J. Appl. Math. & Informatics Vol. 27(2009), No. 3 - 4, pp. 685 - 692
Website: http://www.kcam.biz

ON LINEARIZED VECTOR OPTIMIZATION PROBLEMS WITH PROPER EFFICIENCY

MOON HEE KIM

ABSTRACT. We consider the linearized (approximated) problem for differentiable vector optimization problem, and then we establish equivalence results between a differentiable vector optimization problem and its associated linearized problem under the proper efficiency.

AMS Mathematics Subject Classification : 90C26, 90C30, 90C46.
Key words and phrases : Linearized (approximated) optimization problem, properly efficient solution, V-invex, KKT condition.

1. Introduction and preliminaries

Multiobjective optimization problems consists of conflicting objective functions and constraint sets and are to optimize the objective functions over the constraint sets under some concepts of solution, for example, properly efficient solutions, efficient solutions and weakly efficient solutions. Many authors have studied sufficient and necessary optimality conditions of Karush-Kuhn-Tucker type involving (weakly) efficient solutions for a multiobjective programming problem [7, 8, 15, 19, 20].

In the recent years, Hanson[10] has introduced the concept of differentiable invex function which is a generalization of the differentiable convex function, and he proved the Karush-Kuhn-Tucker sufficient optimality theorem and the Wolfe duality results for a nonlinear scalar mathematical programming problem involving invex functions.

Considerable attention has been given recently to devising new methods which solve the original multiobjective mathematical programming problem and its duals by the help of some associated vector optimization problem. One of a such method is that proposed by Antczak [1]. He introduced a new approach

Received November 20, 2008. Revised March 9, 2009. Accepted March 15, 2009.
This work was supported by the Korea Science and Engineering Foundation (KOSEF) NRL program grant funded by the Korea government(MEST)(No. ROA-2008-000-20010-0).

with a modified objective function for solving a differentiable multiobjective optimization problems involved invex functions. He obtained optimality conditions for Pareto optimality by constructing for a considered multiobjective programming problem and equivalent vector minimization problem and then using an invexity concept in multiobjective programming.

Recently, Antczak [2] considered η-approximated problem associated with a primal differentiable scalar optimization problem, and established equivalence between the primal problem and its associated η-approximated optimization problem under invexity assumptions. Some authors [3, 14, 16] extended the results of Antczak [2] to scalar nonlinear optimization problems under invexity assumptions and vector nonlinear optimization problems. In particular, Antczak [3] gave an equivalence results between the differentiable vector optimization problem and the η-approximated vector optimization problem under the (weak) efficiency.

However Antczak's results [2,3] associated problem are nonlinear optimization problem. But from approaches of Antczak [2] it is clear that we can consider an linearized (approximated) problem associated with a primal differentiable problem which is a linear optimization problem.

Mäkelä and Neittaanm*äki* [18] defined linearizations of a locally Lipschitz functions, which are expressed in terms of generalized Clarke directional derivatives [6] and considered linearized (approximated) problems for locally Lipschitz optimization problems, and showed that such linearized problems provided strong tools for deriving methods for the original locally Lipschitz optimization problems. Their linearization is more efficient and more applicable to many optimization problems than the method of Antczak [2, 3].

In this paper, applying the linearization (approximation) method of M*äkelä* and Neittaanm*äki* [18] to a differentiable vector optimization problem, and then we obtain equivalence results between the original problem and the linearized (approximated) problem under the proper efficiency.

Now we consider the differentiable vector optimization problem:

$$(VP) \qquad \text{Minimize} \quad \Big(f_1(x), \cdots, f_p(x)\Big)$$
$$\text{subject to} \quad g_j(x) \leq 0, \ j = 1, \cdots, m,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, p$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \cdots, m$ are differentiable functions. Further let $S := \Big\{x \in \mathbb{R}^n \mid g_j(x) \leq 0, \ j = 1, \cdots, m\Big\}$ and $I(x) = \Big\{i \mid g_i(x) = 0\Big\}$.

Optimization of (VP) is of finding efficient (or Pareto optimal) solutions defined as follows:

Definition 1. (1) A point $\bar{x} \in S$ is said to be an efficient (or Pareto optimal) solution for (VP) if there exists no other feasible point $x \in S$ such that $f_i(x) \leq f_i(\bar{x})$, for all $i = 1, \cdots, p$, but $f_j(x) < f_j(\bar{x})$ for some $j \neq i$.

(2) [9] A point $\bar{x} \in S$ is called a properly efficient solution of (VP) if it is efficient for (VP) and if there exists a scalar $M > 0$ such that for each $i = 1, \cdots, p$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M$$

for some $j \neq i$ such that $f_j(x) > f_j(\bar{x})$ whenever $x \in S$ and $f_i(x) < f_i(\bar{x})$.

(3) A point $\bar{x} \in S$ is said to be a weakly efficient solution for (VP) if there exists no other feasible point $x \in S$ such that $f_i(x) < f_i(\bar{x})$, for all $i = 1, \cdots, p$.

We denote the set of all efficient solutions of (VP), the set of all properly efficient solutions of (VP) and the set of all weakly efficient solutions of (VP) by Eff(VP), $PrEff$(VP) and $WEff$(VP), respectively.

In the differentiable case, Jeyakumar and Mond [13] defined vector invexity as; (VP) is said to be V-invex at x_0 if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$,

$$f_i(x) - f_i(x_0) - \nabla f_i(x_0)^t \eta(x, x_0) \geq 0,$$
$$g_j(x) - g_j(x_0) - \nabla g_j(x_0)^t \eta(x, x_0) \geq 0.$$

Let x_0 be a feasible solution in (VP). We consider the following linearized (approximated) optimization problem (VP$_L$) given by

$$(\text{VP}_L) \text{ Minimize } \Big(f_1(x_0) + \nabla f_1(x_0)^t(x - x_0), \cdots, f_p(x_0) + \nabla f_p(x_0)^t(x - x_0)\Big)$$
$$\text{subject to } g_j(x_0) + \nabla g_j(x_0)^t(x - x_0) \leq 0, \ j = 1, \cdots, m.$$

We denote the set of all efficient solutions of (VP$_L$), the set of all properly efficient solutions of (VP$_L$) and the set of all weakly efficient solutions of (VP$_L$) by Eff(VP$_L$), $PrEff$(VP$_L$) and $WEff$(VP$_L$), respectively.

Now we introduce the contingent cones which locally approximate sets and is needed for obtaining optimality conditions for properly efficient solutions of (VP).

Definition 2. [4] Let $S \subset \mathbb{R}^n$ and $x \in S$. The contingent cone to the set S at x, denoted by $T(S, x)$, is the set of limits of the form $h = \lim_{n \to \infty} t_n(x_n - x)$, where $\{t_n\}$ is a sequence of positive real numbers and $\{x_n\}$ is a sequence in S with limit x.

Now we give Abadie constraint qualification imposed on the constraint set of (VP) problems to obtain Kuhn-Tucker type necessary optimality theorems for (VP).

Definition 3. [5] The constraint set S of (VP) is said to satisfy Abadie constraint qualification at $x_0 \in S$ if

$$T(S, x_0) = \Big\{d \in \mathbb{R}^n \mid \nabla g_i(x_0)^t d \leq 0 \text{ for any } i \in I(x_0)\Big\}.$$

Now, we give a well-known alternative theorem which is found in [17].

Lemma 1. *(Farkas Lemma) For a given* $m \times n$ *matrix* A *and each given vector* b *in* $\mathbb{R}^n$*,*
either
I. $Ax \leq 0,\ bx > 0$ *has a solution* $x \in \mathbb{R}^n$
or
II. $A^t y = b,\ y \geq 0$ *has a solution* $y \in \mathbb{R}^m$*,*
but never both.

Following the approach of the Proof of Theorem 3.16 in [19], we give the following necessary optimality theorem for a properly efficient solution of (VP), which can be found in [15]:

Theorem 1. *If* x_0 *is a properly efficient solution of (VP), then for each* $i = 1, \cdots, p$*,*

$$\left\{d \in \mathbb{R}^n : \nabla f_i(x_0)^t d < 0\right\} \cap \left\{d \in \mathbb{R}^n : \nabla f_k(x_0)^t d \leq 0,\ k \neq i\right\} \cap T(S, x_0) = \emptyset.$$

2. Equivalence results

From Theorem 1, we obtain the following Kuhn-Tucker type necessary optimality theorem.

Theorem 2. *Let* $x_0 \in PrEff(VP)$ *and suppose that the Abadie constraint qualification satisfies at* x_0*. Then there exist* $\lambda_i > 0,\ i = 1, \cdots, p,\ \mu_j \geq 0,\ j = 1, \cdots, m$ *such that*

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) = 0,$$
$$\sum_{j=1}^{m} \mu_j g_j(x_0) = 0.$$

Proof. Let $x_0 \in PrEff(\text{VP})$. By Theorem 1 and the Abadie constraint qualification, for all i, there exists no $d \in \mathbb{R}^n$ such that

$$\nabla f_i(x_0)^t d < 0,$$
$$\nabla f_k(x_0)^t d \leq 0,\ k \neq i,$$
$$\nabla g_j(x_0)^t d \leq 0,\ j \in I(x_0).$$

By Farkas Lemma, for each i, there exists $\alpha_k^i \geq 0,\ u_j^i \geq 0$ such that

$$-\nabla f_i(x_0) = \sum_{k \neq i} \alpha_k^i \nabla f_k(x_0) + \sum_{j \in I(x_0)} u_j^i \nabla g_j(x_0), \text{ for all } i.$$

Letting $u_j^i = 0$, for any $j \notin I(x_0)$, we have $0 = \nabla f_i(x_0) + \sum_{k \neq i} \alpha_k^i \nabla f_k(x_0) + \sum_{j \in I(x_0)} u_j^i \nabla g_j(x_0)$ and $u_j^i g_j(x_0) = 0$, for $j = 1, \cdots, m$. Summing over i, yields,

$$\left(1 + \sum_{i \neq 1} \alpha_1^i\right) \nabla f_1(x_0) \quad + \quad \left(1 + \sum_{i \neq 2} \alpha_2^i\right) \nabla f_2(x_0) + \cdots + \left(1 + \sum_{i \neq p} \alpha_p^i\right)$$
$$\times \nabla f_p(x_0) + \sum_{j=1}^{m} (u_j^1 + \cdots + u_j^p) \nabla g_j(x_0) = 0,$$

and $\left(u_j^1 + \cdots + u_j^p\right) g_j(x_0) = 0,\ j = 1, \cdots, m$. Letting $\lambda_k = 1 + \sum_{k \neq i} \alpha_k^i,\ k = 1, \cdots, p$ and $\mu_j = u_j^1 + \cdots + u_j^p$ for $j = 1, \cdots, m$, we obtain $\lambda_i > 0,\ i = 1, \cdots, p,\ \mu_j \geq 0, j = 1, \cdots, m$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) = 0,$$
$$\sum_{j=1}^{m} \mu_j g_j(x_0) = 0.$$

□

The following definition on KKT point of (VP) is found in [19]:

Definition 4. x_0 is said to be a KKT point of (VP) if $g_j(x_0) \leq 0,\ j = 1, \cdots, m$ and there exist $\lambda_i > 0,\ i = 1, \cdots, p,\ \ \mu_j \geq 0,\ j = 1, \cdots, m$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) = 0, \tag{1}$$
$$\sum_{j=1}^{m} \mu_j g_j(x_0) = 0.$$

Theorem 3. *If x_0 is a KKT point of (VP), then $x_0 \in PrEff(VP_L)$.*

Proof. Let x_0 be a KKT point of (VP). Suppose to the contrary that x_0 is not a properly efficient solution of (VP$_L$). Since (VP$_L$) is a linear vector optimization, it follows from Iserman's result in [12], x_0 is not an efficient solution of (VP$_L$). Hence there exist $\widetilde{x}$ which is feasible for (VP$_L$) such that

$$f_i(x_0) + \nabla f_i(x_0)^t(\widetilde{x} - x_0) \leq f_i(x_0) + \nabla f_i(x_0)^t(x_0 - x_0) \quad \text{for all} \quad i,$$
$$f_j(x_0) + \nabla f_j(x_0)^t(\widetilde{x} - x_0) < f_j(x_0) + \nabla f_j(x_0)^t(x_0 - x_0)$$

for some j. Thus $\nabla f_i(x_0)^t(\widetilde{x} - x_0) \leq 0$ for all i and $\nabla f_j(x_0)^t(\widetilde{x} - x_0) < 0$ for some j. Since $\lambda_i > 0, \; i = 1, \cdots, p$,

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0)^t(\widetilde{x} - x_0) < 0. \tag{2}$$

Since $\widetilde{x}$ is feasible for (VP_L), $g_j(x_0) + \nabla g_j(x_0)^t(\widetilde{x} - x_0) \leq 0, \; j = 1, \cdots, m$. Since $\mu_j \geq 0$ and $\sum_{j=1}^{m} \mu_j g_j(x_0) = 0$,

$$\sum_{j=1}^{m} \mu_j \nabla g_j(x_0)^t(\widetilde{x} - x_0) \leq 0. \tag{3}$$

Hence from (2) and (3), it follows that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0)^t(\widetilde{x} - x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0)^t(\widetilde{x} - x_0) < 0.$$

This contradicts (1). □

Theorem 4. *If $x_0 \in PrEff$ (VP_L), then x_0 is a KKT point of (VP).*

Proof. Let x_0 be a properly efficient solution of (VP_L). Then it is clear that $g_j(x_0) \leq 0, \; j = 1, \cdots, m$, since $\nabla g_j(x_0)^t(x_0 - x_0) = 0, \; j = 1, \cdots, m$. Since (VP_L) is a linear vector optimization and x_0 is properly efficient solution of (VP_L), then from Theorem 2 in [9], there exist $\lambda_i > 0, \; i = 1, \cdots, p$, such that x_0 is optimal solution of the following scalar optimization problem:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^{p} \lambda_i f_i(x_0) + \sum_{i=1}^{p} \lambda_i \nabla f_i(x_0)^t(x - x_0) \\ \text{subject to} & g_j(x_0) + \nabla g_j(x_0)^t(x - x_0) \leq 0, \; j = 1, \cdots, m. \end{array}$$

From the optimality theorem for a scalar linear optimization problem, there exist $\mu_j \geq 0, \; j = 1, \cdots, m$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) = 0,$$
$$\sum_{j=1}^{m} \mu_j g_j(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0)^t(x_0 - x_0) = 0.$$

Moreover, $\sum_{j=1}^{m} \mu_j g_j(x_0) = 0$ since $\sum_{j=1}^{m} \mu_j \nabla g_j(x_0)^t(x_0 - x_0) = 0,$. Thus x_0 is a KKT point of (VP). □

Theorem 5. *Let $x_0 \in PrEff\ (VP_L)$. If $\left(\sum_{i=1}^{p} \lambda_i f_i, \sum_{j=1}^{m} \mu_j g_j\right)$ are V-invex at x_0, then $x_0 \in PrEff(VP)$.*

Proof. Let x_0 be a properly efficient solution of (VP$_L$). Then by Theorem 4, x_0 is a KKT point of (VP), $\sum_{j=1}^{m} \mu_j g_j(x_0) = 0$. Also, for any $x \in S$, $\sum_{j=1}^{m} \mu_j g_j(x) \leq 0$ and hence $\sum_{j=1}^{m} \mu_j g_j(x) \leq \sum_{j=1}^{m} \mu_j g_j(x_0)$. Since $\sum_{j=1}^{m} \mu_j g_j$ is V-invex at x_0, $\sum_{j=1}^{m} \mu_j \nabla g_j(x_0)^t \eta(x, x_0) \leq 0$. From (1), we obtain $\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0)^t \eta(x, x_0) \geq 0$. Since $\sum_{i=1}^{p} \lambda_i f_i$ is V-invex at x_0, $\sum_{i=1}^{p} \lambda_i f_i(x) \geq \sum_{i=1}^{p} \lambda_i f_i(x_0)$ for all $x \in S$. Thus x_0 is optimal solution of the following scalar optimization problem:

$$\begin{aligned} &\text{Minimize} && \sum_{i=1}^{p} \lambda_i f_i(x) \\ &\text{subject to} && g_j(x) \leq 0,\ j = 1, \cdots, m. \end{aligned}$$

Therefore, by Theorem 1 in [9], x_0 is a properly efficient solution of(VP). □

References

1. T. Antczak, *A new approach to multiobjective programming with a modified objective function*, J. Global Optimization **27** (2003), 485-495.
2. T. Antczak, *An η-approximation approach for nonlinear mathematical programming problems involving invex functions*, Num. Functional Anal. Optimization **25** (2004), 423-438.
3. T. Antczak, *An η-approximation method in nonlinear vector optimization*, Nonlinear Analysis **63** (2005), 225-236.
4. J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkäuser, Boston, 1990.
5. M.R. Bazaraa, H.D. Sherali and C.M. Shetty, *Nonlinear Programming; Theory and Algorithms*, second edition, John Wiley & Sons Inc., New York, 1993.
6. F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.
7. L.N. Das and S. Nanda, *Proper efficiency conditions and duality for multiobjective programming problems involving semilocally invex functions*, Optimization **34** (1995), 43-51.
8. J. Dutta, *On generalized pre-invex functions*, Asia-Pacific J. Oper. Res. **18** (2001), 257-272.
9. A. M. Geoffrion, *Proper efficiency and the theory of vector maximization*, J. Math. Anal. Appl. **22** (1968), 618-630.
10. M.A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981), 545-550.
12. H. Isermann, *Proper efficiency and linear vector maximum problem*, Oper. Res. **22** (1974), 189-191.

13. V. Jeyakumar and B. Mond, *On generalised convex mathematical programming*, J. Austral. Math. Soc. Ser. B **34** (1992), 43-53.
14. D. S. Kim, *Multiobjective fractional programming with a modified objective function*, Commun. Korean Math. Soc. **20** (2005), 837-847.
15. M.H. Kim, *On optimality and duality for multiobjective optimization problems*, Thesis for the Degree of Master of Science, February 1999.
16. Lun Li and Jun Li, *Equivalence and existence of weak Pareto optima for multiobjective optimization problems with cone constraints*, Appl. Math. Lett. **21** (2008), 599-606.
17. O.L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.
18. Marko M. Mäkelä and Pekka Neittaanmäki, *Nonsmooth Optimization: Analysis and Algorithms with Applications to Optimal Control*, World Scientific Publishing Co. Pte. Ltd., 1992.
19. Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press, New York, 1985.
20. T. Weir, B. Mond, B.D. Craven, *On duality for weakly minimized vector valued optimization*, Optimization **17** (1986), 711-721.

Moon Hee Kim received her Ph. D at Pukyong National University under the direction of Professor Gue Myung Lee. Since 2006 she has been at the Tongmyong University. Her research interests focus on vector optimization problem and variational inequality.

Department of Multimedia Engineering, Tongmyong University, Pusan, Korea
e-mail: mooni@tu.ac.kr