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FUZZY PAIRWISE STRONG PRECONTINUOUS MAPPINGS

Kuo Duok Park, Joo Sung Lee* and Young Bin Im

ABSTRACT. We define a (τ_i, τ_j)-fuzzy strongly preopen set on a fuzzy bitopological space and characterize a fuzzy pairwise strong precontinuous mapping and a fuzzy pairwise strong preopen mapping(a fuzzy pairwise strong preclosed mapping) on a fuzzy bitopological space.

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1. Introduction

Singal and Prakash [8] introduced a fuzzy preopen set and studied characteristic properties of a fuzzy precontinuous mapping on a fuzzy topological space. Later, Sampath Kumar [7] defined a (τ_i, τ_j)-fuzzy preopen set and characterized a fuzzy pairwise precontinuous mapping on a fuzzy bitopological space as a natural generalization of a fuzzy topological space.

Krsteska [3, 4] also defined a fuzzy strongly preopen set and studied a fuzzy strong precontinuous mapping(a fuzzy strong preopen mapping) on a fuzzy topological space.

In this paper, we define a (τ_i, τ_j)-fuzzy strongly preopen set and study their properties. And we investigate relationships between the fuzzy pairwise strong precontinuous mappings(the fuzzy pairwise strong preopen mappings) and the

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fuzzy pairwise precontinuous mappings(the fuzzy pairwise preopen mappings) on a fuzzy bitopological space. Then we characterize a fuzzy pairwise strong precontinuous mapping and a fuzzy pairwise strong preopen(preclosed) mapping on a fuzzy bitopological space.

2. Preliminaries

Let X be a set and let τ_1 and τ_2 be fuzzy topologies on X. Then we call (X, τ_1, τ_2) a *fuzzy bitopological space* [*fbts*].

A mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ is *fuzzy pairwise continuous* [*fpc*] if the induced mapping $f : (X, \tau_k) \to (Y, \tau_k^*)$ is fuzzy continuous for $k = 1, 2$.

A mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ is *fuzzy pairwise open* [*fp open*] (*fuzzy pairwise closed* [*fp closed*]) if the induced mapping $f : (X, \tau_k) \to (Y, \tau_k^*)$ is fuzzy open (fuzzy closed) for $k = 1, 2$.

Notations. (1) Throughout this paper, we take an ordered pair (τ_i, τ_j) with $i, j \in \{1, 2\}$ and $i \neq j$.

(2) For simplicity, we abbreviate a τ_i-fuzzy open set μ and a τ_j-fuzzy closed set μ with a $\tau_i - fo$ set μ and a $\tau_j - fc$ set μ respectively. Also, we denote the interior and the closure of μ for a fuzzy topology τ_i with $\tau_i - \mathrm{Int}\,\mu$ and $\tau_i - \mathrm{Cl}\,\mu$ respectively.

Definition 2.1. [7] Let μ be a fuzzy set on a *fbts* X. Then we call μ;

(1) a (τ_i, τ_j)*-fuzzy preopen* [$(\tau_i, \tau_j) - fpo$] *set* on X if

$$\mu \leq \tau_i - \mathrm{Int}(\tau_j - \mathrm{Cl}\,\mu) \quad \text{and}$$

(2) a (τ_i, τ_j)*-fuzzy preclosed* [$(\tau_i, \tau_j) - fpc$] *set* on X if

$$\tau_i - \mathrm{Cl}(\tau_j - \mathrm{Int}\,\mu) \leq \mu.$$

Definition 2.2. [7] Let μ be a fuzzy set on a *fbts* X.

(1) The (τ_i, τ_j)*-preinterior of* μ, $(\tau_i, \tau_j) - pInt\,\mu$ is

$$\bigvee \left\{\nu \,|\, \nu \leq \mu,\ \nu \text{ is a } (\tau_i, \tau_j) - fpo \text{ set}\right\}.$$

(2) The (τ_i, τ_j)*-preclosure of* μ, $(\tau_i, \tau_j) - pCl\,\mu)$ is

$$\bigwedge \left\{\nu \,|\, \nu \geq \mu,\ \nu \text{ is a } (\tau_i, \tau_j) - fpc \text{ set}\right\}.$$

Definition 2.3. [7] Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is called a *fuzzy pairwise precontinuous* [*fppc*] *mapping* if $f^{-1}(\nu)$ is a $(\tau_i, \tau_j) - fpo$ set on X for each $\tau_i^* - fo$ set ν on Y.

Definition 2.4. [7] Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is called;

(1) a *fuzzy pairwise preopen* [*fpp open*] *mapping* if $f(\mu)$ is a $(\tau_i^*, \tau_j^*) - fpo$ set on Y for each $\tau_i - fo$ set μ on X and

(2) a *fuzzy pairwise preclosed* [*fpp closed*] *mapping* if $f(\mu)$ is a $(\tau_i^*, \tau_j^*) - fpc$ set on Y for each $\tau_i - fc$ set μ on X.

Proposition 2.5. [7] *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then the following statements are equivalent:*

(1) *f is fppc.*

(2) *$f((\tau_i, \tau_j) - pCl\mu) \le \tau_i^* - Cl(f(\mu))$ for each fuzzy set μ on X.*

(3) *$f^{-1}(\tau_i^* - Cl\nu) \le (\tau_i, \tau_j) - pCl\Big(f^{-1}(\nu)\Big)$ for each fuzzy set ν on Y.*

(4) *$f^{-1}(\tau_i^* - Int\nu) \le (\tau_i, \tau_j) - pInt\Big(f^{-1}(\nu)\Big)$ for each fuzzy set ν on Y.*

Proposition 2.6. [7] *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a bijection. Then f is fppc if and only if for each fuzzy set μ on X,*

$$\tau_i^* - Int(f(\mu)) \le f\Big((\tau_i, \tau_j) - pInt\mu\Big).$$

Proposition 2.7. [7] *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then the following statements are equivalent:*

(1) f is fpp open.

(2) $f(\tau_i - Int\mu) \le (\tau_i^*, \tau_j^*) - pInt(f(\mu))$ for each fuzzy set μ on X.

(3) $\tau_i - Int(f^{-1}(\nu)) \le f^{-1}\Big((\tau_i, \tau_j) - pInt\nu\Big)$ for each fuzzy set ν on Y.

Proposition 2.8. [7] *A mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ is fpp closed if and only if for each fuzzy set μ on X,*

$$(\tau_i^*, \tau_j^*) - pCl\Big(f(\mu)\Big) \le f(\tau_i - Cl\mu).$$

Proposition 2.9. [7] *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a bijection. Then f is fpp closed if and only if for each fuzzy set ν on Y,*

$$f^{-1}\Big((\tau_i^*, \tau_j^*) - pCl\nu\Big) \le \tau_i - Cl(f^{-1}(\nu)).$$

3. Fuzzy pairwise strong precontinuous mappings

In this section, we introduce a fuzzy pairwise strong precontinuous mapping and a fuzzy pairwise strong preopen mapping which are stronger than a fuzzy pairwise precontinuous mapping and a fuzzy pairwise preopen mapping respectively. And we characterize a fuzzy pairwise strong precontinuous mapping and a fuzzy pairwise preopen mapping.

Definition 3.1. Let μ be a fuzzy set on a *fbts* X. Then we call μ;
(1) a (τ_i,τ_j)-*fuzzy strongly preopen* $[(\tau_i,\tau_j)-fspo]$ *set* on X if

$$\mu \leq \tau_i - \mathrm{Int}\Big((\tau_j,\tau_i) - \mathrm{pCl}\,\mu\Big) \quad \text{and}$$

(2) a (τ_i,τ_j)-*fuzzy strongly preclosed* $\Big[(\tau_i,\tau_j)-fspc\Big]$ *set* on X if

$$\tau_i - \mathrm{Cl}\Big((\tau_j,\tau_i) - \mathrm{pInt}\,\mu\Big) \leq \mu.$$

It is clear that a $\tau_i - fo$ set is a $(\tau_i,\tau_j)-fspo$ set and a $(\tau_i,\tau_j)-fspo$ set is a $(\tau_i,\tau_j)-fpo$ set on a *fbts* X. But the converses are not true in general as the following example shows.

Example 3.2. Let μ_1, μ_2, μ_3 and μ_4 be fuzzy sets on $X=\{a,b,c\}$ with

$$\begin{aligned}
&\mu_1(a)=0.9, \mu_1(b)=0.9, \mu_1(c)=0.9,\\
&\mu_2(a)=0.7, \mu_2(b)=0.7, \mu_2(c)=0.7,\\
&\mu_3(a)=0.6, \mu_3(b)=0.6, \mu_3(c)=0.6 \text{ and}\\
&\mu_4(a)=0.5, \mu_4(b)=0.5, \mu_4(c)=0.5.
\end{aligned}$$

Let $\tau_1=\{0_X,\mu_3,1_X\}, \tau_2=\{0_X,\mu_2,1_X\}$ be fuzzy topologies on X.

Then μ_1 is a $(\tau_i,\tau_j)-fspo$ set but not a τ_i-fo set. And μ_4 is a $(\tau_i,\tau_j)-fpo$ set but not a $(\tau_i,\tau_j)-fspo$ set.

Proposition 3.3. (1) *A union of* $(\tau_i,\tau_j)-fspo$ *sets is a* $(\tau_i,\tau_j)-fspo$ *set.*
(2) *An intersection of* $(\tau_i,\tau_j)-fspc$ *sets is a* $(\tau_i,\tau_j)-fspc$ *set.*

Proof. (1) Let $\{\mu_\lambda\}$ be a family of $(\tau_i,\tau_j)-fspo$ sets on a *fbts* X. Since $\mu_\lambda \leq \tau_i - \mathrm{Int}\Big((\tau_j,\tau_i)-\mathrm{pCl}\mu_\lambda\Big)$ for each λ, we have

$$\bigvee \mu_\lambda \leq \bigvee\Big(\tau_i - \mathrm{Int}((\tau_j,\tau_i)-\mathrm{pCl}\mu_\lambda)\Big) \leq \tau_i - \mathrm{Int}\Big((\tau_j,\tau_i)-\mathrm{pCl}(\bigvee\mu_\lambda)\Big).$$

Hence $\bigvee \mu_\lambda$ is a $(\tau_i, \tau_j) - fspo$ set.

(2) The proof follows easily from complements of (1). □

An intersection of two $(\tau_i, \tau_j) - fspo$ sets need not be a $(\tau_i, \tau_j) - fspo$ set. And a union of two $(\tau_i, \tau_j) - fspc$ sets need not be a $(\tau_i, \tau_j) - fspo$ set as the following example shows.

Example 3.4. Let μ_1, μ_2, μ_3, μ_4 and μ_5 be fuzzy sets on $X = \{a, b, c\}$ with

$$\mu_1(a) = 0.9, \mu_1(b) = 0.5, \mu_1(c) = 0.9,$$
$$\mu_2(a) = 0.5, \mu_2(b) = 0.7, \mu_2(c) = 0.5,$$
$$\mu_3(a) = 0.8, \mu_3(b) = 0.5, \mu_3(c) = 0.8,$$
$$\mu_4(a) = 0.8, \mu_4(b) = 0.5, \mu_4(c) = 0.7 \text{ and}$$
$$\mu_5(a) = 0.3, \mu_5(b) = 0.4, \mu_3(c) = 0.3.$$

Let $\tau_1 = \{0_X, \mu_4, \mu_5, 1_X\}, \tau_2 = \{0_X, \mu_3, \mu_5, 1_X\}$ be fuzzy topologies on X.

Then μ_1 and μ_2 are $(\tau_i, \tau_j) - fspo$ sets but $\mu_1 \wedge \mu_2$ is not a $(\tau_i, \tau_j) - fspo$ set. And μ_1^c and μ_2^c are $(\tau_i, \tau_j) - fspc$ sets but $\mu_1^c \vee \mu_2^c$ is not a $(\tau_i, \tau_j) - fspc$ set. □

Definition 3.5. Let μ be a fuzzy set on a *fbts* X.

(1) The *(τ_i, τ_j)-strongly preinterior of μ, $(\tau_i, \tau_j) - spInt\,\mu$* is

$$\bigvee \Big\{ \nu \,|\, \nu \leq \mu,\ \nu \text{ is a } (\tau_i, \tau_j) - fspo \text{ set} \Big\}.$$

(2) The *(τ_i, τ_j)-strongly preclosure of μ, $(\tau_i, \tau_j) - spCl\,\mu)$* is

$$\bigwedge \Big\{ \nu \,|\, \nu \geq \mu,\ \nu \text{ is a } (\tau_i, \tau_j) - fspc \text{ set} \Big\}.$$

Obviously, $(\tau_i, \tau_j) - \text{spCl}\mu$ is the smallest $(\tau_i, \tau_j) - fspc$ set which contains μ, and $(\tau_i, \tau_j) - \text{spInt}\mu$ is the largest $(\tau_i, \tau_j) - fspo$ set which is contained in μ. Therefore, $(\tau_i, \tau_j) - \text{spCl}\mu = \mu$ for every $(\tau_i, \tau_j) - fspc$ set μ and $(\tau_i, \tau_j) - \text{spInt}\mu = \mu$ for every $(\tau_i, \tau_j) - fspo$ set μ.

Moreover, we have

$$\tau_i - \text{Int}\mu \leq (\tau_i, \tau_j) - \text{spInt}\mu \leq (\tau_i, \tau_j) - \text{pInt}\mu \leq \mu,$$

$$\mu \leq (\tau_i, \tau_j) - \text{pCl}\mu \leq (\tau_i, \tau_j) - \text{spCl}\mu \leq \tau_i - \text{Cl}\mu.$$

We also have the following lemma from the above definition, which will be used later.

Lemma 3.6. *Let μ be a fuzzy set on a fbts X. Then*

$$(\tau_i,\tau_j)-spInt(\mu^c)=((\tau_i,\tau_j)-spCl\mu)^c$$

and

$$(\tau_i,\tau_j)-spCl(\mu^c)=((\tau_i,\tau_j)-spInt\mu)^c.$$

Proof. Let μ be a fuzzy set on a $fbts$ X. Then

$$\begin{aligned}(\tau_i,\tau_j)-\mathrm{spCl}\mu &= \bigwedge\left\{\nu^c \,|\, \nu^c\geq\mu,\ \nu \text{ is a } (\tau_i,\tau_j)-fspo \text{ set}\right\}\\ &= \left(\bigvee\{\mu^c \,|\, \mu^c\geq\mu,\ \nu \text{ is a } (\tau_i,\tau_j)-fspo \text{ set}\}\right)^c\\ &= \left((\tau_i,\tau_j)-\mathrm{spInt}(\mu^c)\right)^c.\end{aligned}$$

Hence $(\tau_i,\tau_j)-\mathrm{spInt}(\mu^c)=\left((\tau_i,\tau_j)-\mathrm{spCl}\mu\right)^c$. Similarly we can prove the second equality. □

Definition 3.7. Let $f:(X,\tau_1,\tau_2)\to(Y,\tau_1^*,\tau_2^*)$ be a mapping. Then f is called a *fuzzy pairwise strong precontinuous* [$fpspc$] *mapping* if $f^{-1}(\nu)$ is a $(\tau_i,\tau_j)-fspo$ set on X for each τ_i^*-fo set ν on Y.

It is clear that every fpc mapping is a $fpspc$ mapping and every $fpspc$ mapping is a $fppc$ mapping on $fbts$. But the converses are not true in general as the following example shows.

Example 3.8. Let μ_1, μ_2, μ_3 and μ_4 be fuzzy sets on $X=\{a,b,c\}$ with

$$\begin{aligned}&\mu_1(a)=0.9,\mu_1(b)=0.9,\mu_1(c)=0.9,\\ &\mu_2(a)=0.7,\mu_2(b)=0.7,\mu_2(c)=0.7,\\ &\mu_3(a)=0.6,\mu_3(b)=0.6,\mu_3(c)=0.6 \text{ and}\\ &\mu_4(a)=0.5,\mu_4(b)=0.5,\mu_4(c)=0.5.\end{aligned}$$

Let

$$\begin{aligned}&\tau_1=\{0_X,\mu_3,1_X\},\quad \tau_2=\{0_X\mu_2,1_X\} \text{ and}\\ &\tau_1^*=\{0_X,\mu_1,1_X\},\quad \tau_2^*=\{0_X,1_X\}.\end{aligned}$$

be fuzzy topologies on X.

Then we can show that the identity mapping $i_X:(X,\tau_1,\tau_2)\to(X,\tau_1^*,\tau_2^*)$ is $fpspc$ but not fpc and μ_1 is a $(\tau_i,\tau_j)-fspo$ set but not a $(\tau_i,\tau_j)-fpo$ set.

Example 3.9. Let μ_1, μ_2, μ_3 and μ_4 be fuzzy sets on $X = \{a, b, c\}$ defined as in Example 3.8. And let

$$\tau_1 = \{0_X, \mu_3, 1_X\}, \tau_2 = \{0_X \mu_2, 1_X\} \text{ and}$$
$$\tau_1^* = \{0_X, \mu_4, 1_X\}, \tau_2^* = \{0_X, 1_X\}.$$

be fuzzy topologies on X.

Then we can show that the identity mapping $i_X : (X, \tau_1, \tau_2) \to (X, \tau_1^*, \tau_2^*)$ is *fppc* but not *fpspc* and μ_4 is a $(\tau_i, \tau_j) - fpo$ set but not a $(\tau_i, \tau_j) - fspo$ set.

Theorem 3.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then the following statements are equivalent:*

(1) *f is fpspc.*

(2) *The inverse image of $\tau_i^* - fc$ set on Y is a $(\tau_i, \tau_j) - fspc$ set on X.*

(3) *$\tau_i - Cl\Big((\tau_j, \tau_i) - pInt(f^{-1}(\nu))\Big) \leq f^{-1}(\tau_i^* - Cl\nu)$ for each fuzzy set ν on Y.*

(4) *$f(\tau_i - Cl\Big((\tau_j, \tau_i) - pInt\mu\Big) \leq \tau_i^* - Cl(f(\mu))$ for each fuzzy set μ on X.*

Proof. (1) implies (2): Let ν be a $\tau_i^* - fc$ set on Y. Then ν^c is a $\tau_i^* - fo$ set on Y. Thus $f^{-1}(\nu^c)$ is a $(\tau_i, \tau_j) - fspo$ set on X. But $f^{-1}(\nu^c) = (f^{-1}(\nu))^c$. Therefore, $f^{-1}(\nu)$ is a $(\tau_i, \tau_j) - fspc$ set on X.

(2) implies (3): Let ν be a fuzzy set on Y. Then $f^{-1}(\tau_i^* - \mathrm{Cl}\nu)$ ia a $(\tau_i, \tau_j) - fspc$ set on X. Hence

$$\begin{aligned}\tau_i - \mathrm{Cl}\Big((\tau_j, \tau_i) - \mathrm{pInt}(f^{-1}(\nu))\Big) &\leq \tau_i - \mathrm{Cl}\Big((\tau_j, \tau_i) - \mathrm{pInt}(f^{-1}(\tau_i^* - \mathrm{Cl}\nu))\Big) \\ &\leq f^{-1}(\tau_i^* - \mathrm{Cl}\nu).\end{aligned}$$

(3) implies (4): Let μ be a fuzzy set on X. Then

$$\tau_i - \mathrm{Cl}\Big((\tau_j, \tau_i) - \mathrm{pInt}(f^{-1}(f(\mu)))\Big) \leq f^{-1}\Big(\tau_i^* - \mathrm{Cl}(f(\mu))\Big).$$

This implies that $f\Big(\tau_i - \mathrm{Cl}((\tau_j, \tau_i) - \mathrm{pInt}\mu)\Big) \leq \tau_i^* - \mathrm{Cl}(f(\mu))$.

(4) implies (1): Let ν be a $\tau_i^* - fo$ set on Y. Then ν^c is a $\tau_i^* - fc$ set. Hence

$$\begin{aligned}f\Big(\tau_i - \mathrm{Cl}((\tau_j, \tau_i) - \mathrm{pInt}(f^{-1}(\nu^c)))\Big) &\leq \tau_i^* - \mathrm{Cl}\Big(f(f^{-1}(\nu^c))\Big) \\ &\leq \tau_i^* - \mathrm{Cl}(\nu^c) \\ &= \nu^c.\end{aligned}$$

Thus $\tau_i - \mathrm{Cl}\Big((\tau_j, \tau_i) - \mathrm{pInt}(f^{-1}(\nu^c))\Big) = (f^{-1}(\nu))^c$ and therefore, $f^{-1}(\nu)$ is a $(\tau_i, \tau_j) - fspo$ set on X. □

Theorem 3.11. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then the following statements are equivalent:*

(1) *f is fpspc.*

(2) *$f\Big((\tau_i, \tau_j) - spCl\mu\Big) \leq \tau_i^* - Cl(f(\mu))$ for each fuzzy set μ on X.*

(3) *$(\tau_i, \tau_j) - spCl\Big(f^{-1}(\nu)\Big) \leq f^{-1}(\tau_i^* - Cl\nu)$ for each fuzzy set ν on Y.*

(4) *$f^{-1}(\tau_i^* - Int\nu) \leq (\tau_i, \tau_j) - spInt(f^{-1}(\nu))$ for each fuzzy set ν on Y.*

Proof. (1) implies (2): Let μ be a fuzzy set on X. Then $f^{-1}\Big(\tau_i^* - \mathrm{Cl}(f(\mu))\Big)$ is a $(\tau_i, \tau_j) - fspc$ set on X. Thus

$$\begin{aligned}(\tau_i, \tau_j) - \mathrm{spCl}\mu &\leq (\tau_i, \tau_j) - \mathrm{spCl}\Big(f^{-1}(f(\mu))\Big) \\ &\leq (\tau_i, \tau_j) - \mathrm{spCl}\Big(f^{-1}(\tau_i^* - \mathrm{Cl}(f(\mu)))\Big) \\ &= f^{-1}\Big(\tau_i^* - \mathrm{Cl}(f(\mu))\Big).\end{aligned}$$

Hence

$$\begin{aligned}f\Big((\tau_i, \tau_j) - \mathrm{spCl}\mu\Big) &\leq f\Big(f^{-1}(\tau_i^* - \mathrm{Cl}(f(\mu)))\Big) \\ &\leq \tau_i^* - \mathrm{Cl}(f(\mu)).\end{aligned}$$

(2) implies (3): Let ν be a fuzzy set on Y. Then

$$f\Big((\tau_i, \tau_j) - \mathrm{spCl}(f^{-1}(\nu))\Big) \leq \tau_i^* - \mathrm{Cl}\Big(f(f^{-1}(\nu))\Big) \leq \tau_i^* - \mathrm{Cl}\nu.$$

Hence

$$\begin{aligned}(\tau_i, \tau_j) - \mathrm{spCl}(f^{-1}(\nu)) &\leq f^{-1}\Big((\tau_i, \tau_j) - \mathrm{spCl}(f^{-1}(\nu)))\Big) \\ &\leq f^{-1}(\tau_i^* - \mathrm{Cl}\nu).\end{aligned}$$

(3) implies (4): Let ν be a fuzzy set on Y. Then

$$(\tau_i, \tau_j) - \mathrm{spCl}(f^{-1}(\nu^c)) \leq f^{-1}\Big(\tau_i^* - \mathrm{Cl}(\nu^c)\Big).$$

Hence, by Lemma 3.6,

$$\begin{aligned}f^{-1}(\tau_i^* - \mathrm{Int}\nu) &= f^{-1}\Big((\tau_i^* - \mathrm{Cl}(\nu^c)^c\Big) \\ &\leq \Big((\tau_i, \tau_j) - \mathrm{spCl}(f^{-1}(\nu^c))\Big)^c \\ &= (\tau_i, \tau_j) - \mathrm{spInt}(f^{-1}(\nu)).\end{aligned}$$

(4) implies (1): Let ν be a $\tau_i^* - fo$ set on Y. Then

$$f^{-1}(\nu) = f^{-1}(\tau_i^* - \mathrm{Int}\nu) \leq (\tau_i, \tau_j) - \mathrm{spInt}(f^{-1}(\nu)).$$

Hence $f^{-1}(\nu)$ is a $(\tau_i, \tau_j) - fspo$ set on X and therefore, f is *fpspc*. □

Theorem 3.12. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a bijection. f is fpspc if and only if $\tau_i^* - Int(f(\mu)) \leq f\Big((\tau_i, \tau_j) - spInt\mu\Big)$ for each fuzzy set μ on X.*

Proof. Let μ be a fuzzy set on X. Then, by Theorem 3.11,

$$f^{-1}\Big(\tau_i^* - \text{Int}(f(\mu))\Big) \leq (\tau_i, \tau_j) - \text{spInt}\Big(f^{-1}(f(\mu))\Big).$$

Since f is a bijection,

$$\tau_i^* - \text{Int}(f(\mu)) = f\Big(f^{-1}(\tau_i^* - \text{Int}(f(\mu)))\Big) \leq f((\tau_i, \tau_j) - \text{spInt}\mu).$$

Conversely, let ν be a fuzzy set on Y. Then

$$\tau_i^* - \text{Int}\Big(f(f^{-1}(\nu))\Big) \leq f\Big((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu))\Big).$$

Recall that f is a bijection. Hence

$$\tau_i^* - \text{Int}\nu = \tau_i^* - \text{Int}\Big(f(f^{-1}(\nu))\Big) \leq f\Big((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu))\Big).$$

and

$$\begin{aligned} f^{-1}(\tau_i^* - \text{Int}\nu) &\leq f^{-1}\Big(f((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)))\Big) \\ &= (\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)). \end{aligned}$$

Therefore, by Theorem 3.11, f is *fpspc*. □

Definition 3.13. Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is called;

(1) a *fuzzy pairwise strong preopen [fpspopen] mapping* if $f(\mu)$ is a $(\tau_i^*, \tau_j^*) - fspo$ set on Y for each $\tau_i - fo$ set μ on X and

(2) a *fuzzy pairwise strong preclosed [fpspclosed] mapping* if $f(\mu)$ is a $(\tau_i^*, \tau_j^*) - fspc$ set on Y for each $\tau_i - fc$ set μ on X.

It is clear that every *fp open(fp closed)* mapping is a *fpsp open(fpsp closed)* mapping and every *fpsp open(fpsp closed)* mapping is a *fpp open(fpp closed)* mapping on *fbts*. But the converses are not true in general as the following example shows.

Example 3.14. In Example 3.8, the identity mapping $i_X : (X, \tau_1^*, \tau_2^*) \to (X, \tau_1, \tau_2)$ is *fpsp open(fpsp closed)* but not *fp open(fp closed)*.

Example 3.15. In Example 3.9, the identity mapping $i_X : (X, \tau_1^*, \tau_2^*) \to (X, \tau_1^*, \tau_2^*)$ is *fpp open(fpp closed)* but not *fpsp open(fpsp closed)*.

Theorem 3.16. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then the following statements are equivalent:*

(1) *f is fpsp open.*

(2) *$f(\tau_i^* - Int\mu) \leq (\tau_i, \tau_j) - spInt(f(\mu))$ for each fuzzy set μ on X.*

(3) *$\tau_i^* - (f^{-1}(\nu)) \leq f^{-1}\Big((\tau_i^*, \tau_j^*) - spInt\nu\Big)$ for each fuzzy set ν on Y.*

Proof. (1) implies (2): Let μ be a fuzzy set on X. Then $\tau_i - \mathrm{Int}\mu$ is a $\tau_i - fo$ set on X. Since f is *fpsp open*, $f(\tau_i - \mathrm{Int}\mu)$ is a $(\tau_i^*, \tau_j^*) - fspo$ set on Y. We also have $f(\tau_i - \mathrm{Int}\mu) \leq f(\mu)$. Hence

$$\begin{aligned} f(\tau_i - \mathrm{Int}\mu) &= (\tau_i^*, \tau_j^*) - \mathrm{spInt}\Big(f(\tau_i - \mathrm{Int}\mu)\Big) \\ &\leq (\tau_i^*, \tau_j^*) - \mathrm{spInt}(f(\mu)). \end{aligned}$$

(2) implies (3): Let ν be a fuzzy set on Y. Then $f^{-1}(\nu)$ is a fuzzy set on X. Thus

$$\begin{aligned} f\Big(\tau_i - \mathrm{Int}(f^{-1}(\nu)\Big) &\leq (\tau_i^*, \tau_j^*) - \mathrm{spInt}\Big(f(f^{-1}(\nu))\Big) \\ &\leq (\tau_i^*, \tau_j^*) - \mathrm{spInt}\nu. \end{aligned}$$

Hence

$$\begin{aligned} \tau_i - \mathrm{Int}(f^{-1}(\nu)) &\leq f^{-1}\Big(f(\tau_i - \mathrm{Int}(f^{-1}(\nu)))\Big) \\ &\leq f^{-1}\Big((\tau_i^*, \tau_j^*) - \mathrm{spInt}\nu\Big). \end{aligned}$$

(3) implies (1): Let μ be a $\tau_i - fo$ set on X. Then $\tau_i - \mathrm{Int}\mu = \mu$ and $f(\mu)$ is a fuzzy set on Y. Since

$$\begin{aligned} \mu = \tau_i - \mathrm{Int}\mu &\leq \tau_i - \mathrm{Int}\Big(f^{-1}(f(\mu))\Big) \\ &\leq f^{-1}\Big((\tau_i^*, \tau_j^*) - \mathrm{spInt}(f(\mu))\Big), \end{aligned}$$

we have

$$f(\mu) \leq f\Big(f^{-1}\Big((\tau_i^*, \tau_j^*) - \mathrm{spInt}(f(\mu))\Big)\Big) \leq (\tau_i^*, \tau_j^*) - \mathrm{spInt}(f(\mu)).$$

Hence $f(\mu)$ is a $(\tau_i^*, \tau_j^*) - fspo$ set on Y and therefore, f is *fpsp open*. □

Theorem 3.17. *A mapping* $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ *is fpsp closed if and only if* $(\tau_i^*, \tau_j^*) - spCl(f(\mu)) \leq f(\tau_i - Cl\mu)$ *for each fuzzy set* μ *on* X.

Proof. Let f be a *fpsp closed* mapping and let μ be a fuzzy set on X. Then $f(\tau_i - \mathrm{Cl}\mu)$ is a $(\tau_i^*, \tau_j^*) - fspc$ set such that $f(\mu) \leq f(\tau_i - \mathrm{Cl}\mu)$. Hence

$$(\tau_i^*, \tau_j^*) - \mathrm{spCl}(f(\mu)) \leq (\tau_i^*, \tau_j^*) - \mathrm{spCl}(f(\tau_i - \mathrm{Cl}\mu) = f(\tau_i - \mathrm{Cl}\mu).$$

Conversely, let μ be a $\tau_i - fc$ set on X. Then

$$f(\mu) \leq (\tau_i^*, \tau_j^*) - \mathrm{spCl}(f(\mu)) \leq f(\tau_i - \mathrm{Cl}\mu).$$

This implies that $f(\mu) = (\tau_i^*, \tau_j^*) - \mathrm{spCl}(f(\mu))$. Hence $f(\mu)$ is a $(\tau_i^*, \tau_j^*) - fspc$ set on Y. Therefore, f is *fpsp closed.* □

Theorem 3.18. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ *be a bijection. f is fpsp closed if and only if* $f^{-1}((\tau_i^*, \tau_j^*) - spCl\nu) \leq \tau_i Cl(f^{-1}(\nu))$ *for each fuzzy set* ν *on* Y.

Proof. Let ν be a fuzzy set on Y. Then, by Theorem 3.17,

$$(\tau_i^*, \tau_j^*) - \mathrm{spCl}\nu \leq f\Big(\tau_i - \mathrm{Cl}(f^{-1}(\nu))\Big).$$

Since f is a bijection,

$$\begin{aligned} f^{-1}\Big((\tau_i^*, \tau_j^*) - \mathrm{spCl}\nu\Big) &= f^{-1}\Big((\tau_i^*, \tau_j^*) - \mathrm{spCl}(f(^{-1}(\nu)))\Big) \\ &\leq f^{-1}\Big(f(\tau_i - \mathrm{Cl}(f^{-1}(\nu)))\Big) \\ &= \tau_i - \mathrm{Cl}(f^{-1}(\nu)). \end{aligned}$$

Conversely, let μ be a fuzzy set on X. Then

$$\begin{aligned} (\tau_i^*, \tau_j^*) - \mathrm{spCl}(f(\mu)) &= f\Big(f^{-1}((\tau_i^*, \tau_j^*) - \mathrm{spCl}(f(\mu)))\Big) \\ &\leq f\Big(\tau_i - \mathrm{Cl}(f^{-1}(f(\mu)))\Big) \\ &= f(\tau_i - \mathrm{Cl}\mu). \end{aligned}$$

Therefore, by Theorem 3.11, f is *fpsp closed.* □

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Kuo Duok Park received his B.S. and Ph.D. at Yonsei University. Since 1979 he has been a professor at Dongguk University. His research interests are fuzzy topological space and fuzzy measure.

Dept. of Mathematics, Dongguk University, Seoul 100-715, Korea
e-mail : kdpark@dongguk.edu

Joo Sung Lee received his B.S. from Dongguk University and Ph.D. at University of Florida under the direction of Professor B. Brechner. Since 1995 he has been a professor at Dongguk University. His research interests are topological dynamics and fuzzy theory.

Dept. of Mathematics, Dongguk University, Seoul 100-715, Korea
e-mail : jsl@dongguk.edu

Young Bin Im received his B.S. and Ph.D. at Dongguk University under the direction of Professor K. D. Park. Since 2009 he has been a professor at Dongguk University. His research interests are fuzzy topological space and fuzzy matrix.

Office of General Education, Dongguk University, Seoul 100-715, Korea
e-mail : philpen@naver.com