J. Appl. Math. & Informatics Vol. 27(2009), No. 3 - 4, pp. 603 - 619
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MULTIOBJECTIVE CONTINUOUS PROGRAMMING CONTAINING SUPPORT FUNCTIONS

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ABSTRACT. Wolfe and Mond-Weir type dual to a nondifferentiable continuous programming containing support functions are formulated and duality is investigated for these two dual models under invexity and generalized invexity. A close relationship of our duality results with those of nondifferentiable nonlinear programming problem is also pointed out.

AMS Mathematics Subject Classification : Primary 90C30, Secondary 90C11, 90C20, 90C26.

Key words and phrases : Multiobjective continuous programming; support functions; Wolfe type duality; Mond and Weir duality; invexity.

1. Introduction

Chandra, Craven and Husain [3] obtained necessary optimality conditions for a constrained continuous programming having term with a square root of a quadratic form in the objective function, and using these optimality conditions formulated Wolfe type dual and established weak, strong and Huard [11] type converse duality theorems under convexity of functions. Subsequently, for the problems of [3], Bector, Chandra and Husain [2] constructed a Mond-Weir type dual which allows weakening of convexity hypotheses of [3] and derived various duality results under generalized convexity of functionals.

Recently, Husain and Jabeen [7] derived optimality conditions for a nondifferentiable continuous programming problem in which nondifferentiable enters due to appearance of support functions in the integrand of the objective functional as well as in each constraint function. As an application of these optimality conditions, the authors in [7] formulated both Wolfe and Mond-Weir type duals to the nondifferentiable continuous programming problem and established various duality results under invexity and generalized invexity.

Received July 20, 2007. Revised April 20, 2008. Accepted May 10, 2008. *Corresponding author.

There exist an extensive literature relating to optimality and duality in multiobjective nonlinear programming. But the status of continuous programming for optimality and duality is not very accomplished. Duality and optimality for multiobjective variational problems which can be referred to as continuous programming problems have been studied by a number of authors notably Bector and Husain [1], Chen [5] and many others cited in these references.

Generally any real world problems can be identified as multiple conflicting criteria, e.g., the problems of oil refinery scheduling, production planning, portfolio selection and many others can be modeled as multiobjective programming problems. Motivated with this observation in this exposition we study duality for a class of nondifferentiable continuous programming problems containing support functions. The close relationship of our duality results with those of nonlinear programming is also witnessed.

2. Invexity and generalized invexity

Invexity was introduced for functions in variational problems by Mond, Chandra and Husain [10] while Mond and Smart [8] defined invexity for functionals instead of functions. Here we introduce extended forms of definitions of invexity and various generalized invexity for functional in variational problems involving higher order derivatives.

Let $I = [a, b]$ be a real interval; $\phi : I \times R^n \times R^n \to R$ and $g : I \times R^n \times R^n \to R^m$ be continuously differentiable function. In order to consider $\phi(t, x(t), \dot{x}(t))$ where $x = (x^1, x^2, \ldots, x^n)^T$, the gradient vector of ϕ is differentiable with derivative $\dot{x}$, denote the partial derivatives of ϕ to x and $\dot{x}$ respectively denoted by

$$\phi_x = \left[\frac{\partial \phi}{\partial x^1}, \ldots, \frac{\partial \phi}{\partial x^n}\right]^T \quad \text{and} \quad \phi_{\dot{x}} = \left[\frac{\partial \phi}{\partial \dot{x}^1}, \ldots, \frac{\partial \phi}{\partial \dot{x}^n}\right]^T$$

Let X designates the space of piecewise functions $x : I \to R^n$ possessing derivatives x and $\dot{x}$ with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Rightarrow x(t) = \alpha + \int_a^t u(s)ds,$$

where α is given boundary value; thus $D \equiv \frac{d}{dt}$ except at discontinuities.

In the results to follow, we use $C(I, R^m)$ to denote the space of continuous functions $\phi : I \to R^k$ with the uniform norm $\|\phi\| = \sup |\phi|_{t \in I}$; Let $C_+(I, R^m)$ denote the cone of non-negative function in $C(I, R^m)$. The partial derivatives of g are defined using $m \times n$ matrices; superscript T denotes matrix transpose.

The following definitions are required for further analysis.

Definition 1 (Invexity). If there exists vector function $\eta(t, u, \dot{u}, x, \dot{x}) \in R^n$ with $\eta = 0$ and $x(t) = u(t), t \in I$ such that for a scalar function $\phi(t, x, \dot{x})$, the functional $\Phi(x, \dot{x}) = \int_I \phi(t, x, \dot{x})dt$ satisfies

$$\Phi(x, \dot{u}) - \Phi(x, \dot{x}) \geqq \int_I \{\eta \phi_x(t, x, \dot{x}) + (D\eta)^T \phi_{\dot{x}}(t, x, \dot{x})\}dt,$$

Φ is said to be invex in x and $\dot{x}$ on I with respect to η.

Definition 2 (Pseudoinvexity). Φ is said to be pseudoinvex in x and $\dot{x}$ with respect to η if

$$\int_I \{\eta^T \phi_x(t,x,\dot{x}) + (D\eta)^T \phi_{\dot{x}}(t,x,\dot{x})\}dt \geqq 0$$

implies $\Phi(x,\dot{u}) \geqq \Phi(x,\dot{x})$.

Definition 3 (Quasi-invex). The functional Φ is said to quasi-invex in x and $\dot{x}$ with respect to η if

$$\Phi(x,\dot{u}) \leqq \Phi(x,\dot{x})$$

implies

$$\int_I \{\eta^T \phi_x(t,x,\dot{x}) + (D\eta)^T \phi_{\dot{x}}(t,x,\dot{x})\}dt \leqq 0.$$

Definition 4 (Support function). Let K be a compact set in R^n, then the support function of K is defined by

$$s(x(t)|K) = \max\{x(t)^T v(t) : v(t) \in K, t \in I\}$$

A support function, being convex everywhere finite, has a subdifferential in the sense of convex analysis. From [8] subdifferential of $s(x(t)|K)$ is given by

$$\partial s(x(t)|K) = \{z(t) \in K, t \in I \ \text{ such that } \ |x(t)^T z(t) = s(x(t)|K)\}.$$

Definition 5 (Efficient Solution). A feasible solution $\bar{x}$ is efficient for (VPE) if there exist no other feasible x for (VPE) such that for some $i \in P = \{1,2,\ldots,p\}$,

$$\int_I (f^i(x) + S(x|C^i))dt < \int_I (f^i(\bar{x}) + S(\bar{x}|C^i))dt \ \text{ for all } \ i \in P.$$

and

$$\int_I (f^j(x) + S(x|C^j))dt \leqq \int_I (f^j(\bar{x}) + S(\bar{x}|C^j))dt \ \text{ for all } \ j \in P, j \neq i.$$

3. Variational problem and optimality conditions

Before stating our variational problem and deriving its necessary optimality condition, we mention the following conventions for vectors x and y in n-dimensional Euclidian space R^n to be used throughout the analysis of this research.

$$\begin{aligned}
&x < y, && \Leftrightarrow && x_i < y_i, && i = 1,2,\ldots,n.\\
&x \leqq y, && \Leftrightarrow && x_i \leqq y_i, && i = 1,2,\ldots,n.\\
&x \leq y, && \Leftrightarrow && x_i \leq y_i, && i = 1,2,\ldots,n, \text{ but } x \neq y\\
&x \nleq y, && \text{is the negation of } x \leq y
\end{aligned}$$

For $x, y \in R$, $x \leq y$ and $x < y$ have the usual meaning.

We present the following nondifferentiable continuous programming problem containing support function.

$$\text{(CP)} \qquad \text{Minimize}\left(\int_I (f^1(t,x,\dot{x}) + S(x|C^1))dt, \ldots, \int_I (f^p(t,x,\dot{x}) + S(x|C^p))dt\right)$$

Subject to

$x(a) = \alpha, \quad x(b) = \beta$

$g^j(t, x, \dot{x}) + S(x|D^j) \leqq 0, \quad j = 1, 2, \ldots, m, \ t \in I$

where $f^i : I \times R^n \times R^n \to R, (i = 1, 2, \ldots, p), g : I \times R^n \times R^n \to R^m, j = 1, 2, \ldots, m$ are continuously differentiable function, and for each $C^i, i = 1, \ldots, p$ and $D^j, j = 1, \ldots, m$ are compact convex set in R^n.

In order to validate the strong duality theorem, we will require the following lemma of Chankong and Haimes [4].

Lemma 1. *A point $\bar{x}$ is an efficient for (CP) if and only if $\bar{x}$ is an optimal solution for all*

$$(P_k(\bar{x})) \text{ Minimize} \left(\int_I (f^k(t, x, \dot{x}) + S(x|C^k))dt \right)$$

Subject to

$$x(a) = \alpha, \quad x(b) = \beta$$

$$\int_I (f^i(t, x, \dot{x}) + S(x|C^i))dt \nleq \int_I (f^i(t, \bar{x}, \dot{\bar{x}}) + S(\bar{x}|C^i))dt, \text{ for all } i \neq k$$

$$g^j(t, x, \dot{x}) + S(x|D^j) \leqq 0, \quad j = 1, 2, \ldots, m, \ t \in I$$

4. WOLFE duality

The following problem is formulatd as Wolfe type dual for the problem (CP).
(WCD) Maximize

$$\left(\int_I (f^1(t, u, \dot{u}) + u(t)^T z^1(t)) + \sum_{j=1}^m y^j(t)^T (g^j(t, u, \dot{u}) + u(t)^T w^j(t)) \right) dt$$

$$, \ldots, \int_I \left(f^p(t, u, \dot{u}) + u(t)^T z^p(t) + \sum_{j=1}^m y^j(t)^T (g^j(t, u, \dot{u}) + u(t)^T w^j(t)) \right) dt \Bigg)$$

Subject to

$$u(a) = \alpha, u(b) = \beta \tag{4.1}$$

$$\sum_{i=1}^p \lambda^i (f_x^i + z^i(t)) + \sum_{j=1}^m y^j(t)(g_x^j + w^j(t)) = D(\lambda^T f_{\dot{x}}^i + y(t) g_{\dot{x}}), t \in I \tag{4.2}$$

$$z^i(t) \in C^i, \quad i = 1, 2, \ldots, p \tag{4.3}$$

$$w^j(t) \in D^j, \quad j = 1, 2, \ldots, m \tag{4.4}$$

$$\lambda > 0, \sum_{i=1}^p \lambda^i = 1 \tag{4.5}$$

Theorem 1 (Weak Duality). *Let $\bar{x}$ be feasible for* (CP) *and $(u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)$ be feasible for* (WCD). *If for all feasible $(x, u, y, z^1, \ldots, z^p, w^1, \ldots,$*

$w^m, \lambda)$ and with respect to $\eta \equiv \eta(t, x, u)$,

$$\int_I \left(\sum_{i=1}^{p} \lambda^i (f^i(t,.,.) + (\cdot)^T z^i(t))dt + \sum_{j=1}^{m} y^j(t)(g^j(t,.,.) + (\cdot)^T w^j(t)) \right) dt$$

is pseudoinvex then the following cannot hold.

$$\begin{aligned} \int_I (f^i(t,x,\dot{x}) + S(x(t)|C^i))dt \quad &\leq \quad \int_I (f^i(t,u,\dot{u}) + u(t)^T z^i(t)dt \\ &\quad + \sum_{j=1}^{m} y^j(t)^T (g^j(t,u,\dot{u}) + u(t)^T w^j(t)))dt \end{aligned}$$

for all $i \in \{1, \ldots, p\}$, and

$$\begin{aligned} \int_I (f^r(t,x,\dot{x}) + S(x(t)|C^r))dt \quad &< \quad \int_I (f^r(t,u,\dot{u}) + u(t)^T z^r(t)dt \\ &\quad + \sum_{j=1}^{m} y^j(t)^T (g^j(t,u,\dot{u}) + u(t)^T w^j(t)))dt \end{aligned}$$

for some $r \in \{1, 2, \ldots, p\}$.

Proof. Suppose that the conclusion of the theorem hold. With the feasibility of the problems (CP) and (WCD), together with $x^T(t)z^i(t) \leqq S(x(t)|C^i), i = 1, 2, \ldots, p$, we have

$$\begin{aligned} &\int_I \left(f^i(t,x,\dot{x}) + x(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t,x,\dot{x}) + x(t)^T w^j(t)) \right) dt \\ &\quad \leq \int_I \left(f^i(t,u,\dot{u}) + u(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t,u,\dot{u}) + u(t)^T w^j(t)) \right) dt \end{aligned}$$

for all $i \in \{1, \ldots, p\}$

$$\begin{aligned} &\int_I (f^r(t,x,\dot{x}) + x(t)^T z^r(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t,x,\dot{x}) + x(t)^T w^j(t)))dt \\ &\quad < \int_I (f^r(t,u,\dot{u}) + u(t)^T z^r(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t,u,\dot{u}) + u(t)^T w^j(t)))dt \end{aligned}$$

for some $r \in \{1, 2, \ldots, p\}$.

Now in view of $\lambda > 0$ and $\sum\limits_{i=1}^{p} \lambda^i = 1$, these inequalities yield

$$\begin{aligned} &\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x}) + x(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t,x,\dot{x}) + x(t)^T w^j(t)) \right) dt \\ &\quad < \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,u,\dot{u}) + u(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t,u,\dot{u}) + u(t)^T w^j(t)) \right) dt \end{aligned}$$

This in view of the pseudoinvexity of

$$\int_I \left(\sum_{i=1}^{p} \lambda^i (f^i(t,.,.) + (\cdot)^T z^i(t))dt + \sum_{j=1}^{m} y^j(t)(g^j(t,.,.) + (\cdot)^T w^j(t)) \right) dt$$

gives,

$$\begin{aligned} 0 &> \int_I \Big\{ \eta^T \sum_{i=1}^{p} \lambda^i (f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t))) \\ &\qquad + (D\eta)^T (\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) \Big\} dt \\ &= \int_I \eta^T \Big\{ \sum_{i=1}^{p} \lambda^i (f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t))) \\ &\qquad - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) \Big\} dt + \eta^T (\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) \Big|_{t=a}^{t=b} \\ &\qquad\qquad \text{(by integration by parts)} \end{aligned}$$

Using the boundary conditions which at $t = a$, $t = b$ give $\eta = 0$, we have

$$\begin{aligned} &= \int_I \eta^T \Big\{ \sum_{i=1}^{p} \lambda^i (f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t))) \\ &\qquad - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) \Big\} dt < 0 \end{aligned} \tag{4.6}$$

From the equality constraint of the (WCD), we have

$$\int_I \eta^T \Big\{ \sum_{i=1}^{p} \lambda^i (f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t))) - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) \Big\} dt = 0$$

This relation (6) contradicts the equality constraint. Hence the conclusion of the theorem is true.

Theorem 2 (Strong Duality). *Let $\bar{x}$ be a feasible solution of* (CP) *and for at least one i, $i \in \{1, 2, \ldots, p\}, \bar{x}$ satisfies the regularity condition* [3] *for* $(P_i(x))$. *Then there exist $\bar{\lambda} \in R^p$ with $\bar{\lambda}^T = (\bar{\lambda}^1, \ldots, \bar{\lambda}^i, \ldots, \bar{\lambda}^p)$, $z^i(t) \in C^i$, $i = 1, 2, \ldots, p$, $w^j(t) \in D^j$, $j = 1, 2, \ldots, m$ piecewise smooth $\bar{v} : I \to R^m$ with $\bar{v}^T = (\bar{v}^1, \ldots, \bar{v}^i, \ldots, \bar{v}^m)$, such that $(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda})$ is feasible for* (WCD) *and the objective values of* (CP) *and* (WCD) *are equal, and*

$$\sum_{j=1}^{m} \int_I \bar{y}^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}(t)^j) dt = 0.$$

Further, if the hypothesis of Theorem 1 *is met, then $(x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)$ is an efficient solution of* (WCD).

Proof. Since $\bar{x}$ is an efficient solution of (CP), by Lemma 1, $\bar{x}$ is an optimal solution of $(P_k(\bar{x}))$. Consequently, by Theorem1 [7] there exists $\tau \in R^p$ with $\tau^T = (\tau^1, \dots, \tau^i, \dots, \tau^p)$, $z^i(t) \in C^i$, $i = 1, 2, \dots, p$ and $w^j(t) \in D^j$, $j = 1, 2, \dots, m$ and piecewise smooth $\bar{v} : I \to R^m$ with $\bar{v}^T = (\bar{v}^1, \dots, \bar{v}^i, \dots, \bar{v}^m)$ such that the following optimality conditions [3] hold:

$$\tau^k(f_x^k + \bar{z}^k(t) - Df_{\dot{x}}^k) + \sum_{j=1}^{m} \bar{v}^j(t)(g_x^j + \bar{w}^j(t)) - Dv(t)^T g_{\dot{x}}$$

$$+ \sum_{\substack{i=1 \\ i \neq k}}^{p} \tau^i(f_x^i + \bar{z}^i(t) - Df_{\dot{x}}^i) = 0 \tag{4.7}$$

$$\sum_{j=1}^{m} y^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t))dt = 0 \tag{4.8}$$

$$\bar{x}(t)^T \bar{z}^i(t) = S(\bar{x}(t)|C^i), \quad i = 1, 2, \dots, p, t \in I \tag{4.9}$$

$$\bar{x}(t)^T \bar{w}^j(t) = S(\bar{x}(t)|D^j), \quad j = 1, 2, \dots, m, t \in I \tag{4.10}$$

$$z^i(t) \in C^i, \quad i = 1, 2, \dots, p \tag{4.11}$$

$$w^j(t) \in D^j, \quad j = 1, 2, \dots, m \tag{4.12}$$

$$\tau > 0, v(t) \geqq 0, t \in I \tag{4.13}$$

Dividing (7), (8) and (13) by $\sum_{i=1}^{p} \tau^i \neq 0$, and setting $\lambda^i = \dfrac{\tau^i}{\sum_{i=1}^{p} \tau^i}$ and $y^i(t) = \dfrac{v^i(t)}{\sum_{i=1}^{p} \tau^i}$, we have

$$\sum_{j=1}^{m} \lambda^i(f_x^i + \bar{z}^i(t)) + \sum_{j=1}^{m} y^j(t)(g_x^j + \bar{x}(t)\bar{w}^j(t)) = D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) \tag{4.14}$$

$$\sum_{j=1}^{m} \bar{y}^j(t)(g^j(t, x, \dot{x}) + \bar{x}(t)^T \bar{w}^j(t)) = 0, t \in I \tag{4.15}$$

$$\lambda > 0, y(t) \geqq 0, t \in I, \sum_{i=1}^{p} \lambda^i = 1 \tag{4.16}$$

Consequently from (11), (12), (14), (15) and (16), the feasibility of $(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \dots, \bar{z}^p, \bar{w}^1, \dots, \bar{w}^m, \bar{\lambda})$ for (WCD) follows.

In view of (9), (10) and (15), we have for each $i = 1, \dots, p$.

$$\int_I (f^i(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{z}^i(t) + \sum_{j=1}^{m} \bar{y}^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)))dt = 0$$

$$= \int_I f^i(t,\bar{x},\dot{\bar{x}}) + S(\bar{x}(t)|C^i)dt, \quad i = 1,2,\ldots,p.$$

This in view of Theorem 1, yields the efficiency of $(x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)$ for (WCD)

For the converse duality, we make the assumption that X denote the space of the piecewise differentiable function $x : I \to R^n$ for which $x(a) = 0 = x(b)$ equipped with the norm $\|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$.

(WCD) may be rewritten in the following form.

$$\text{Minimize}\Bigg(\int_I -(f^1(t,u,\dot{u}) + u(t)^T z^1(t) + \sum_{j=1}^m y^j(t)^T(g^j(t,u,\dot{u}) + u(t)^T w^j(t)))dt,$$
$$\ldots, \int_I -(f^p(t,u,\dot{u}) + u(t)^T z^p(t) + \sum_{j=1}^m y^j(t)^T(g^j(t,u,\dot{u}) + u(t)^T w^j(t)))dt\Bigg)$$

Subject to

$$u(a) = \alpha, u(b) = \beta; \quad \theta(t,x,\dot{x},y,\lambda) = 0;$$

$$z^i(t) \in C^i, i = 1,2,\ldots,p\ ; \quad w^j(t) \in D^j, j = 1,2,\ldots,m; \quad \lambda > 0, \sum_{i=1}^p \lambda^i = 1$$

where

$$\begin{aligned}\theta &= \theta(t,x(t),\dot{x}(t),y(t),\lambda)\\ &= \sum_{i=1}^p \lambda^i(f^i_x + z^i(t)) + \sum_{j=1}^m y^j(t)(g^j_x + w^j(t)) - D(\lambda^T f^i_{\dot{x}} + y(t)g_{\dot{x}}), t \in I\end{aligned}$$

with $\ddot{x} = D^2x(t)$ and $\dot{y} = Dy(t)$

Consider $\theta(t,x(\cdot),\dot{x}(\cdot),\ddot{x}(\cdot),y(\cdot),\dot{y}(\cdot),\lambda) = 0$ as defining a mapping $\psi : I \times X \times Y \times R^p \to B$ where Y is a space of piecewise differentiable function and B is the Banach Space. In order to apply results of Craven [6] to the problem (WCD), the infinite dimensional inequality must be restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $[\psi_x(x,y,\lambda), \psi_y(x,y,\lambda), \psi_\lambda(x,y,\lambda)]$.

Theorem 3 (Converse Duality). *Let* $(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda})$ *be an efficient solution for* (WCD). *Assume that*

(H_1) *The Frèchèt derivative* ψ' *has a (weak*) closed range,*

(H_2) f *and* g *are twice continuously differentiable and*

(H_3) $(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}})\beta(t) = 0, \Rightarrow \beta(t) = 0, t \in I$

Further, if the assumptions of Theorem 1 *are satisfied, then* $\bar{x}$ *is an efficient solution of* (CP).

Proof. Since $(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda})$ with ψ' having a (weak*) closed range ,is an efficient solution of (WD), then there exist $\alpha \in R^p$ with $\alpha^T = (\alpha^1,\ldots,\alpha^i,\ldots,\alpha^p)$, piecewise smooth $\beta : I \to R^n$ and $\mu : I \to R^m$ with

$\mu(t)^T = (\mu^1(t), \ldots, \mu^m(t))$, $\eta \in R^p$ and $\kappa \in R$ such that the following Fritz-John conditions [6] holds,

$$-\sum_{i=1}^{p} \alpha^i((f_x^i + z^i(t)) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t)) - D(\alpha^T f_{\dot{x}} + y(t)^T g_{\dot{x}}))$$

$$+\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2 \beta(t)^T \theta_{\ddot{x}} = 0, t \in I \quad (4.17)$$

$$-(\alpha^T e)(g^j + \bar{x}(t)^T \bar{w}^j(t)) + \beta(t)^T \theta_{y^j} - D\beta(t)^T \theta_{\dot{y}^j} - \mu^j(t) = 0, t \in I \quad (4.18)$$

$$j = 1, 2, \ldots, m$$

$$(f_x^i + z^i(t) - Df_{\dot{x}}^i)\beta(t) - \eta^i + \kappa = 0 \; i = 1, \ldots, p \quad (4.19)$$

$$(\beta(t)\lambda^i - \alpha^i \bar{x}(t)) \in N_{C^i}(\bar{z}^i(t)), i = 1, \ldots, p, t \in I \quad (4.20)$$

$$(\beta(t) - (\alpha^T e)\bar{x}(t))\bar{y}^j(t) \in N_{D^j}(\bar{w}^j(t)), \; j = 1, \ldots, m, t \in I \quad (4.21)$$

$$\eta^T \bar{\lambda} = 0 \quad (4.22)$$

$$\kappa(\sum_{i=1}^{p} \lambda^i - 1) = 0 \quad (4.23)$$

$$\mu(t)^T \bar{y}(t) = 0, t \in I \quad (4.24)$$

$$(\alpha, \eta, \kappa, \mu(t)) \geqq 0 \quad (4.25)$$

$$(\alpha, \beta(t), \eta, \kappa, \mu(t)) \neq 0 \quad (4.26)$$

Since $\lambda > 0$, (22) implies $\eta = 0$. Consequently (19) implies

$$(f_x^i + z^i(t) - Df_{\dot{x}}^i)\beta(t) = -\kappa \quad (4.27)$$

From the equality constraint of (WCD), we have

$$\begin{aligned}(\sum_{j=1}^{m} y^j(t)(g_x^j + \bar{w}^j(t) - Dy(t)^T g_{\dot{x}}))\beta(t) &= -\sum_{i=1}^{p} \lambda^i (f_x^i + \bar{z}^i(t) + Df_{\dot{x}}^i)\beta(t) \\ &= -\sum_{i=1}^{p} \lambda^i(-\kappa) = \kappa \end{aligned} \quad (4.28)$$

From (17) have

$$-\sum_{i=1}^{p} \alpha^i (f_x^i + z^i(t) + Df_{\dot{x}}^i)\beta(t) - (\alpha^T e)\left(\sum_{j=1}^{m} \bar{y}^j(t)(g_x^j + \bar{w}^j(t) - D\bar{y}(t)^T + g_{\dot{x}})\right)\beta(t)$$

$$+(\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2 \beta(t)^T \theta_{\ddot{x}})\beta(t) = 0, \quad t \in I$$

Using (27) and (28) in this relation, we have

$$(\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2 \beta(t)^T \theta_{\ddot{x}} = 0)\beta(t), \quad t \in I$$

This because of the hypothesis (H_3), gives

$$\beta(t) = 0, t \in I \quad (4.29)$$

Suppose $\alpha = 0$, then from (18) we have $\mu^j(t) = 0, j = 1, 2, \ldots, m$, and from (19) we get $\kappa = 0$. Consequently $(\alpha, \beta(t), \eta, \kappa, \mu(t)) = 0$. This contradicts (26). Hence $\alpha > 0$.
From (20) and (21) in view of (29) implies,

$$\bar{x}(t)^T \bar{z}^i(t) = S(\bar{x}(t)|C^i), \quad i = 1, 2, \ldots, p, t \in I \tag{4.30}$$

$$\bar{x}(t)^T \bar{w}^j(t) = S(\bar{x}(t)|D^j), \quad j = 1, 2, \ldots, m, t \in I \tag{4.31}$$

From (18) along with (25), (29) and (31), we have

$$g^j(t, x, \dot{x}) + S(.\bar{x}(t)|D^j) \leqq 0, j = 1, 2, \ldots, m, t \in I$$

This implies $\bar{x}$ is feasible for (CP).
The relation (18) along with (29) and (24) gives

$$\sum_{j=1}^{m} \bar{y}^j(t)(g^j(t, x, \dot{x}) + \bar{x}(t)^T \bar{w}^j(t)) = 0, t \in I \tag{4.32}$$

Now for each $i \in \{1, \ldots, p\}$, in view of (30) and (32), we have

$$\int_I (f^i(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T w^j(t)))dt = 0$$

$$= \int_I (f^i(t, \bar{x}, \dot{\bar{x}}) + S(\bar{x}|C^i))dt, i = 1, 2, \ldots, p$$

This along with the requirements of Theorem 1 yields the efficiency of $\bar{x}$ for (CP).

5. MOND-WEIR type duality

We further weaken the invexity requirements by formulating Mond-Weir type dual to the problem (CP).

(M-WCD) $\quad$ Maximize$\left(\int_I (f^1(t, u, \dot{u}) + u(t)^T z^1(t))dt, \ldots, \int_I (f^p(t, u, \dot{u}) + u(t)^T z^p(t))dt \right)$

Subject to

$$u(a) = \alpha, u(b) = \beta \tag{5.33}$$

$$\sum_{i=1}^{p} \lambda^i (f_x^i + z^i(t)) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t)) = D(\lambda^T f_{\dot{x}}^i + y(t)^T g_{\dot{x}}), t \in I \tag{5.34}$$

$$z^i(t) \in C^i, i = 1, 2, \ldots, p, \tag{5.35}$$

$$w^j(t) \in D^j, j = 1, 2, \ldots, m, \tag{5.36}$$

$$y(t) \geqq 0, t \in I \tag{5.37}$$

$$\sum_{j=1}^{m}\int_I y^j(t)^T(g^j(t,u,\dot{u})+u(t)^T w^j(t))dt \geqq 0, t\in I \tag{5.38}$$

$$\lambda > 0 \tag{5.39}$$

Theorem 4 (Weak Duality). *Let $\bar{x}$ be feasible for* (CP) *and* $(u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)$ *be feasible for* (M-WCD). *If for all feasible* $(x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)$ *with respect to* $\eta \equiv \eta(t, x, u)$,

(i) $\sum\limits_{i=1}^{p} \lambda^i(f^i(t,.,.)+(\cdot)^T z^i(t))dt$ *is pseudoinvex and*

(ii) $\sum\limits_{j=1}^{m} \int_I y^j(t)(g^j(t,.,.)+(\cdot)^T w^j(t))dt$ *is quasi-invex with respect to same* $\eta \equiv \eta(t,x,u)$ *following cannot hold.*

$$\int_I (f^i(t,x,\dot{x})+S(x(t)|C^i))dt \leq \int_I (f^i(t,u,\dot{u})+u(t)^T z^i(t)dt \tag{5.40}$$

for all $i \in \{1, \ldots, p\}$, *and*

$$\int_I (f^r(t,x,\dot{x})+S(x(t)|C^r))dt < \int_I (f^r(t,u,\dot{u})+u(t)^T z^r(t)dt \tag{5.41}$$

for some $r \in \{1, 2, \ldots, p\}$.

Proof. Suppose that (40) and (41) hold, then in view of $\lambda > 0$ and $x(t)^T z^i(t) \leqq S(x(t)|C^i), i = 1, 2, \ldots, p$ we have

$$\sum_{i=1}^{p}\lambda^i\int_I (f^i(t,x,\dot{x})+x(t)^T z^i(t))dt < \sum_{i=1}^{p}\lambda^i\int_I (f^i(t,u,\dot{u})+u(t)^T z^i(t))dt$$

This in view of the pseudoinvexity of $\sum\limits_{i=1}^{p} \lambda^i(f^i(t,.,.)+(\cdot)^T z^i(t))dt$ yields,

$$\int_I \{\eta^T(\sum_{i=1}^{p}\lambda^i(f_x^i+z^i(t)))+(D\eta)^T(\lambda^T f_{\dot{x}})\}dt < 0$$

This on integration by parts gives

$$= \int_I \eta^T(\sum_{i=1}^{p}\lambda^i(f_x^i+z^i(t))-D(\lambda^T f_{\dot{x}}))dt+\eta^T(\lambda^T f_{\dot{x}})|_{t=a}^{t=b}$$

Using the boundary conditions which at $t = a, t = b$ gives $\eta = 0$, we have

$$\int_I \eta^T(\sum_{i=1}^{p}\lambda^i(f_x^i+z^i(t))-D(\lambda^T f_{\dot{x}}))dt < 0$$

From the feasibility requirements of (CP) and (M-WCD) together with $x(t)^T z^i(t) \leqq S(x(t)|C^i)$, we have

$$\sum_{j=1}^{m} \int_I y^j(t)(g^j(t,x,\dot{x}) + x(t)^T w^j(t))dt$$
$$\leqq \sum_{j=1}^{m} \int_I y^j(t)(g^j(t,u,\dot{u}) + u(t)^T w^j(t))dt$$

By quasi-invexity of $\sum_{j=1}^{m} \int_I y^j(t)(g^j(t,.,.) + (\cdot)^T w^j(t))dt$, implies

$$\int_I [\eta^T \sum_{j=1}^{m} y^j(t)(g^j_x + w^j(t)) + (D\eta)^T y^T(t) g^j_{\dot{x}}]dt \leqq 0 \tag{5.42}$$

This, by integration by parts, as earlier gives

$$\int_I \eta^T [\sum_{j=1}^{m} y^j(t)(g^j_x + w^j(t)) - Dy^T(t) g^j_{\dot{x}}]dt \leqq 0 \tag{5.43}$$

Combining (42) and (43), we have

$$\int_I \eta^T \{\sum_{i=1}^{p} \lambda^i (f^i_x + z^i(t) + \sum_{j=1}^{m} y^j(t)(g^j_x + w^j(t))) - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}})\}dt < 0$$

which contradicts (34), this establishes the conclusion of the theorem.

The following strong duality can be proved on the lines of Theorem 2 with slight modification.

Theorem 5 (Strong Duality). *Let $\bar{x}$ be a efficient solution of* (CP) *and for at least one i , $i \in \{1,2,\dots,p\}$, $\bar{x}$ satisfies the regularity condition* [3] *for $(P_i(\bar{x}))$. Then there exist $\bar{\lambda} \in R^p$ with $\bar{\lambda}^T = (\bar{\lambda}^1,\dots,\bar{\lambda}^i,\dots,\bar{\lambda}^p)$ and piecewise smooth $\bar{y} : I \to R^m$ with $\bar{y}^T = (\bar{y}^1,\dots,\bar{y}^i,\dots,\bar{y}^m)$, $z^i(t) \in C^i$, $i = 1,2,\dots,p$ and $w^j(t) \in D^j$, $j = 1,2,\dots,m$ such that $(\bar{x},\bar{u},\bar{y},\bar{z}^1,\dots,\bar{z}^p,\bar{w}^1,\dots,\bar{w}^m,\bar{\lambda})$ is feasible for* (M-WCD) *and the objective values of* (CP) *and* (M-WCD) *are equal.*

Further, if the hypothesis of Theorem 4 is met, then $(\bar{x},\bar{u},\bar{y},\bar{z}^1,\dots,\bar{z}^p,\bar{w}^1,\dots,\bar{w}^m,\bar{\lambda})$ is an efficient solution of (M-WCD).

(M-WCD) can be rewritten in the following form:

$$\text{Minimize} - \left(\int_I (f^1(t,u,\dot{u}) + u(t)^T z^1(t))dt, \dots, \int_I (f^p(t,u,\dot{u}) + u(t)^T z^p(t))dt \right)$$

Subject to

$u(a) = \alpha$, $u(b) = \beta$; $\qquad \theta(t,x(t),\dot{x}(t),y(t),\lambda) = 0$;

$z^i(t) \in C^i, i = 1, 2, \ldots, p \; ; \qquad w^j(t) \in D^j, j = 1, 2, \ldots, m \; ;$
$y(t) \geqq 0, t \in I;$

$$\sum_{j=1}^{m} \int_I y^j(t)^T (g^j(t, u, \dot{u}) + u(t)^T w^j(t))dt \geqq 0, t \in I; \qquad \lambda > 0$$

where

$$\begin{aligned} \theta &= \theta(t, x(t), \dot{x}(t), y(t), \lambda) \\ &= \sum_{i=1}^{p} \lambda^i (f_x^i + z^i(t)) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t)) - D(\lambda^T f_{\dot{x}}^i + y(t) g_{\dot{x}}), t \in I \end{aligned}$$

Theorem 6 (Converse Duality). *Let $(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda})$ be an efficient solution for (M-WCD). Assume that*
(A_1) The Frèchèt derivative ψ' has a (weak) closed range,*
(A_2) f and g are twice continuously differentiable,
(A_3) $f_x^i + \bar{z}^i(t) - Df_{\dot{x}}^i, i \in \{1, 2, \ldots, p\}$ are linearly independent and
(A_4) $(\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2 \beta(t)^T \theta_{\ddot{x}})\beta(t) = 0, \Rightarrow \beta(t) = 0, t \in I$
Further, if the hypotheses of Theorem 4 are met then $\bar{x}$ is an efficient solution of (CP)

Proof. Since $(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda})$ with ψ' having a (weak$*$) closed range, is an efficient solution of (MWD), then there exist $\alpha \in R^p$ with $\alpha^T = (\alpha^1, \ldots, \alpha^i, \ldots, \alpha^p)$, piecewise smooth $\beta : I \to R^n$ and $\mu : I \to R^m, \eta \in R^p$ with $\eta^T = (\eta^1, \ldots, \eta^p)$ satisfying the following Fritz-John conditions [6],

$$\begin{aligned} &-\sum_{i=1}^{p} \alpha^i (f_x^i + z^i(t) - Df_{\dot{x}}^i) - \gamma(\sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t)) - Dy(t)^T g_{\dot{x}}) \\ &+\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2 \beta(t)^T \theta_{\ddot{x}} = 0, t \in I \end{aligned} \tag{5.44}$$

$$-\gamma(g^j + \bar{x}(t)\bar{w}^j(t)) + \beta(t)^T \theta_{y^j} - D\beta(t)^T \theta_{\dot{y}^j} - \mu^j(t) = 0, t \in I, \quad j = 1, 2, \ldots, m \tag{5.45}$$

$$(f_x^i + z^i(t) - Df_{\dot{x}}^i)\beta(t) - \eta^i = 0, i = 1, \ldots, p \tag{5.46}$$

$$(\beta(t)\lambda^i - \alpha^i \bar{x}(t)) \in N_{C^i}(\bar{z}^i(t)), i = 1, \ldots, p, t \in I \tag{5.47}$$

$$(\beta(t) - \gamma\bar{x}(t))\bar{y}^j(t) \in N_{D^j}(\bar{w}^j(t)), j = 1, \ldots, m, t \in I \tag{5.48}$$

$$\gamma \sum_{j=1}^{m} \int_I y^j(t)(g^j + \bar{x}(t)^T \bar{w}^j(t))dt = 0 \tag{5.49}$$

$$\eta^T \lambda = 0 \tag{5.50}$$

$$\bar{\mu}^T(t)\bar{y}(t) = 0, t \in I \tag{5.51}$$

$$(\alpha, \mu(t), \eta, \gamma) \geqq 0 \tag{5.52}$$

$$(\alpha, \beta(t), \mu(t), \eta, \gamma \neq 0 \tag{5.53}$$

Since $\lambda > 0$, (50) implies $\eta = 0$. Consequently (46) implies

$$(f_x^i + z^i(t) - Df_{\dot{x}}^i)\beta(t) = 0, i = 1, \ldots, p \tag{5.54}$$

Using the equality constraint of (M-WCD) in (44), we have

$$-\sum_{i=1}^{p}(\alpha^i - \gamma\lambda^i)(f_x^i + z^i(t) - Df_{\dot{x}}^i) + \beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} \\ +D^2\beta(t)^T\theta_{\ddot{x}} = 0, t \in I \tag{5.55}$$

Using (54) and (55) in (44), we have

$$(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}})\beta(t) = 0, t \in I$$

which because of the hypothesis (A_4) implies

$$\beta(t) = 0, t \in I \tag{5.56}$$

The relation (55) along with (56), gives $\sum\limits_{i=1}^{p}(\alpha^i - \gamma\lambda^i)(f_x^i + z^i(t) - Df_{\dot{x}}^i) = 0$. This, due to the hypothesis (A_3) gives,

$$\alpha^i = \gamma\lambda^i = 0, i = 1, 2, \ldots, p \tag{5.57}$$

Suppose $\gamma = 0$; Then from (57) we have $\alpha = 0$. From (45), we have $\mu(t) = 0, t \in I$.
Consequently, $(\alpha, \beta(t), \mu(t), \eta, \gamma) = 0$ contradicting the Fritz-John condition (53).

Hence $\gamma > 0$ and from (57), $\alpha > 0$. In view of (56) together with $\gamma > 0$ and $\alpha > 0$, (47) and (48), respectively imply

$$\bar{x}(t)^T\bar{z}^i(t) = S(\bar{x}(t)|C^i), \quad i = 1, 2, \ldots, p, t \in I \tag{5.58}$$

$$\bar{x}(t)^T\bar{w}^j(t) = S(\bar{x}(t)|D^j), \quad j = 1, 2, \ldots, m, t \in I \tag{5.59}$$

The relation (45) along with $\gamma > 0$ and $y(t) \geqq 0, t \in I$ and (59) imply

$$g^j(t, \bar{x}, \dot{\bar{x}}) + S(\bar{x}(t)|D^j) \leqq 0, \quad j = 1, 2, \ldots, m$$

This implies the feasibility of $\bar{x}$ for (CP). In view of (58), we have

$$f^i(\bar{x}) + \bar{x}^T\bar{z}^i = f^i(\bar{x}) + S(\bar{x}|C^i), \quad i = 1, 2, \ldots, p$$

This in view of the hypothesis of Theorem 4, gives the efficiency of $\bar{x}$ for (CP).

6. Related problems

It is possible to extend duality theorems established in the previous sections to the corresponding multiobjective variational problem containing support function with natural boundary values rather than fixed end points.

PRIMAL(CP)$_0$:

$$\text{Minimize}\Big(\int_I (f^1(t,x,\dot{x}) + S(x(t)|C^1))dt, \ldots, \int_I (f^p(t,x,\dot{x}) + S(x(t)|C^p))dt\Big)$$

Subject to

$$g^j(t,x,\dot{x}) + S(x(t)|D^j) \leqq 0, j = 1,2,\ldots,m, t \in I$$

DUAL (WDP)$_0$:

$$\text{Maximize}\Big(\int_I (f^1(t,u,\dot{u}) + u(t)^T z^1(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t,u,\dot{u}) + u(t)^T w^j(t)))dt$$

$$,\ldots, \int_I (f^p(t,u,\dot{u}) + u(t)^T z^p(t) + \sum_{j=1}^{m} y^j(t)^T (g^j(t,u,\dot{u}) + u(t)^T w^j(t)))dt\Big)$$

Subject to

$$\sum_{i=1}^{p} \lambda^i (f_x^i + z^i(t)) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t)) = D(\lambda^T f_{\dot{x}}^i + y(t) g_{\dot{x}}), t \in I$$

$$\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} = 0, \text{ at } t = a \text{ and } t = b;$$

$$z^i(t) \in C^i, i = 1,2,\ldots,p \ ; \quad w^j(t) \in D^j, j = 1,2,\ldots,m;$$

$$\lambda > 0, \sum_{i=1}^{p} \lambda^i = 1.$$

DUAL (M-WD)$_0$:

$$\text{Maximize}\Big(\int_I (f^1(t,u,\dot{u}) + u(t)^T z^1(t))dt, \ldots, \int_I (f^p(t,u,\dot{u}) + u(t)^T z^p(t))dt\Big)$$

Subject to

$$\sum_{i=1}^{p} \lambda^i (f_x^i + z^i(t)) + \sum_{j=1}^{m} y^j(t)(g_x^j + w^j(t))$$

$$= D(\lambda^T f_{\dot{x}}^i + y(t)^T g_{\dot{x}}), t \in I$$

$$\lambda^T f_{\dot{x}} = 0 = \lambda^T y(t) g_{\dot{x}}, \text{ at } t = a \text{ and } t = b;$$

$$z^i(t) \in C^i, i = 1,2,\ldots,p \ ; \quad w^j(t) \in D^j, j = 1,2,\ldots,m;$$

$$y(t) \geqq 0, t \in I$$

$$\sum_{j=1}^{m}\int_I y^j(t)^T(g^j(t,u,\dot{u})+u(t)^T w^j(t))dt \geqq 0, t\in I$$
$$\lambda > 0.$$

If the functions in the problem mentioned in Section 5 are independent of t, they will reduce to the following.

PRIMAL(NP):

Minimize $(f^1(x)+S(x|C^1),\ldots,f^p(x)+S(x|C^p))$

Subject to

$$g^j(x)+S(x|D^j) \leqq 0, j=1,2,\ldots,m.$$

DUAL (WNP):

$$\text{Maximize } (f^1(u)+u^Tz^1+\sum_{j=1}^{m}y^j(g^j(u)+u^Tw^j).,\ldots,$$
$$f^p(u)+u^Tz^p+\sum_{j=1}^{m}y^j(g^j(u)+u^Tw^j))$$

Subject to

$$\sum_{i=1}^{p}\lambda^i(f_x^i+z^i)+\sum_{j=1}^{m}y^j(g_x^j+w^j)=0;$$
$$z^i\in C^i, i=1,2,\ldots,p\,; \quad w^j\in D^j, j=1,2,\ldots,m;$$
$$\lambda>0, \sum_{i=1}^{p}\lambda^i=1.$$

DUAL (M-WNP):

Maximize$(f^1(u)+u^Tz^1,\ldots,f^p(u)+u^Tz^p)$

Subject to

$$\sum_{i=1}^{p}\lambda^i(f_x^i+z^i)+\sum_{j=1}^{m}y^j(g_x^j+w^j)=0$$
$$z^i\in C^i, i=1,2,\ldots,p\,; \quad w^j\in D^j, j=1,2,\ldots,m; \quad y\geqq 0, t\in I;$$
$$\sum_{j=1}^{m}y^j(g^j(u)+u^Tw^j)\geqq 0;$$
$$\lambda>0.$$

REFERENCES

1. C.R.Bector and I.Husain, "Duality for Multiobjective variational problems, Journal of Math. Anal. and Appl. 166 (1992), no.1, 214-224.
2. C.R.Bector, S.Chandra and I.Husain, "Generalized concavity and nondifferentiable continuous programming duality", Research Report # 85-7 (1985), Faculty of administrative studies, The University of Manitoba, Winnipeg, Canada R3T 2N2.
3. S.Chandra, B.D.Craven and I.Husain, "A class of nondifferentiable continuous programming problems", J.Math. Anal. Appl. 107 (1985), 122-131.
4. V.Chankong and Y.Y.Haimes, "Multiobjective Decision Making: Theory and Methodology", North-Holland, New York, (1983).
5. X-h Chen, "Duality for multiobjective variational problems with Invexity" J. Math. Anal. Appl. 203 (1996) 236-253.
6. Craven, B. D. (1977),Lagrangian conditions and quasiduality, Bulletin of Australian Mathematical Society 16 pp 325-339.
7. I.Husain and Z.Jabeen, "Continuous programming containing support function", (To appear in Journal of Applied Mathematics and Informatics).
8. B.Mond and I.Smart, "Duality with Invexity for a class of nondifferentiable static and continuous programming problems", J. Math. Anal. Appl. 136 (1988) 325-333.
9. B.Mond and M.Schechter, "Nondifferentiable symmetric duality', Bull. Autral. Math. Soc. 53 (1996), 177-188.
10. B.Mond and S.Chandra and I.Husain, "Duality of variational problems with Invexity", J. Math. Anal. Appl. 134 (1988) 322-328.
11. O.L.Mangasarian, "Nonlinear programming", Mcgraw Hill, New York, (1969).

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