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THREE-STEP MEAN VALUE ITERATIVE SCHEME FOR VARIATIONAL INCLUSIONS AND NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we present the three-step mean value iterative scheme and prove that the iteration sequence converge strongly to a common element of the set of fixed points of a nonexpansive mappings and the set of the solutions of the variational inclusions under some mild conditions. The results presented in this paper extend, generalize and improve the results of Noor and Huang and some others.

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1. Introduction

Variational inclusions involving two operators are useful and important extension and generalization of the variational inequalities with a wide range of applications in industry, mathematical finance, economics, decision sciences, ecology, mathematical and engineering sciences,see[1-10] and the references therein. By using the technique of the resolvent operators, one can show that the variational inclusions are equivalent to the fixed point problems. This alternative equivalent formulation has played very crucial role in developing some very efficient methods for solving the variational inclusions and related optimization problems. Using the technique of updating the solution, Noor[6,7] suggested and analyzed several three-step iterative methods for solving different classes of variational inequalities. The main idea in this technique is to modify the resolvent method by performing an additional step forward and a resolvent at each iteration. It has been shown[2,3,8] that three-step schemes are

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numerically better than two-step and one-step methods. Recently, Noor and Huang[9] have considered the following three-step iterative method for finding the common element of the set of fixed points of a nonexpansive mappings and the set of the solutions of the variational inclusions:

For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes:

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n SJ_A[x_n - \rho T x_n], \\ y_n &= (1-b_n)x_n + b_n SJ_A[z_n - \rho T z_n], \\ x_{n+1} &= (1-a_n)x_n + a_n SJ_A[y_n - \rho T y_n], \end{aligned}$$

where $a_n, b_n, c_n \in [0,1]$, for all $n \geq 0$. $T, A : H \to H$ be two nonlinear operators and S be a nonexpansive operator and J_A is the resolvent operator.

Motivated and inspired by Noor and Huang, we establish three-step mean value iterative schemes by

$$\begin{aligned} z_n =&(1-c_n)x_n + c_n SJ_A[x_n - \rho T x_n], \\ y_n =&(1-a_n-b_n)x_n + a_n SJ_A[x_n - \rho T x_n] + b_n SJ_A[z_n - \rho T z_n], \\ x_{n+1} =&(1-\alpha_n-\beta_n-\gamma_n)x_n + \alpha_n SJ_A[x_n - \rho T x_n] \\ &+ \beta_n SJ_A[z_n - \rho T z_n] + \gamma_n SJ_A[y_n - \rho T y_n], \end{aligned}$$

which is more general than Noor's iteration and to study the convergence of $\{x_n\}$, we will prove that if the coefficients satisfy appropriate conditions, the iteration sequence $\{x_n\}$ converge strongly to a common element of the set of fixed points of a nonexpansive mappings and the set of the solutions of the variational inclusions. The results presented in this paper extend, generalize and improve the results of Noor's and Noor and Huang's and some others.

2. Preliminaries

Let K be a nonempty closed and convex set in a real Hilbert space , whose with inner product $\langle .,. \rangle$ and norm $\|.\|$ respectively. Let $T, A : H \to H$ be two nonlinear operators and S be a nonexpansive operator.

We consider the problem of finding $u \in H$ such that:

$$0 \in Tu + A(u), \tag{1}$$

which is known as the variational inclusion. Problem (1) is also known as finding the zero of the sum of two (or more) monotone operators. Variational inclusions and related problems are being studied extensively by many authors and have important applications in operations research, optimization, mathematical finance, decision sciences and other several branches of pure and applied sciences, see[1-10] and the references therein.

If $A(\cdot) \equiv \partial\varphi(\cdot)$ is the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ defined by

$$\partial\varphi(x) = \{f \in H^* : \varphi(y) - \varphi(x) \geq \langle f, y-x \rangle \quad \forall y \in H\},$$

then the problem (1) reduces to finding $u \in H$ such that:

$$0 \in Tu + \partial\varphi(u),$$

or equivalently, finding $u \in H$ such that for some $f \in \partial\varphi(u)$ and $f = -Tu$, from the definition of $\partial\varphi(u)$, that is finding $u \in H$ such that

$$\varphi(v) - \varphi(u) \geq \langle -Tu, v - u\rangle \ \ \forall v \in H.$$

which equivalent to

$$\langle Tu, v - u\rangle + \varphi(v) - \varphi(u) \geq 0 \ \ \forall v \in H. \tag{2}$$

The inequality (2) is called the mixed variational inequality or the variational inequality of the second kind. It has been shown that a wide class of linear and nonlinear problems arising in various branches of pure and applied sciences can be studied in the unified framework of mixed variational inequalities, see[1-10].

We note that if φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & if u \in K \\ +\infty, & otherwise, \end{cases}$$

then the mixed variational inequality (2) is equivalent to finding $u \in K$ such that:

$$\langle Tu, v - u\rangle \geq 0, \ \ \forall v \in K, \tag{3}$$

which is called the classical variational inequality introduced and studied by Stampacchia[10] in 1964. For the recent trends and developments in variational inclusions and inequalities, see[1-10] and the references therein.

We also need the following well known concepts and results.

Definition 2.1[4]. If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by:

$$J_A(u) = (I + \rho A)^{-1}(u), \quad for\ all\ u \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent opeerator is defined everywhere. In addition, the resolvent operator is a single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\|J_A(u) - J_A(v)\| \leq \|u - v\|.$$

Remark 2.1. It is well known that the subdifferential $\partial\varphi$ of a proper, convex and lower-semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ is a maximal monotone operator, we denote by:

$$J_\varphi(u) = (I + \rho\partial\varphi)^{-1}(u), \quad for\ all\ u \in H$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on H. In particular, the resolvent operator J_φ has the following interesting characterization.

Lemma 2.1[4]. *For a given $z \in H, u \in H$ satisfies the inequality:*

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \text{for all } v \in H,$$

if and only if:

$$u = J_{\varphi}z, \tag{4}$$

where $J_{\varphi} = (I+\rho\partial\varphi)^{-1}$ is the resolvent operator. This property of the resolvent operator J_{φ} plays an important part in developing the numerical methods for solving the mixed variational inequalities.

If the function $\varphi(\cdot)$ is the indicator function of a closed convex set K in H, then it is well knowen that $J_{\varphi} = P_K$, the projection operator of H onto the closed convex set K. In this case Lemma 2.1 reduces to the well known result, see[1].

Lemma 2.2. *For a given $z \in H, u \in H$ satisfies the inequality:*

$$\langle u - z, v - u\rangle \geq 0, \quad \text{for all } v \in K,$$

if and only if

$$u = P_K(z),$$

where P_K is the projection of H onto the closed convex set K. It is also known that the projection operator P_K is nonexpansive.

Using the definition of the resolvent operator J_A, one can easily prove the following well known result.

Lemma 2.3[9]. *The function $u \in H$ is a solution of the variational inclusion (1) if and only if $u \in H$ satisfies the relation:*

$$u = J_A(u - \rho Tu), \tag{5}$$

where $\rho > 0$ is a constant and $J_A = (I + \rho A)^{-1}$ is the resolvent operator associated with the maximal monotone operator.

It is clear from Lemma 2.3 that variational inclusion (1) and the fixed point problems are equivalent. This alterative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Let S be a nonexpansive mapping. We denote the set of the fixed points of S by $F(S)$ and the set of the solutions of the variational inclusion (1) by $VI(H,T)$. We can characterize the problem. If $x^* \in F(S) \cap VI(H,T)$, then $x^* \in F(S)$ and $x^* \in VI(H,T)$. Thus, from Lemma 2.2, it follows that:

$$x^* = Sx^* = J_A(x^* - \rho Tx^*) = SJ_A(x^* - \rho Tx^*), \tag{6}$$

where $\rho > 0$ is a constant.

Now we propose the following three-step iterative methods for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings S and the variational inclusion (1).

Algorithm 2.1. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes:

$$\begin{aligned} z_n =&(1-c_n)x_n + c_n SJ_A[x_n - \rho T x_n], \\ y_n =&(1-a_n-b_n)x_n + a_n SJ_A[x_n - \rho T x_n] + b_n SJ_A[z_n - \rho T z_n], \\ x_{n+1} =&(1-\alpha_n-\beta_n-\gamma_n)x_n + \alpha_n SJ_A[x_n - \rho T x_n] \\ &+ \beta_n SJ_A[z_n - \rho T z_n] + \gamma_n SJ_A[y_n - \rho T y_n], \end{aligned} \tag{7}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n+b_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n+\beta_n+\gamma_n\} \in [0,1]$, for all $n \geq 0$ and S is the nonexpansive operator.

Note that for $a_n = \alpha_n = \beta_n \equiv 0$, Algorithm 2.1 reduces to:

Algorithm 2.2. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes:

$$\begin{aligned} z_n =&(1-c_n)x_n + c_n SJ_A[x_n - \rho T x_n], \\ y_n =&(1-b_n)x_n + b_n SJ_A[z_n - \rho T z_n], \\ x_{n+1} =&(1-\gamma_n)x_n + \gamma_n SJ_A[y_n - \rho T y_n], \end{aligned} \tag{8}$$

where $\gamma_n, b_n, c_n \in [0,1]$, for all $n \geq 0$ and S is the nonexpansive operator.This algorithm is called the Noor iterative scheme , see [9].

For $c_n \equiv 0$, Algorithm 2.2 reduces to:

Algorithm 2.3. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes:

$$\begin{aligned} y_n =&(1-b_n)x_n + b_n SJ_A[x_n - \rho T x_n], \\ x_{n+1} =&(1-\gamma_n)x_n + \gamma_n SJ_A[y_n - \rho T y_n], \end{aligned} \tag{9}$$

where $\gamma_n, b_n \in [0,1]$, for all $n \geq 0$ and S is the nonexpansive operator.

For $\gamma_n = b_n = 1$, Algorithm 2.3 reduces to:

Algorithm 2.4. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes:

$$y_n = SJ_A[x_n - \rho T x_n], \quad x_{n+1} = SJ_A[y_n - \rho T y_n].$$

Remark 2.2. It should be remarked that our Algorithm 2.4 is a two-step method, which may be regarded as a predictor-corrector method. Moreover, Algorithm 2.2 covers the case in Algorithm 2.4 whenever $a_n \equiv 1$, for all $n \geq 0$. Algorithm 2.4 can be written as:

$$x_{n+1} = SJ_A(SJ_A(x_n - \rho T x_n) - \rho T SJ_A(x_n - \rho T x_n)),$$

which is called extraresolvent algorithm and is mainly due to Noor[5].

For $a_n \equiv 0, b_n \equiv 0, c_n \equiv 0$, Algorithm 2.1 collapses to the following iterative method, which is known as the Mann iteration or one-step method for solving the variational inclusion(1).

Algorithm 2.5. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes:

$$x_{n+1} = (1 - t_n)x_n + t_n S J_A[x_n - \rho T x_n], \tag{10}$$

where $t_n = \alpha_n + \beta_n + \gamma_n \in (0, 1]$.

We now discuss some special cases of Algorithm 2.1 for solving the mixed variational inequalities (2) and the classical variational inequalities (3):

1. If $A(\cdot) \equiv \partial\varphi(\cdot)$, the subdifferential of a proper lower-semicontinuous and convex function φ, then $J_A = J_\varphi = (I + \rho\partial\varphi)^{-1}$ and consequently Algorithm 2.1 collapses to:

Algorithm 2.6. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes:

$$\begin{aligned} z_n =&(1 - c_n)x_n + c_n S J_\varphi[x_n - \rho T x_n], \\ y_n =&(1 - a_n - b_n)x_n + a_n S J_\varphi[x_n - \rho T x_n] + b_n S J_\varphi[z_n - \rho T z_n], \\ x_{n+1} =&(1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n S J_\varphi[x_n - \rho T x_n] \\ &+ \beta_n S J_\varphi[z_n - \rho T z_n] + \gamma_n S J_\varphi[y_n - \rho T y_n], \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n + b_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n + \beta_n + \gamma_n\} \in [0, 1]$, for all $n \geq 0$ and S is the nonexpansive operator. This algorithm is also a three-step method for solving the mixed variational inequalities(2).

2. If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto the closed convex set K. In this case Algorithm 2.6 reduces to the following method.

Algorithm 2.7. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes:

$$\begin{aligned} z_n =&(1 - c_n)x_n + c_n S P_K[x_n - \rho T x_n], \\ y_n =&(1 - a_n - b_n)x_n + a_n S P_K[x_n - \rho T x_n] + b_n S P_K[z_n - \rho T z_n], \\ x_{n+1} =&(1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n S P_K[x_n - \rho T x_n] \\ &+ \beta_n S P_K[z_n - \rho T z_n] + \gamma_n S P_K[y_n - \rho T y_n], \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n + b_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n + \beta_n + \gamma_n\} \in [0, 1]$, for all $n \geq 0$ and S is the nonexpansive operator.This algorithm is a three-step method for solving the classical variational inequalities(3).

From the above discussion, it is clear that Algorithm 2.1 is quite general and it includes several new and previously known algorithms for solving variational inequalities and related optimization problems.

Definition 2.2. A mapping $T : H \to H$ is called μ-Lipschitzian if for all $x, y \in H$, there exists a constant $\mu > 0$ such that:

$$\|Tx - Ty\| \leq \mu\|x - y\|.$$

Definition 2.3. A mapping $T : H \to H$ is called α-inverse strongly monotonic if for all $x, y \in H$, there exists a constant $\alpha > 0$ such that:

$$\langle Tx - Ty, x - y\rangle \leq \alpha\|Tx - Ty\|^2.$$

Definition 2.4. A mapping $T : H \to H$ is called r-strongly monotonic if for all $x, y \in H$, there exists a constant $r > 0$ such that:

$$\langle Tx - Ty, x - y\rangle \leq r\|x - y\|^2.$$

Definition 2.5. A mapping $T : H \to H$ is called γ, r-cocoercive if for all $x, y \in H$, there exists a constant $\gamma > 0$ and $r > 0$ such that:

$$\langle Tx - Ty, x - y\rangle \leq -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2.$$

Remark 2.3. Clearly a r-strongly monotonic mapping, or a γ-inverse strongly monotonic mapping must be a relaxed γ, r-cocoercive mapping, but the converse is not true. Therefore, the class of the relaxed γ, r-cocoercive mapping is the most general class, and hence Definition 2.5 includes both the Definition 2.3 and the Definition 2.4 as special cases.

Lemma 2.4[11]. *Suppose $\{\delta_k\}_{k=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{k+1} \leq (1 - \lambda_k)\delta_k + \sigma_k, \quad k \geq 0$$

with $\lambda_k \in [0, 1], \sum_{k=0}^{\infty} \lambda_k = \infty$, and $\sigma_k = o(\lambda_k)$. Then $\lim_{k\to\infty} \delta_k = 0$.

3. Main results

Theorem 3.1 *Let T be a relaxed γ, r-cocoercive and μ-Lipschitzian mapping of H into H, and S be a nonexpansive mapping of H into H such that $F(S) \cap VI(H, T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by Algorithm 2.1, for any initial point $x_0 \in H$, with conditions:*

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2, \quad \gamma\mu^2 < r, \tag{11}$$

$\{a_n\}, \{b_n\}, \{c_n\}, \{a_n + b_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n + \beta_n + \gamma_n\} \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n + \beta_n + \gamma_n = \infty$, then x_n obtained from Algorithm 2.1 converges strongly to $x^* \in F(S) \cap VI(H, T)$.

Proof. Let $x^* \in H$ be the solution of $F(S) \cap VI(H,H)$. Then:

$$\begin{aligned}x^* =&(1-c_n)x^* + c_n SJ_A[x^* - \rho Tx^*],\\ =&(1-a_n-b_n)x^* + a_n SJ_A[x^* - \rho Tx^*] + b_n SJ_A[x^* - \rho Tx^*],\\ =&(1-\alpha_n-\beta_n-\gamma_n)x^* + \alpha_n SJ_A[x^* - \rho Tx^*]\\ &+ \beta_n SJ_A[x^* - \rho Tx^*] + \gamma_n SJ_A[x^* - \rho Tx^*].\end{aligned}$$

From (9) and the nonexpansive property of the J_A and S, we have:

$$\begin{aligned}&\|x_{n+1} - x^*\|\\ =&\|(1-\alpha_n-\beta_n-\gamma_n)x_n + \alpha_n SJ_A[x_n - \rho Tx_n] + \beta_n SJ_A[z_n - \rho Tz_n]\\ &+ \gamma_n SJ_A[y_n - \rho Ty_n] - (1-\alpha_n-\beta_n-\gamma_n)x^* - \alpha_n SJ_A[x^* - \rho Tx^*]\\ &- \beta_n SJ_A[x^* - \rho Tx^*] - \gamma_n SJ_A[x^* - \rho Tx^*]\|\\ \leq&(1-\alpha_n-\beta_n-\gamma_n)\|x_n - x^*\| + \alpha_n\|SJ_A[x_n - \rho Tx_n]\\ &- SJ_A[x^* - \rho Tx^*]\| + \beta_n\|SJ_A[z_n - \rho Tz_n] - SJ_A[x^* - \rho Tx^*]\|\\ &+ \gamma_n\|SJ_A[y_n - \rho Ty_n] - SJ_A[x^* - \rho Tx^*]\|\\ \leq&(1-\alpha_n-\beta_n-\gamma_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - \rho(Tx_n - Tx^*)\|\\ &+ \beta_n\|[z_n - \rho Tz_n] - [x^* - \rho Tx^*]\| + \gamma_n\|[y_n - \rho Ty_n] - [x^* - \rho Tx^*]\|.\end{aligned} \tag{12}$$

From the relaxed γ, r-cocoercive and μ-Lipschitzian definition on T,

$$\begin{aligned}&\|y_n - x^* - \rho(Ty_n - Tx^*)\|^2\\ =&\|y_n - x^*\|^2 - 2\rho\langle Ty_n - Tx^*, y_n - x^*\rangle + \rho^2\|Ty_n - Tx^*\|^2\\ \leq&\|y_n - x^*\|^2 - 2\rho[-\gamma\|Ty_n - Tx^*\|^2 + r\|y_n - x^*\|^2] + \rho^2\|Ty_n - Tx^*\|^2\\ \leq&\|y_n - x^*\|^2 + 2\rho\gamma\mu^2\|y_n - x^*\|^2 - 2\rho r\|y_n - x^*\|^2 + \rho^2\mu^2\|y_n - x^*\|^2\\ =&[1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|y_n - x^*\|^2\\ =&\theta^2\|y_n - x^*\|^2,\end{aligned} \tag{13}$$

where:

$$\theta = \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}.$$

It follows from (11) that $\theta < 1$. In a similar way, it follows that

$$\|z_n - x^* - \rho(Tz_n - Tx^*)\|^2 \leq \theta^2\|z_n - x^*\|^2, \tag{14}$$

and

$$\|x_n - x^* - \rho(Tx_n - Tx^*)\|^2 \leq \theta^2\|x_n - x^*\|^2. \tag{15}$$

At the same time, we calculate

$$
\begin{aligned}
&\|y_n - x^*\| \\
=&\|(1-a_n-b_n)x_n + a_n SJ_A[x_n - \rho T x_n] + b_n SJ_A[z_n - \rho T z_n] \\
&- (1-a_n-b_n)x^* - a_n SJ_A[x^* - \rho T x^*] - b_n SJ_A[x^* - \rho T x^*]\| \\
\le&(1-a_n-b_n)\|x_n - x^*\| + a_n\|SJ_A[x_n - \rho T x_n] - SJ_A[x^* - \rho T x^*]\| \\
&+ b_n\|SJ_A[z_n - \rho T z_n] - SJ_A[x^* - \rho T x^*]\| \\
\le&(1-a_n-b_n)\|x_n - x^*\| + a_n\|[x_n - \rho T x_n] - [x^* - \rho T x^*]\| \\
&+ b_n\|[z_n - \rho T z_n] - [x^* - \rho T x^*]\|,
\end{aligned} \tag{16}
$$

put (14) and (15) in (16), we have:

$$
\|y_n - x^*\| \le (1-a_n-b_n)\|x_n - x^*\| + a_n\theta\|x_n - x^*\| + b_n\theta\|z_n - x^*\|. \tag{17}
$$

Furthermore, we have

$$
\begin{aligned}
\|z_n - x^*\| \le&(1-c_n)\|x_n - x^*\| + c_n\theta\|x_n - x^*\| \\
=&[1 - c_n(1-\theta)]\|x_n - x^*\| \le \|x_n - x^*\|.
\end{aligned} \tag{18}
$$

So,

$$
\begin{aligned}
\|y_n - x^*\| \le& (1-a_n-b_n)\|x_n - x^*\| + a_n\theta\|x_n - x^*\| + b_n\theta\|x_n - x^*\| \\
\le& \|x_n - x^*\|.
\end{aligned} \tag{19}
$$

From (12)-(19), we obtain that:

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
\le&(1-\alpha_n-\beta_n-\gamma_n)\|x_n - x^*\| + \alpha_n\theta\|x_n - x^*\| + \beta_n\theta\|z_n - x^*\| \\
&+ \gamma_n\theta\|y_n - x^*\| \\
\le&[1 - (\alpha_n+\beta_n+\gamma_n)(1-\theta)]\|x_n - x^*\|.
\end{aligned} \tag{20}
$$

Hence by Lemma 2.4, $\lim_{n\to\infty}\|x_n - x^*\| = 0$, completing the proof. □

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