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SYSTEM OF MIXED VARIATIONAL INEQUALITIES IN REFLEXIVE BANACH SPACES

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ABSTRACT. In this paper, we introduce and study a system of mixed variational inequalities in Banach spaces. By using J-proximal mapping and its Lipschitz continuity for a nonconvex, lower semicontinuous, subdifferentiable, proper functional, an iterative algorithm for computing the approximate solutions of system of mixed variational inequalities is suggested and analyzed. The convergence criteria of the iterative sequences generated by iterative algorithm is also discussed.

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1. Introduction

In the recent past, much attention has been focused on developing the theory of system of variational inequalities and variational inclusions. Ansari and Yao [1] studied a system of variational inequalities by using a fixed point theorem. Verma [8] studied some system of variational inequalities with single-valued mappings and suggested some iterative algorithms to compute approximate solutions of these systems in Hilbert spaces. Agarwal et al. [2] studied sensitivity analysis for a system of generalized nonlinear mixed quasi-variational inclusions with single-valued mappings.

In 2002, Ding and Xia [5] introduced the notion of J-proximal mapping for a lower semicontinuous subdifferentiable proper (may not be convex) functional on a Banach spaces. They have shown the existence and Lipschitz continuity of J-proximal mappings in Banach spaces.

Inspired and motivated by the research work going on this field, in this paper, we introduce and study a system of mixed variational inequalities involving J-proximal mapping in a reflexive Banach spaces. The Lipschitz continuity of J-proximal mapping for a nonconvex, lower semicontinuous, subdifferentiable,

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proper functional is used to established an existence and convergence result for the system of mixed variational inequalities.

2. Formulation and preliminaries

Throughout the paper, we assume that E is a real Banach space with its norm $\|.\|$, $E^\star$ is the topological dual of E, d is the metric induced by the norm $\|.\|$, $CB(E)$ (respectively, 2^E) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A,B)=\max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\Big\}.$$

where $d(x,B)=\inf_{y\in B} d(x,y)$ and $d(A,y)=\inf_{x\in A} d(x,y)$.

We also assume that $\langle\cdot,\cdot\rangle$ is the duality pairing between E and $E^\star$ and $\mathcal{F}: E\to 2^{E^\star}$ is the normalized duality mapping defined by

$$\mathcal{F}(x)=\{f\in E^\star:\langle x,f\rangle=\|x\|\|f\|,\ \textit{and}\ \|x\|=\|f\|\},\ \textit{for all}\ x\in E.$$

Definition 2.1. Let $H: E\to CB(E^*)$ be a set-valued mapping, $J: E\to E^*$ and $f: E\to E$ be single-valued mappings.

(1) H is said to be D-Lipschitz continuous with constant $\lambda_{D_H}\geq 0$ if

$$D(H(x),H(y))\leq\lambda_{D_H}\|x-y\|,\ \textit{for all}\ x,y\in E;$$

(2) J is said to be α-strongly accretive ($\alpha>0$) if

$$\langle J(x)-J(y),x-y\rangle\geq\alpha\|x-y\|^2,\ \textit{for all}\ x,y\in E;$$

(3) f is said to be strongly accretive if there exist a constant $\delta_f>0$ such that

$$\langle f(x)-f(y),j(x-y)\rangle\geq\delta_f\|x-y\|^2,\ \textit{for all}\ x,y\in E;$$

(4) f is said to be Lipschitz continuous with constant $\lambda_f>0$ if

$$\|f(x)-f(y)\|\leq\lambda_f\|x-y\|,\ \textit{for all}\ x,y\in E.$$

Definition 2.2. Let $\varphi: E\to R\cup\{+\infty\}$ be a proper functional, φ is said to be subdifferential at a point $x\in E$ if there exists a point $f^\star\in E^\star$ such that

$$\varphi(y)-\varphi(x)\geq\langle f^\star,y-x\rangle,\ \textit{for all}\ y\in E;$$

where $f^\star$ is called a subgradient of φ at x. The set of all subgradient of φ at x is denoted by $\partial\varphi(x)$.

The mapping $\partial\varphi: E\to 2^{E^\star}$ defined by

$$\partial\varphi(x)=\{f^\star\in E^\star:\varphi(y)-\varphi(x)\geq\langle f^\star,y-x\rangle,\ \textit{for all}\ y\in E\}$$

is said to be subdifferential of φ at x.

Definition 2.3 [5]. Let E be a Banach space with the dual space $E^\star$, $\varphi : E \to R \cup \{+\infty\}$ be a proper subdifferentiable (may not convex) functional and $J : E \to E^\star$ be a mapping. If for any given point $x^\star \in E^\star$ and $\rho > 0$, there is unique point $x \in E$ satisfying

$$\langle Jx - x^\star, y - x\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \ for\ all\ y \in E.$$

The mapping $x^\star \to x$, denoted by $J_\rho^{\partial\varphi}(x^\star)$, is said to be J-proximal mapping of φ. We have $x^\star - Jx \in \rho\partial\phi(x)$, it follows that $J_\rho^{\partial\phi}(x^\star) = (J + \rho\partial\phi)^{-1}(x^\star)$.

Lemma 2.1(Ding and Tan [4]). *Let D be a nonempty convex subset of a topological vector space and $f : D \times D \to R \cup \{+\infty\}$ such that*

(i) *for any $x \in D$, $y \to f(x, y)$ is lower semicontinuous on each compact subset of D,*

(ii) *for each finite set $\{x_1, \cdots, x_n\} \in D$ and for each $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i \equiv 1$, $\min_{1\leq i\leq n} f(x_i, y) \leq 0$,*

(iii) *there exists a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D\backslash K$, there is an $x \in c_0(D_0 \cup \{y\})$ satisfying $f(x, y) > 0$, then there exists $\hat{y} \in D$ such that $f(x, \hat{y}) \leq 0$, for all $x \in D$.*

Theorem 2.1 [5]. *Let E be a reflexive Banach space with the dual space E^* and $\varphi : E \to R \cup \{+\infty\}$ be a lower semicontinuous, subdifferentiable, proper functional which may not be convex. Let $J : E \to E^*$ be an α-strongly accretive continuous mapping. Then for any $\rho > 0$ and any $x^* \in E^*$, there exists a unique $x \in E$ such that*

$$\langle Jx - x^*, y - x\rangle + \rho\varphi(y) - \rho\varphi(x) \ \geq \ 0, \ for\ all\ y \in E.$$

That is $x = J_\rho^{\partial\varphi}(x^*)$ and so the J-proximal mapping $J_\rho^{\partial\varphi}$ of φ is well defined and is $1/\alpha$-Lipschitz continuous.

Proposition 2.1 [6]. *Let E be a real Banach space and $\mathcal{F} : E \to 2^{E^*}$ be a normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \ for\ all\ j(x + y) \in \mathcal{F}(x + y).$$

Let E_1 and E_2 be any two real Banach spaces. Let $S : E_1 \times E_2 \to E_1^*$, $T : E_1 \times E_2 \to E_2^*$, $f_1 : E_1 \to E_1$ and $f_2 : E_2 \to E_2$ be the single-valued mappings, $H : E_1 \to CB(E_1^*)$ and $F : E_2 \to CB(E_2^*)$ be set-valued mappings. Let $\varphi_1 : E_1 \times E_1 \to R \cup \{+\infty\}$ be lower semicontinuous, subdifferentiable (may not be convex), proper functional on E_1 satisfying $f_1(x) \in dom(\partial\varphi_1(\cdot, x))$ and $\varphi_2 : E_2 \times E_2 \to R \cup \{+\infty\}$ be lower semicontinuous, subdifferentiable (may not be convex), proper functional on E_2 satisfying $f_2(y) \in dom(\partial\varphi_2(\cdot, y))$, where

$\partial\varphi_1(\cdot,x)$ is subdifferential of $\varphi_1(\cdot,x)$ and $\partial\varphi_2(\cdot,y)$ is subdifferential of $\varphi_2(\cdot,y)$. We consider the following system of mixed variational inequalities:
Find $(x,y)\in E_1\times E_2$, $u\in H(x)$ and $v\in F(y)$ such that

$$\begin{aligned}\langle S(x,v), a-f_1(x)\rangle &\geq \varphi_1(f_1(x),x)-\varphi_1(a,x),\ \textit{for all } a\in E_1\\ \langle T(u,y), b-f_2(y)\rangle &\geq \varphi_2(f_2(y),y)-\varphi_2(b,y),\ \textit{for all } b\in E_2\end{aligned}\tag{2.1}$$

If $E_1=H_1$, $E_2=H_2$, where H_1 and H_2 are Hilbert spaces, $f_1=f_2=I$, where I is the identity mapping, H and F are single-valued mapping, $\varphi_1(x,\cdot)=\varphi_1(x)$ and $\varphi_2(y,\cdot)=\varphi_2(y)$, then the problem (2.1) reduces to the following problem:
Find $(x,y)\in H_1\times H_2$ such that

$$\begin{aligned}\langle S(x,F(y)), a-x\rangle+\varphi_1(a)-\varphi_1(x)\geq 0,\ \textit{for all } a\in H_1\\ \langle T(H(x),y), b-y\rangle+\varphi_2(b)-\varphi_2(y)\geq 0,\ \textit{for all } b\in H_2\end{aligned}\tag{2.2}$$

which is called a system of nonlinear mixed variational inequalities. Some special cases of the problem (2.2) can be found in [7]. Further, if $F=H=I$, then the problem (2.2) reduces to the system of nonlinear variational inequalities problem considered by Cho et al. [3].

3. Proximal point algorithm and convergence result

Theorem 3.1. *(x,y,u,v) with $(x,y)\in E_1\times E_2$, $u\in H(x)$ and $v\in F(y)$ is a solution of the system of mixed variational inequalities (2.1) if and only if it satisfies*

$$f_1(x)=J_\rho^{\partial\varphi_1(\cdot,x)}[J_1(f_1(x))-\rho S(x,v)],$$
$$f_2(y)=J_\gamma^{\partial\varphi_2(\cdot,y)}[J_2(f_2(y))-\gamma T(u,y)],$$

where $J_1:E_1\to E_1^*$, $J_2:E_2\to E_2^*$, $J_\rho^{\partial\varphi_1(\cdot,x)}=(J_1+\rho\partial\varphi_1(\cdot,x))^{-1}$, $J_\gamma^{\partial\varphi_2(\cdot,y)}=(J_2+\gamma\partial\varphi_2(\cdot,y))^{-1}$ and $\rho>0$, $\gamma>0$ are constants.

Proof. The conclusion follows directly from Definition 2.3. □

The above fixed point formulation enables us to suggest the following proximal point algorithm.

Algorithm 3.1. *For any given $(x_o,y_o)\in E_1\times E_2$, we choose $u_o\in H(x_o)$, $v_o\in F(y_o)$ and compute $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by iterative schemes*

$$f_1(x_{n+1})=J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_n))-\rho S(x_n,v_n)],\tag{3.1}$$

$$f_2(y_{n+1})=J_\gamma^{\partial\varphi_2(\cdot,y_n)}[J_2(f_2(y_n))-\gamma T(u_n,y_n)],\tag{3.2}$$

and choose $u_{n+1}\in H(x_{n+1})$ and $v_{n+1}\in F(y_{n+1})$ such that

$$\|u_{n+1}-u_n\|\leq\Big(1+\frac{1}{n+1}\Big)D(H(x_{n+1}),H(x_n)),\tag{3.3}$$

$$\|v_{n+1}-v_n\|\leq\Big(1+\frac{1}{n+1}\Big)D(F(y_{n+1}),F(y_n)),\tag{3.4}$$

where $\rho>0$ and $\gamma>0$ are constants and $n=0,1,2,\ldots\ldots$

Theorem 3.2. *Let E_1 and E_2 be two reflexive Banach spaces with their duals E_1^* and E_2^*, respectively. Let $S : E_1 \times E_2 \to E_1^*$ and $T : E_1 \times E_2 \to E_2^*$ are Lipschitz continuous in both the arguments with constants λ_{S_1}, λ_{S_2} and λ_{T_1}, λ_{T_2}, respectively. For $i = 1, 2$, let $f_i : E_i \to E_i$ is Lipschitz continuous with constants λ_{f_i} and strongly accretive with constants δ_{f_i} such that $f(E_i) = E_i$, $J_i : E_i \to E_i^*$ be Lipschitz continuous with constants λ_{J_i} and strongly accretive with constants α_i. Let $\varphi_1 : E_1 \times E_1 \to R \cup \{+\infty\}$ be lower semicontinuous, subdifferential (may not be convex), proper functional on E_1 satisfying $f_1(x) \in dom(\partial\varphi_1(\cdot, x))$ and $\varphi_2 : E_2 \times E_2 \to R \cup \{+\infty\}$ be lower semicontinuous, subdifferential (may not be convex), proper functional on E_2 satisfying $f_2(y) \in dom(\partial\varphi_2(\cdot, y))$, for all $x \in E_1$ and $y \in E_2$. Let $H : E_1 \to CB(E_1^*)$ and $F : E_2 \to CB(E_2^*)$ be D-Lipschitz continuous mappings with constants λ_{D_H} and λ_{D_F}, respectively. If there exists constants $\rho > 0$ and $\gamma > 0$ such that*

$$\|J_\rho^{\partial\varphi_1(\cdot,x_n)}(x^*) - J_\rho^{\partial\varphi_1(\cdot,x_{n-1})}(x^*)\| \le \mu^*\|x_n - x_{n-1}\| \tag{3.5}$$

for any $x_n, x_{n-1} \in E_1$, $x^* \in E_1^*$,

$$\|J_\gamma^{\partial\varphi_2(\cdot,y_n)}(y^*) - J_\gamma^{\partial\varphi_2(\cdot,y_{n-1})}(y^*)\| \le \mu^{**}\|y_n - y_{n-1}\| \tag{3.6}$$

*for any $y_n, y_{n-1} \in E_2$, $y^* \in E_2^*$
and the following condition is satisfied:*

$$\begin{cases} 0 < \sqrt{\dfrac{4(\lambda_{J_1}\lambda_{f_1})^2 + 8\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}} + \sqrt{\dfrac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2}{(2\delta_{f_2}+3)\alpha_2^2}} < 1 \\ 0 < \sqrt{\dfrac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2}{(2\delta_{f_1}+3)\alpha_1^2}} + \sqrt{\dfrac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}} < 1 \end{cases} \tag{3.7}$$

Then problem (2.1) admits a solution (x, y, u, v) with $(x, y) \in E_1 \times E_2$, $u \in H(x)$, $v \in F(y)$ and the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ converge to x, y, u, and v, respectively, where $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are the sequences generated by Algorithm (3.1).

Proof. We can write

$$\|x_{n+1} - x_n\|^2 = \|f_1(x_{n+1}) - f_1(x_n) - f_1(x_{n+1}) + f_1(x_n) - x_{n+1} + x_n\|^2.$$

By Proposition 2.1, we have

$$\|x_{n+1} - x_n\|^2 \le \|f_1(x_{n+1}) - f_1(x_n)\|^2 - 2\langle f_1(x_{n+1}) - f_1(x_n) + x_{n+1} - x_n, j(x_{n+1} - x_n)\rangle. \tag{3.8}$$

By (3.1), we have $f_1(x_{n+1}) = J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)]$. Thus

$$\|f_1(x_{n+1}) - f_1(x_n)\|^2 = \|J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)] - J_\rho^{\partial\varphi_1(\cdot,x_{n-1})}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})]\|^2.$$

Since $\|x + y\|^2 \le 2(\|x\|^2 + \|y\|^2)$, by the assumption (3.5) and Theorem 2.1, we have

$$\frac{1}{2}\|f_1(x_{n+1}) - f_1(x_n)\|^2 \le \|J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)] - J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})]\|^2$$

$$+\|J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_{n-1}))-\rho S(x_{n-1},v_{n-1})]$$
$$-J_\rho^{\partial\varphi_1(\cdot,x_{n-1})}[J_1(f_1(x_{n-1}))-\rho S(x_{n-1},v_{n-1})]\|^2$$
$$\leq \frac{1}{\alpha_1^2}\|[J_1(f_1(x_n))-\rho S(x_n,v_n)]-[J_1(f_1(x_{n-1}))$$
$$-\rho S(x_{n-1},v_{n-1})]\|^2+\mu^{*2}\|x_n-x_{n-1}\|^2$$
$$\leq \frac{2}{\alpha_1^2}\|J_1(f_1(x_n))-J_1(f_1(x_{n-1}))\|^2+\frac{2\rho^2}{\alpha_1^2}\|S(x_n,v_n)$$
$$-S(x_{n-1},v_{n-1})\|^2+\mu^{*2}\|x_n-x_{n-1}\|^2. \tag{3.9}$$

By the Lipschitz continuity of J_1 and f_1, we have

$$\|J_1(f_1(x_n))-J_1(f_1(x_{n-1}))\|\leq \lambda_{J_1}\lambda_{f_1}\|x_n-x_{n-1}\|. \tag{3.10}$$

By the Lipschitz continuity of $S(\cdot,\cdot)$ in both the arguments, (3.4) and D-Lipschitz continuity of F, we have

$$\|S(x_n,v_n)-S(x_n,v_{n-1})\|\leq \lambda_{S_2}\|v_n-v_{n-1}\|$$
$$\leq \lambda_{S_2}\Big(1+\frac{1}{n}\Big)D(F(y_n),F(y_{n-1}))$$
$$\leq \lambda_{S_2}\Big(1+\frac{1}{n}\Big)\lambda_{D_F}\|y_n-y_{n-1}\|. \tag{3.11}$$

$$\|S(x_n,v_{n-1})-S(x_{n-1},v_{n-1})\|\leq \lambda_{S_1}\|x_n-x_{n-1}\|. \tag{3.12}$$

Using (3.11) and (3.12), it follows that

$$\|S(x_n,v_n)-S(x_{n-1},v_{n-1})\|^2\leq 2\|S(x_n,v_n)-S(x_n,v_{n-1})\|^2$$
$$+2\|S(x_n,v_{n-1})-S(x_{n-1},v_{n-1})\|^2$$
$$\leq 2(\lambda_{S_2}\lambda_{D_F})^2\Big(1+\frac{1}{n}\Big)^2\|y_n-y_{n-1}\|^2$$
$$+2\lambda_{S_1}^2\|x_n-x_{n-1}\|^2. \tag{3.13}$$

By (3.10) and (3.13), (3.9) becomes

$$\|f_1(x_{n+1})-f_1(x_n)\|^2\leq\Big[\frac{4}{\alpha_1^2}(\lambda_{J_1}\lambda_{f_1})^2+\frac{8}{\alpha_1^2}\rho^2\lambda_{S_1}^2+2\mu^{*2}\Big]\|x_n-x_{n-1}\|^2$$
$$+\frac{8}{\alpha_1^2}\rho^2(\lambda_{S_2}\lambda_{D_F})^2\Big(1+\frac{1}{n+1}\Big)^2\|y_n-y_{n-1}\|^2. \tag{3.14}$$

By using the strong accretiveness of f_1 with constant δ_{f_1} and (3.14), (3.8) becomes

$$\|x_{n+1}-x_n\|^2$$
$$\leq \|f_1(x_{n+1})-f_1(x_n)\|^2-2\langle f_1(x_{n+1})-f_1(x_n)+x_{n+1}-x_n, j(x_{n+1}-x_n)\rangle$$
$$\leq\Big[\frac{4}{\alpha_1^2}(\lambda_{J_1}\lambda_{f_1})^2+\frac{8}{\alpha_1^2}\rho^2\lambda_{S_1}^2+2\mu^{*2}\Big]\|x_n-x_{n-1}\|^2$$
$$+\frac{8}{\alpha_1^2}\rho^2(\lambda_{S_2}\lambda_{D_F})^2\Big(1+\frac{1}{n}\Big)^2\|y_n-y_{n-1}\|^2-(2\delta_{f_1}+2)\|x_{n+1}-x_n\|^2. \tag{3.15}$$

It follows that

$$\|x_{n+1}-x_n\|^2 \leq \Big[\frac{4(\lambda_{J_1}\lambda_{f_1})^2}{(2\delta_{f_1}+3)\alpha_1^2}+\frac{8\rho^2\lambda_{S_1}^2}{(2\delta_{f_1}+3)\alpha_1^2}+\frac{2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}\Big]\|x_n-x_{n-1}\|^2$$
$$+\frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2\Big(1+\frac{1}{n}\Big)^2}{(2\delta_{f_1}+3)\alpha_1^2}\|y_n-y_{n-1}\|^2$$
$$=\theta_1\|x_n-x_{n-1}\|^2+\theta_2\|y_n-y_{n-1}\|^2$$
$$\leq \theta_1\|x_n-x_{n-1}\|^2+\theta_2\|y_n-y_{n-1}\|^2+2\sqrt{\theta_1}\sqrt{\theta_2}\|x_n-x_{n-1}\| \times\|y_n-y_{n-1}\|$$
$$=(\sqrt{\theta_1}\|x_n-x_{n-1}\|+\sqrt{\theta_2}\|y_n-y_{n-1}\|)^2.$$

Thus, we have

$$\|x_{n+1}-x_n\| \leq \sqrt{\theta_1}\|x_n-x_{n-1}\|+\sqrt{\theta_2}\|y_n-y_{n-1}\|, \tag{3.16}$$

where

$$\theta_1=\frac{4(\lambda_{J_1}\lambda_{f_1})^2}{(2\delta_{f_1}+3)\alpha_1^2}+\frac{8\rho^2\lambda_{S_1}^2}{(2\delta_{f_1}+3)\alpha_1^2}+\frac{2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}$$

and

$$\theta_2=\frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2\Big(1+\frac{1}{n}\Big)^2}{(2\delta_{f_1}+3)\alpha_1^2}.$$

We can also write

$$\|y_{n+1}-y_n\|^2=\|f_2(y_{n+1})-f_2(y_n)-f_2(y_{n+1})+f_2(y_n)-y_{n+1}+y_n\|^2.$$

By Proposition 2.1, we have

$$\|y_{n+1}-y_n\|^2 \leq \|f_2(y_{n+1})-f_2(y_n)\|^2-2\langle f_2(y_{n+1})-f_2(y_n)+y_{n+1}-y_n, j(y_{n+1}-y_n)\rangle. \tag{3.17}$$

By (3.2), we have $f_2(y_{n+1})=J_\gamma^{\partial\varphi_2(\cdot,y_n)}[J_2(f_2(y_n))-\gamma T(u_n,y_n)]$. Thus

$$\|f_2(y_{n+1})-f_2(y_n)\|^2=\|J_\gamma^{\partial\varphi_2(\cdot,y_n)}[J_2(f_2(y_n))-\gamma T(u_n,y_n)]$$
$$-J_\gamma^{\partial\varphi_2(\cdot,y_{n-1})}[J_2(f_2(y_{n-1}))-\gamma T(u_{n-1},y_{n-1})]\|^2.$$

Using the same argument as for (3.9), we have

$$\frac{1}{2}\|f_2(y_{n+1})-f_2(y_n)\|^2 \leq \frac{2}{\alpha_2^2}\|J_2(f_2(y_n))-J_2(f_2(y_{n-1}))\|^2+\frac{2\gamma^2}{\alpha_2^2}\|T(u_n,y_n)$$
$$-T(u_{n-1},y_{n-1})\|^2+\mu^{**2}\|y_n-y_{n-1}\|^2. \tag{3.18}$$

By the Lipschitz continuity of J_2 and f_2, we have

$$\|J_2(f_2(y_n))-J_2(f_2(y_{n-1}))\| \leq \lambda_{J_2}\lambda_{f_2}\|y_n-y_{n-1}\|. \tag{3.19}$$

By the Lipschitz continuity of $T(\cdot,\cdot)$ in both the arguments, (3.3) and D-Lipschitz continuity of H, we have

$$\|T(u_n,y_n)-T(u_n,y_{n-1})\| \leq \lambda_{T_2}\|y_n-y_{n-1}\|. \tag{3.20}$$

$$\begin{aligned}\|T(u_n, y_{n-1}) - T(u_{n-1}, y_{n-1})\| &\leq \lambda_{T_1}\|u_n - u_{n-1}\| \\ &\leq \lambda_{T_1}\Big(1+\frac{1}{n}\Big)D(H(x_n), H(x_{n-1})) \\ &\leq \lambda_{T_1}\lambda_{D_F}\Big(1+\frac{1}{n}\Big)\|x_n - x_{n-1}\|. \end{aligned} \tag{3.21}$$

Using (3.20) and (3.21), it follows that

$$\begin{aligned}\|T(u_n, y_n) - T(u_{n-1}, y_{n-1})\|^2 &\leq 2\|T(u_n, y_n) - T(u_n, y_{n-1})\|^2 \\ &\quad +2\|T(u_n, y_{n-1}) - T(u_{n-1}, y_{n-1})\|^2 \\ &\leq 2\lambda_{T_2}^2\|y_n - y_{n-1}\|^2 \\ &\quad +2(\lambda_{T_1}\lambda_{D_H})^2\Big(1+\frac{1}{n}\Big)^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.22}$$

By (3.19) and (3.22), (3.18) becomes

$$\begin{aligned}\|f_2(y_{n+1}) - f_2(y_n)\|^2 \leq& \Big[\frac{4}{\alpha_2^2}(\lambda_{J_2}\lambda_{f_2})^2 + \frac{8}{\alpha_2^2}\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\Big]\|y_n - y_{n-1}\|^2 \\ &+\frac{8}{\alpha_2^2}\gamma^2(\lambda_{T_1}\lambda_{D_H})^2(1+\frac{1}{n})^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.23}$$

By using the strong accretiveness of f_2 with constant δ_{f_2} and (3.23), (3.17) becomes

$$\begin{aligned}\|y_{n+1}-y_n\|^2 &\leq \|f_2(y_{n+1})-f_2(y_n)\|^2 - 2\langle f_2(y_{n+1})-f_2(y_n)+y_{n+1}-y_n, j(y_{n+1}-y_n)\rangle \\ &\leq \Big[\frac{4}{\alpha_2^2}(\lambda_{J_2}\lambda_{f_2})^2 + \frac{8}{\alpha_2^2}\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\Big]\|y_n - y_{n-1}\|^2 \\ &\quad +\frac{8}{\alpha_2^2}\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\Big(1+\frac{1}{n}\Big)^2\|x_n - x_{n-1}\|^2 - (2\delta_{f_2}+2)\|y_{n+1} - y_n\|^2. \end{aligned} \tag{3.24}$$

It follows that

$$\begin{aligned}\|y_{n+1} - y_n\|^2 &\leq \Big[\frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}\Big]\|y_n - y_{n-1}\|^2 \\ &\quad +\frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\Big(1+\frac{1}{n}\Big)^2}{(2\delta_{f_2}+3)\alpha_2^2}\|x_n - x_{n-1}\|^2 \\ &= \theta_3\|y_n - y_{n-1}\|^2 + \theta_4\|x_n - x_{n-1}\|^2 \\ &\leq \theta_3\|y_n - y_{n-1}\|^2 + \theta_4\|x_n - x_{n-1}\|^2 + 2\sqrt{\theta_3}\sqrt{\theta_4}\|y_n - y_{n-1}\|\|x_n - x_{n-1}\| \\ &= (\sqrt{\theta_3}\|y_n - y_{n-1}\| + \sqrt{\theta_4}\|x_n - x_{n-1}\|)^2. \end{aligned}$$

Thus, we have

$$\|y_{n+1} - y_n\| \leq \sqrt{\theta_3}\|y_n - y_{n-1}\| + \sqrt{\theta_4}\|x_n - x_{n-1}\|, \tag{3.25}$$

where

$$\theta_3 = \frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\tau_2^2\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}$$

and

$$\theta_4 = \frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}.$$

By (3.16) and (3.25), we have

$$\begin{aligned}\|x_{n+1}-x_n\| + \|y_{n+1}-y_n\| &\le (\sqrt{\theta_1}+\sqrt{\theta_4})\|x_n - x_{n-1}\| \\ &\quad +(\sqrt{\theta_2}+\sqrt{\theta_3})\|y_n - y_{n-1}\| \\ &= \theta_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \qquad (3.26)\end{aligned}$$

where

$$\begin{aligned}\theta_n = max\Bigg\{&\sqrt{\frac{4(\lambda_{J_1}\lambda_{f_1})^2 + 8\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}} \\ &+\sqrt{\frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}}, \sqrt{\frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_1}+3)\alpha_1^2}} \\ &+\sqrt{\frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}}\Bigg\}.\end{aligned}$$

Let

$$\begin{aligned}\theta = max\Bigg\{&\sqrt{\frac{4(\lambda_{J_1}\lambda_{f_1})^2 + 8\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}} \\ &+\sqrt{\frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2}{(2\delta_{f_2}+3)\alpha_2^2}}, \sqrt{\frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2}{(2\delta_{f_1}+3)\alpha_1^2}} \\ &+\sqrt{\frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}}\Bigg\}.\end{aligned}$$

Then $\theta_n \to \theta$ as $n \to \infty$. By (3.7), we know that $0 < \theta < 1$ and so (3.26) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus there exists $x \in E_1$ and $y \in E_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we prove that $u_n \to u \in H(x)$ and $v_n \to v \in F(y)$. In fact, it follows from the D-Lipschitz continuity of H and F and (3.3), (3.4)

$$\|u_n - u_{n-1}\| \le \left(1+\frac{1}{n}\right)\lambda_{D_H}\|x_n - x_{n-1}\|, \qquad (3.27)$$

$$\|v_n - v_{n-1}\| \le \left(1+\frac{1}{n}\right)\lambda_{D_F}\|y_n - y_{n-1}\|. \qquad (3.28)$$

From (3.27) and (3.28), we know that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. We can assume that $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Further

$$\begin{aligned}d(u, H(x)) &\le \|u - u_n\| + d(u_n, H(x)) \\ &\le \|u - u_n\| + D(H(x_n), H(x)) \\ &\le \|u - u_n\| + \lambda_{D_H}\|x_n - x\| \to 0, \ as\ n \to +\infty.\end{aligned}$$

Hence $d(u, H(x)) = 0$ and therefore $u \in H(x)$. Similarly, we can show that $v \in F(y)$. By continuity of f_1, f_2, J_1, J_2, S, T, $J_\rho^{\partial\varphi_1}$, $J_\gamma^{\partial\varphi_2}$, φ_1, φ_2, H, F, and Algorithm (3.1), we know that x, y, u and v satisfy the following relations

$$f_1(x) = J_\rho^{\partial\varphi_1(\cdot,x)}[J_1(f_1(x)) - \rho S(x,v)],$$

$$f_2(y) = J_\gamma^{\partial\varphi_2(\cdot,y)}[J_2(f_2(y)) - \gamma T(u,y)].$$

By Theorem 3.1, $(x,y) \in E_1 \times E_2$, $u \in H(x)$ and $v \in F(y)$ is a solution of problem (2.1). This completes the proof. □

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