^*(\mathscr{C}(A,\alpha,\beta),Y)$.

(ii) $G_*\Big(\mathscr{C}(A,\alpha,\beta), X\Big)G_*(\mathscr{C}(A,\alpha,\beta),Y) \subseteq G_*(\mathscr{C}(A,\alpha,\beta),XY)$.

Proof. (i) Let c be any element of $G^*\Big(\mathscr{C}(A,\alpha,\beta), XY\Big)$. Then there exists $x \in G$ such that $x \in XY$ and $x \in [c]^A_{(\alpha,\beta)}$. Thus $x = ab$ for some $a \in X$ and $b \in Y$. Since $c \in [x]^A_{(\alpha,\beta)} = [ab]^A_{(\alpha,\beta)} = [a]^A_{(\alpha,\beta)}[b]^A_{(\alpha,\beta)}$, we have $c = yz$ for some $y \in [a]^A_{(\alpha,\beta)}$ and $z \in [b]^A_{(\alpha,\beta)}$. Then $a \in [y]^A_{(\alpha,\beta)}$ and so $[y]^A_{(\alpha,\beta)} \cap X \neq \emptyset$. Thus $y \in G^*(\mathscr{C}(A,\alpha,\beta),X)$. Similarly, $z \in G^*(\mathscr{C}(A,\alpha,\beta),Y)$. Hence $c = yz \in G^*\Big(\mathscr{C}(A,\alpha,\beta), X\Big)G^*\Big(\mathscr{C}(A,\alpha,\beta), Y\Big)$.

Conversely, assume that $c \in G^*\Big(\mathscr{C}(A,\alpha,\beta), X\Big)G^*\Big(\mathscr{C}(A,\alpha,\beta), Y\Big)$. Then $c = ab$ for some $a \in G^*\Big(\mathscr{C}(A,\alpha,\beta), X\Big)$ and $b \in G^*\Big(\mathscr{C}(A,\alpha,\beta), Y\Big)$. Thus there exists $x \in G$ such that $x \in [a]^A_{(\alpha,\beta)}$ and $x \in X$; and there exists $y \in G$ such that $y \in Y$ and $y \in [b]^A_{(\alpha,\beta)}$. It follows that $xy \in Y$ and $xy \in [a]^A_{(\alpha,\beta)}[b]^A_{(\alpha,\beta)} = [ab]^A_{(\alpha,\beta)}$ so that $[ab]^A_{(\alpha,\beta)} \cap Y \neq \emptyset$. Hence $c = ab \in G^*\Big(\mathscr{C}(A,\alpha,\beta), XY\Big)$.

(ii) Let c be any element of $G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)G_*\Big(\mathscr{C}(A,\alpha,\beta),Y\Big)$. Then $c = ab$ for some $a \in G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)$ and $b \in G_*\Big(\mathscr{C}(A,\alpha,\beta),Y\Big)$. Hence $[a]^A_{(\alpha,\beta)} \subseteq X$ and $[b]^A_{(\alpha,\beta)} \subseteq Y$, and so $[c]^A_{(\alpha,\beta)} = [ab]^A_{(\alpha,\beta)} = [a]^A_{(\alpha,\beta)}[b]^A_{(\alpha,\beta)} \subseteq XY$, that is, $x \in G_*\Big(\mathscr{C}(A,\alpha,\beta),XY\Big)$. Therefore (ii) is valid. □

Theorem 3.7. *Let $A = (G;\mu_A,\gamma_A)$ and $B = (G;\mu_B,\gamma_B)$ be intuitionistic fuzzy subgroups of G and let $\alpha,\beta \in [0,1]$ be such that $\alpha+\beta \le 1$. For any nonempty subsets X of G, we have*

(i) $G^*\Big(\mathscr{C}(A\cap B,\alpha,\beta),X\Big) = G^*\Big(\mathscr{C}(A,\alpha,\beta)\cap\mathscr{C}(B,\alpha,\beta),X\Big)$
$= G^*(\mathscr{C}(A,\alpha,\beta),X)\cap G^*(\mathscr{C}(B,\alpha,\beta),X)$.

(ii) $G_*\Big(\mathscr{C}(A\cap B,\alpha,\beta),X\Big) = G_*(\mathscr{C}(A,\alpha,\beta),X)\cap G_*(\mathscr{C}(B,\alpha,\beta),X)$.

Proof. (i) Using Lemma 3.3, we have

$$G^*\Big(\mathscr{C}(A\cap B,\alpha,\beta),X\Big) = G^*(\mathscr{C}(A,\alpha,\beta)\cap\mathscr{C}(B,\alpha,\beta),X).$$

If $x \in G^*(\mathscr{C}(A\cap B,\alpha,\beta),X)$, then $[x]^{A\cap B}_{(\alpha,\beta)} \cap X \neq \emptyset$ and so there exists $a \in G$ such that $a \in X$ and $a \in [x]^{A\cap B}_{(\alpha,\beta)}$. It follows from Proposition 3.4(iv) that $a \in X$, $a \in [x]^A_{(\alpha,\beta)}$ and $a \in [x]^B_{(\alpha,\beta)}$ so that $a \in [x]^A_{(\alpha,\beta)} \cap X$ and $a \in [x]^B_{(\alpha,\beta)} \cap X$. Hence $x \in G^*\Big(\mathscr{C}(A,\alpha,\beta),X\Big) \cap G^*\Big(\mathscr{C}(B,\alpha,\beta),X\Big)$.

Now assume that $x \in G^*\Big(\mathscr{C}(A,\alpha,\beta),X\Big) \cap G^*\Big(\mathscr{C}(B,\alpha,\beta),X\Big)$. Then $x \in G^*(\mathscr{C}(A,\alpha,\beta),X)$ and $x \in G^*(\mathscr{C}(B,\alpha,\beta),X)$, and so $[x]^A_{(\alpha,\beta)} \cap X \neq \emptyset \neq [x]^B_{(\alpha,\beta)} \cap X$. Hence there exists $a \in X$, $a \in [x]^A_{(\alpha,\beta)}$ and $a \in [x]^B_{(\alpha,\beta)}$. It follows that $(x,a) \in \mathscr{C}(A,\alpha,\beta)\cap\mathscr{C}(B,\alpha,\beta)$ so that

$$(\mu_A \wedge \mu_B)(xa^{-1}) = \min\Big\{\mu_A(xa^{-1}),\mu_B(xa^{-1})\Big\} \ge \alpha,$$

$$(\gamma_A \vee \gamma_B)(xa^{-1}) = \max\Big\{\gamma_A(xa^{-1}),\gamma_B(xa^{-1})\Big\} \le \beta.$$

Therefore $(x,a) \in U(\mu_A\wedge\mu_B,\alpha)\cap L(\gamma_A\vee\gamma_B,\beta) = \mathscr{C}(A\cap B,\alpha,\beta)$, and thus $a \in [x]^{A\cap B}_{(\alpha,\beta)}$. Since $a \in X$, we have $[x]^{A\cap B}_{(\alpha,\beta)} \cap X \neq \emptyset$, that is, $x \in G^*\Big(\mathscr{C}(A\cap B,\alpha,\beta),X\Big)$. This proves (i).

(ii) Assume that $x \in G_*\Big(\mathscr{C}(A\cap B,\alpha,\beta),X\Big)$. Then $[x]^{A\cap B}_{(\alpha,\beta)} \subseteq X$, and so $[x]^A_{(\alpha,\beta)} \subseteq X$ and $[x]^B_{(\alpha,\beta)} \subseteq X$ by Proposition 3.4(iv). It follows that

$$x \in G_*(\mathscr{C}(A,\alpha,\beta),X)\cap G_*(\mathscr{C}(B,\alpha,\beta),X).$$

Conversely, if $y \in G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)\cap G_*\Big(\mathscr{C}(B,\alpha,\beta),X\Big)$, then $[y]^A_{(\alpha,\beta)} \subseteq X$ and $[y]^B_{(\alpha,\beta)} \subseteq X$, and therefore

$$[y]^{A\cap B}_{(\alpha,\beta)} = [y]^A_{(\alpha,\beta)} \cap [y]^B_{(\alpha,\beta)} \subseteq X.$$

Consequently, $y \in G_*(\mathscr{C}(A \cap B, \alpha, \beta), X)$. This completes the proof. □

Theorem 3.8. *Let $A = (G; \mu_A, \gamma_A)$ be an intuitionistic fuzzy normal subgroups of G and let $\alpha, \beta \in [0,1]$ be such that $\alpha + \beta \leq 1$. If X is a (normal) subgroup of G, then $G^*\left(\mathscr{C}(A, \alpha, \beta), X\right)$ is a (normal) subgroup of G.*

Proof. Assume that X is a subgroup of G. Obviously, $e \in G^*\left(\mathscr{C}(A, \alpha, \beta), X\right)$. Let $x, y \in G^*\left(\mathscr{C}(A, \alpha, \beta), X\right)$. Then there exists $a_x, a_y \in G$ such that $a_x \in [x]^A_{(\alpha,\beta)} \cap X$ and $a_y \in [y]^A_{(\alpha,\beta)} \cap X$. It follows that $a_x \in [x]^A_{(\alpha,\beta)}$, $a_y \in [y]^A_{(\alpha,\beta)}$, and $a_x, a_y \in X$. Since $\mathscr{C}(A, \alpha, \beta)$ is a congruence relation on G,

$$a_x a_y \in [x]^A_{(\alpha,\beta)}[y]^A_{(\alpha,\beta)} = [xy]^A_{(\alpha,\beta)}.$$

Also, since X is a subgroup of G, we have $a_x a_y \in X$ and so $a_x a_y \in [xy]^A_{(\alpha,\beta)} \cap X$. This implies that $xy \in G^*\left(\mathscr{C}(A, \alpha, \beta), X\right)$.

Let x be any element of $G^*\left(\mathscr{C}(A, \alpha, \beta), X\right)$. Then $a_x \in [x]^A_{(\alpha,\beta)} \cap X$ for some $a_x \in G$. Thus $a_x \in [x]^A_{(\alpha,\beta)}$ and $a_x \in X$. Since X is a subgroup of G, $a_x^{-1} \in X$. Since $\mathscr{C}(A, \alpha, \beta)$ is a congruence relation on G,

$$(a_x, x) \in \mathscr{C}(A, \alpha, \beta) \implies (x^{-1}, a_x^{-1}) = (x^{-1}a_x a_x^{-1}, x^{-1}x a_x^{-1}) \in \mathscr{C}(A, \alpha, \beta).$$

It follows that $a_x^{-1} \in [x^{-1}]^A_{(\alpha,\beta)}$ so that $[x^{-1}]^A_{(\alpha,\beta)} \cap X \neq \emptyset$, i.e.,

$$x^{-1} \in G^*(\mathscr{C}(A, \alpha, \beta), X).$$

Hence $G^*\left(\mathscr{C}(A, \alpha, \beta), X\right)$ is a subgroup of G. Suppose that X is a normal subgroup of G. Let $x \in G^*(\mathscr{C}(A, \alpha, \beta), X)$. Then there exists $a_x \in G$ such that $a_x \in [x]^A_{(\alpha,\beta)} \cap X$. Since X is normal, $(\forall g \in G)\,(g a_x g^{-1} \in gXg^{-1} \subseteq X)$. Since $\mathscr{C}(A, \alpha, \beta)$ is a congruence relation on G,

$$(a_x, x) \in \mathscr{C}(A, \alpha, \beta) \implies (g a_x g^{-1}, gxg^{-1}) \in \mathscr{C}(A, \alpha, \beta).$$

Hence $g a_x g^{-1} \in [gxg^{-1}]^A_{(\alpha,\beta)}$, and so $g a_x g^{-1} \in [gxg^{-1}]^A_{(\alpha,\beta)} \cap X$. Therefore $gxg^{-1} \in G^*(\mathscr{C}(A, \alpha, \beta), X)$, which shows that $G^*\left(\mathscr{C}(A, \alpha, \beta), X\right)$ is a normal subgroup of G. □

Theorem 3.9. *Let $A = (G; \mu_A, \gamma_A)$ be an intuitionistic fuzzy normal subgroups of G and let $\alpha, \beta \in [0,1]$ be such that $\alpha + \beta \leq 1$. If X is a (normal) subgroup of G and $[e]^A_{(\alpha,\beta)} \subseteq X$, then $G_*\left(\mathscr{C}(A, \alpha, \beta), X\right)$ is a (normal) subgroup of G.*

Proof. Suppose that X is a subgroup of G. Since $[e]^A_{(\alpha,\beta)} \subseteq X$, we have $e \in G_*(\mathscr{C}(A, \alpha, \beta), X)$. Let $x, y \in G_*\left(\mathscr{C}(A, \alpha, \beta), X\right)$. Then $[x]^A_{(\alpha,\beta)} \subseteq X$ and $[y]^A_{(\alpha,\beta)} \subseteq X$. Since $\mathscr{C}(A, \alpha, \beta)$ is a congruence relation on G,

$$[xy]^A_{(\alpha,\beta)} = [x]^A_{(\alpha,\beta)}[y]^A_{(\alpha,\beta)} \subseteq XX \subseteq X.$$

Hence $xy \in G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)$. Let x be any element of $G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)$. Then $[x]^A_{(\alpha,\beta)} \subseteq X$. Let a_x be any element of $[x^{-1}]^A_{(\alpha,\beta)}$. Then $(x^{-1},a_x) \in \mathscr{C}(A,\alpha,\beta)$, and so $(x,a_x^{-1}) \in \mathscr{C}(A,\alpha,\beta)$. Thus $a_x^{-1} \in [x]^A_{(\alpha,\beta)} \subseteq X$. Since X is a subgroup of G, it follows that $a_x \in X$ so that $[x^{-1}]^A_{(\alpha,\beta)} \subseteq X$. This shows that $x^{-1} \in G_*(\mathscr{C}(A,\alpha,\beta),X)$. Hence $G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)$ is a subgroup of G. Now assume that X is normal. Let $x \in G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)$. Then $[x]^A_{(\alpha,\beta)} \subseteq X$. Let $a,z \in G$ be such that $z \in [axa^{-1}]^A_{(\alpha,\beta)}$. Then $(axa^{-1},z) \in \mathscr{C}(A,\alpha,\beta)$, which implies that $(x,a^{-1}za) \in \mathscr{C}(A,\alpha,\beta)$ since $\mathscr{C}(A,\alpha,\beta)$ is a congruence relation on G. Hence $a^{-1}za \in [x]^A_{(\alpha,\beta)} \subseteq X$, and thus $a^{-1}za = b$ for some $b \in X$. Since X is normal, $z = aba^{-1} \in aXa^{-1} \subseteq X$. Therefore $[axa^{-1}]^A_{(\alpha,\beta)} \subseteq X$, i.e., $axa^{-1} \in G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)$. Consequently, $G_*\Big(\mathscr{C}(A,\alpha,\beta),X\Big)$ is a normal subgroup of G. □

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