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CONVERGENCE OF RELAXED TWO-STAGE MULTISPLITTING METHOD USING M-SPLITTINGS OR SOR MULTISPLITTING†

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ABSTRACT. In this paper, we study the convergence of relaxed two-stage multisplitting method using M-splittings or SOR multisplitting as inner splittings and an outer splitting for solving a linear system whose coefficient matrix is an M-matrix. We also provide numerical experiments for the convergence of the relaxed two-stage multisplitting method.

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1. Introduction

In this paper, we consider relaxed two-stage multisplitting method for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse M-matrix. Multisplitting method was introduced by O'Leary and White [7] and was further studied by many authors [4, 5, 6, 8, 10, 11].

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *monotone* if $A^{-1} \geq 0$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if A is monotone and $a_{ij} \leq 0$ for $i \neq j$. A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, and *M-splitting* of A if M is an M-matrix and $N \geq 0$.

A collection of triples (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, is called a *multisplitting* of A if $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's, called

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weighting matrices, are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$. The *relaxed two-stage multisplitting method* with a relaxation parameter $\beta > 0$ using $A = M_k - N_k$ as outer splittings and $M_k = B_k - C_k$ as inner splittings is as follows.

ALGORITHM 1: RELAXED TWO-STAGE MULTISPLITTING METHOD
Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ
    $y_{k,0} = x_{i-1}$
    For $j = 1$ to s
      $y_{k,j} = \beta {B_k}^{-1}(C_k y_{k,j-1} + N_k x_{i-1} + b) + (1-\beta) y_{k,j-1}$
  $x_i = \sum_{k=1}^{\ell} E_k y_{k,s}$

In Algorithm 1, it is assumed to be $s \geq 1$. Bru et al [2] showed that if $0 < \beta \leq 1$, then Algorithm 1 converges for a monotone matrix A under the assumption that the outer splittings $A = M_k - N_k$ are regular and the inner splittings $M_k = B_k - C_k$ are weak regular.

In 1991, Wang [10] studied the convergence of relaxed multisplitting method associated with AOR multisplitting for solving the linear system (1). In this paper, we study the convergence of relaxed two-stage multisplitting method using M-splittings or SOR multisplitting as inner splittings and an outer splitting for solving the linear system (1). This paper is organized as follows.

In Section 2, we present some notation and well-known results. In Section 3, we provide convergence results of relaxed two-stage multisplitting method using M-splittings or SOR multisplitting as inner splittings and an outer splitting. In Section 4, we provide numerical experiments for the convergence of the relaxed two-stage multisplitting method.

2. Preliminaries

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. These definitions carry immediately over to matrices. Let $\text{diag}(A)$ denote a diagonal matrix whose diagonal part coincides with the diagonal part of A, and let $\rho(A)$ denote the *spectral radius* of a square matrix A. Varga [9] showed that for any square matrices A and B, $|A| \leq B$ implies $\rho(A) \leq \rho(B)$. It is well-known that if $A = M - N$ is a weak regular splitting, then $A^{-1} \geq 0$ if and only if $\rho(M^{-1}N) < 1$ [1, 9].

It is also well-known that if $A \geq 0$ and there exists a vector $x > 0$ and an $\alpha \geq 0$ such that $Ax \leq \alpha x$, then $\rho(A) \leq \alpha$ [1].

Theorem 2.1 ([2]). *Let $A \in \mathbb{R}^{n\times n}$ be a monotone matrix. Assume that the outer splittings $A = M_k - N_k$ are regular and the inner splittings $M_k = B_k - C_k$ are weak regular. If $0 < \beta \leq 1$, then the relaxed two-stage multisplitting method converges to the exact solution of $Ax = b$ for any initial vector x_0.*

The SOR multisplitting to be used in this paper is defined as follows.

Definition 2.1. *Let $0 < \omega < 2$ and $A = D - L_k - U_k$ for $k = 1, 2, \ldots, \ell$, where $D = diag(A)$, L_k's are strictly lower triangular matrices, and U_k's are general matrices. $\Big(M_k(\omega), N_k(\omega), E_k\Big)$, $k = 1, 2, \ldots, \ell$, is called the* SOR multisplitting *of A if $\Big(M_k(\omega), N_k(\omega), E_k\Big)$, $k = 1, 2, \ldots, \ell$, is a multisplitting of A, $M_k(\omega) = \frac{1}{\omega}(D - \omega L_k)$, and $N_k(\omega) = \frac{1}{\omega}\left((1-\omega)D + \omega U_k\right)$.*

If $\omega = 1$ in Definition 2.1, then the SOR multisplitting of A is called the *Gauss-Seidel multisplitting* of A. In this case, $M_k(\omega) = D - L_k$ and $N_k(\omega) = U_k$.

3. Convergence results of relaxed two-stage multisplitting method

In this section, we consider convergence of relaxed two-stage multisplitting method (Algorithm 1) with a relaxation parameter $\beta > 0$ using an outer splitting $A = M - N$ and inner splittingss $M = B_k - C_k$. Then, Algorithm 1 can be written as $x_i = H_\beta x_{i-1} + P_\beta b$, $i = 1, 2, \ldots$, where

$$H_\beta = \sum_{k=1}^{\ell} E_k (R_{\beta,k})^s + \beta \sum_{k=1}^{\ell} E_k \left(\sum_{j=0}^{s-1} (R_{\beta,k})^j \right) B_k^{-1} N,$$

$$P_\beta = \beta \sum_{k=1}^{\ell} E_k \left(\sum_{j=0}^{s-1} (R_{\beta,k})^j \right) B_k^{-1},$$

where $R_{\beta,k} = \beta B_k^{-1} C_k + (1-\beta) I$. The H_β is called an *iteration matrix* for the relaxed two-stage multisplitting method with a relaxation parameter $\beta > 0$ and s inner iterations. Then, it can be shown that $P_\beta A = I - H_\beta$ and the relaxed two-stage multisplitting method with a relaxation parameter $\beta > 0$ converges to the exact solution of $Ax = b$ for any initial vector x_0 if and only if $\rho(H_\beta) < 1$. First, we provide convergence result of the relaxed two-stage multisplitting method using M-splittings.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix and $A = M - N$ be an M-splitting of A. Let $M = B_k - C_k$ be an M-splitting such that $diag(B_k) = diag(M) = D$ for each $1 \leq k \leq \ell$, and let $B = D - M$. Then, the relaxed two-stage multisplitting method using an outer splitting $A = M - N$ and inner splittings $M = B_k - C_k$,*

$k = 1, 2, \ldots, \ell$, converges to the exact solution of $Ax = b$ for any initial vector x_0 if $0 < \beta < \dfrac{2}{1+\alpha}$, where $\alpha = \rho(D^{-1}(B+N))$.

Proof. For $0 < \beta \leq 1$, this theorem follows directly from Theorem 2.1 since M-splitting implies regular or weak regular splittings. Now we consider the case of $1 < \beta < \dfrac{2}{1+\alpha}$. Let

$$\begin{aligned} H_\beta &= \sum_{k=1}^{\ell} E_k H_{\beta,k}\,, \quad H_{\beta,k} = (R_{\beta,k})^s + \beta \sum_{j=0}^{s-1} (R_{\beta,k})^j B_k^{-1} N, \\ \tilde{H}_\beta &= \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,k}\,, \quad \tilde{H}_{\beta,k} = (\tilde{R}_{\beta,k})^s + \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j B_k^{-1} N, \end{aligned}$$

where $R_{\beta,k} = \beta B_k^{-1} C_k + (1-\beta) I$ and $\tilde{R}_{\beta,k} = \beta B_k^{-1} C_k + (\beta - 1) I$. Let

$$\tilde{A}_{\beta,k} = \frac{2-\beta}{\beta} B_k - C_k - N \text{ and } \tilde{A}_\beta = \frac{2-\beta}{\beta} D - B - N.$$

Since $|R_{\beta,k}| \leq \tilde{R}_{\beta,k}$, $|H_{\beta,k}| \leq \tilde{H}_{\beta,k}$ and thus

$$|H_\beta| \leq \tilde{H}_\beta. \tag{2}$$

Since $A = D - (B + N) = B_k - (C_k + N)$ are regular splittings of A and $B_k^{-1} \geq D^{-1}$, $\rho\Big(B_k^{-1}(C_k + N)\Big) \leq \alpha < 1$. Since $\beta < \dfrac{2}{1+\alpha}$, $\dfrac{\beta\alpha}{2-\beta} < 1$ and thus $I - \tilde{R}_{\beta,k} = (2-\beta) I - \beta B_k^{-1} C_k$ is nonsingular. Hence, one obtains

$$\begin{aligned} \tilde{H}_{\beta,k} &= (\tilde{R}_{\beta,k})^s + \beta\Big(I - (\tilde{R}_{\beta,k})^s\Big)(I - \tilde{R}_{\beta,k})^{-1} B_k^{-1} N \\ &= I - \Big(I - (\tilde{R}_{\beta,k})^s\Big)\Big(I - \beta(I - \tilde{R}_{\beta,k})^{-1} B_k^{-1} N\Big) \\ &= I - \Big(I - (\tilde{R}_{\beta,k})^s\Big)(I - \tilde{R}_{\beta,k})^{-1}\Big(I - \tilde{R}_{\beta,k} - \beta B_k^{-1} N\Big) \\ &= I - \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j \Big((2-\beta) I - \beta B_k^{-1} C_k - \beta B_k^{-1} N\Big) \qquad (3) \\ &= I - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j B_k^{-1} \left(\frac{2-\beta}{\beta} B_k - C_k - N\right) \\ &= I - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j B_k^{-1} \tilde{A}_{\beta,k}. \end{aligned}$$

Let $B_k = D - E_k$ for each k, where $E_k \geq 0$. Since $1 < \beta < \dfrac{2}{1+\alpha}$, one obtains

$$\begin{aligned}
\tilde{A}_{\beta,k} &= \frac{2-\beta}{\beta} B_k - C_k - N \\
&= \frac{2-\beta}{\beta} D - \frac{2-\beta}{\beta} E_k - C_k - N \\
&\geq \frac{2-\beta}{\beta} D - E_k - C_k - N \\
&= \frac{2-\beta}{\beta} D - B - N \\
&= \tilde{A}_\beta
\end{aligned} \tag{4}$$

for each k. Since $\tilde{A}_\beta = \dfrac{2-\beta}{\beta} D - (B+N)$ is a regular splitting of $\tilde{A}_\beta$ and $\dfrac{\beta\alpha}{2-\beta} < 1$, $\tilde{A}_\beta^{-1} \geq 0$. Since $\tilde{R}_{\beta,k}$ and B_k^{-1} are nonnegative, from (3) and (4) one obtains

$$\tilde{H}_{\beta,k} \leq I - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j B_k^{-1} \tilde{A}_\beta. \tag{5}$$

Let $e = (1, 1, \ldots, 1)^T$ and $v = \tilde{A}_\beta^{-1} e$. Then $v > 0$ and $B_k^{-1} e > 0$. Using these relations and (5),

$$\tilde{H}_{\beta,k} v \leq v - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j B_k^{-1} e \leq v - \beta B_k^{-1} e < v. \tag{6}$$

From (6), there exists a $\theta_{\beta,k} \in [0, 1)$ such that

$$\tilde{H}_{\beta,k} v \leq \theta_{\beta,k} v \tag{7}$$

for each k. Let $\theta_\beta = \max\left\{\theta_{\beta,k} \mid 1 \leq k \leq \ell\right\}$. It is clear that $\theta_\beta < 1$. From (2) and (7),

$$|H_\beta| v \leq \tilde{H}_\beta v = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,k} v \leq \sum_{k=1}^{\ell} \theta_{\beta,k} E_k v \leq \theta_\beta v. \tag{8}$$

From (8), $\rho(|H_\beta|) \leq \theta_\beta < 1$ and hence $\rho(H_\beta) \leq \theta_\beta < 1$ for $1 < \beta < \dfrac{2}{1+\alpha}$. Therefore, the proof is complete. □

In Theorem 3.1, notice that $\dfrac{2}{1+\alpha} > 1$ since $\alpha < 1$. It means that Theorem 3.1 can be viewed as an extension of Theorem 2.1. We next provide convergence results of the relaxed two-stage multisplitting method using SOR multisplitting.

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix and $A = M - N$ be an M-splitting of A. Let $M = D - B = D - L_k - U_k$ $(1 \leq k \leq \ell)$, where $D = diag(M)$,*

$L_k \geq 0$ is a strictly lower triangular matrix, and $U_k \geq 0$ is a general matrix, and let $\left(B_k(\omega), C_k(\omega), E_k\right)$, $k = 1, 2, \ldots, \ell$, be the SOR multisplitting of M.

Let $\alpha = \max\left\{ \max\{\rho\left((B_k(\omega))^{-1}C_k(\omega)\right) \mid 1 \leq k \leq \ell\}, \rho(D^{-1}(B+N)) \right\}$. Then, the relaxed two-stage multisplitting method using an outer splitting $A = M - N$ and inner splittings $M = B_k(\omega) - C_k(\omega)$, $k = 1, 2, \ldots, \ell$, converges to the exact solution of $Ax = b$ for any initial vector x_0 if $0 < \omega \leq 1$ and $0 < \beta < \dfrac{2}{2 - \omega(1-\alpha)}$.

Proof. Since $0 < \omega \leq 1$, it is clear that $\alpha < 1$. For $0 < \omega \leq 1$ and $0 < \beta \leq 1$, this theorem follows directly from Theorem 2.1 since both $A = M - N$ and $M = B_k(\omega) - C_k(\omega)$ are regular splittings. Now we consider the case of $0 < \omega \leq 1$ and $1 < \beta < \dfrac{2}{2 - \omega(1-\alpha)}$. Let

$$R_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}C_k(\omega) + (1-\beta)I,$$

$$\tilde{R}_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}C_k(\omega) + (\beta-1)I,$$

$$H_{\beta,\omega} = \sum_{k=1}^{\ell} E_k H_{\beta,\omega,k}, \quad H_{\beta,\omega,k} = (R_{\beta,\omega,k})^s + \beta \sum_{j=0}^{s-1} (R_{\beta,\omega,k})^j (B_k(\omega))^{-1} N,$$

$$\tilde{H}_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,\omega,k}, \quad \tilde{H}_{\beta,\omega,k} = (\tilde{R}_{\beta,\omega,k})^s + \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,\omega,k})^j (B_k(\omega))^{-1} N.$$

Let

$$\tilde{A}_{\beta,\omega,k} = \frac{2-\beta}{\beta} B_k(\omega) - C_k(\omega) - N \text{ and } \tilde{A}_{\beta,\omega} = \frac{2 - 2\beta + \beta\omega}{\beta\omega} D - B - N.$$

Since $|R_{\beta,\omega,k}| \leq \tilde{R}_{\beta,\omega,k}$, $|H_{\beta,\omega,k}| \leq \tilde{H}_{\beta,\omega,k}$ and thus

$$|H_{\beta,\omega}| \leq \tilde{H}_{\beta,\omega}. \tag{9}$$

Since $\beta < \dfrac{2}{2-\omega(1-\alpha)} \leq \dfrac{2}{1+\alpha}$, $\dfrac{\beta\alpha}{2-\beta} < 1$ and thus $I - \tilde{R}_{\beta,\omega,k} = (2-\beta)I - \beta(B_k(\omega))^{-1}C_k(\omega)$ is nonsingular. Hence, one obtains

$$\begin{aligned}
\tilde{H}_{\beta,\omega,k} &= (\tilde{R}_{\beta,\omega,k})^s + \beta\Big(I - (\tilde{R}_{\beta,\omega,k})^s\Big)(I - \tilde{R}_{\beta,\omega,k})^{-1}(B_k(\omega))^{-1}N \\
&= I - \Big(I - (\tilde{R}_{\beta,\omega,k})^s\Big)\Big(I - \beta(I - \tilde{R}_{\beta,\omega,k})^{-1}(B_k(\omega))^{-1}N\Big) \\
&= I - \Big(I - (\tilde{R}_{\beta,\omega,k})^s\Big)(I - \tilde{R}_{\beta,\omega,k})^{-1}\Big(I - \tilde{R}_{\beta,\omega,k} - \beta(B_k(\omega))^{-1}N\Big) \\
&= I - \sum_{j=0}^{s-1} (\tilde{R}_{\beta,\omega,k})^j((2-\beta)I - \beta(B_k(\omega))^{-1}C_k(\omega) - \beta(B_k(\omega))^{-1}N)
\end{aligned} \tag{10}$$

$$= I - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,\omega,k})^j (B_k(\omega))^{-1} \left(\frac{2-\beta}{\beta} B_k(\omega) - C_k(\omega) - N \right)$$
$$= I - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,\omega,k})^j (B_k(\omega))^{-1} \tilde{A}_{\beta,\omega,k}.$$

Since $\beta > 1$, $B_k(\omega) = \frac{1}{\omega}(D - \omega L_k)$ and $C_k(\omega) = \frac{1}{\omega}((1-\omega)D + \omega U_k)$, one obtains

$$\begin{aligned} \tilde{A}_{\beta,\omega,k} &= \frac{2-2\beta+\beta\omega}{\beta\omega} D - \frac{2-\beta}{\beta} L_k - U_k - N \\ &\geq \frac{2-2\beta+\beta\omega}{\beta\omega} D - L_k - U_k - N \\ &= \frac{2-2\beta+\beta\omega}{\beta\omega} D - B - N \\ &= \tilde{A}_{\beta,\omega} \end{aligned} \tag{11}$$

for each k. Since $\beta < \frac{2}{2-\omega(1-\alpha)}$, $2 - 2\beta + \beta\omega > 0$ and $\frac{\beta\omega\alpha}{2-2\beta+\beta\omega} < 1$. It follows that $\tilde{A}_{\beta,\omega} = \frac{2-2\beta+\beta\omega}{\beta\omega} D - (B+N)$ is a regular splitting of $\tilde{A}_{\beta,\omega}$ and thus $\tilde{A}^{-1}_{\beta,\omega} \geq 0$. Since $\tilde{R}_{\beta,\omega,k}$ and $(B_k(\omega))^{-1}$ are nonnegative, from (10) and (11) one obtains

$$\tilde{H}_{\beta,\omega,k} \leq I - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,\omega,k})^j (B_k(\omega))^{-1} \tilde{A}_{\beta,\omega}. \tag{12}$$

Let $e = (1, 1, \ldots, 1)^T$ and $v = \tilde{A}^{-1}_{\beta,\omega} e$. Then $v > 0$ and $(B_k(\omega))^{-1} e > 0$. Using these relations and (12),

$$\tilde{H}_{\beta,\omega,k} v \leq v - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,\omega,k})^j (B_k(\omega))^{-1} e \leq v - \beta (B_k(\omega))^{-1} e < v. \tag{13}$$

From (13), there exists a $\theta_{\beta,\omega,k} \in [0, 1)$ such that

$$\tilde{H}_{\beta,\omega,k} v \leq \theta_{\beta,\omega,k} v \tag{14}$$

for each k. Let $\theta_{\beta,\omega} = \max\left\{\theta_{\beta,\omega,k} \mid 1 \leq k \leq \ell\right\}$. It is clear that $\theta_{\beta,\omega} < 1$. From (9) and (14),

$$|H_{\beta,\omega}| v \leq \tilde{H}_{\beta,\omega} v = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,\omega,k} v \leq \sum_{k=1}^{\ell} \theta_{\beta,\omega,k} E_k v \leq \theta_{\beta,\omega} v. \tag{15}$$

From (15), $\rho(|H_{\beta,\omega}|) \leq \theta_{\beta,\omega} < 1$ and hence $\rho(H_{\beta,\omega}) \leq \theta_{\beta,\omega} < 1$ for $0 < \omega \leq 1$ and $1 < \beta < \frac{2}{2-\omega(1-\alpha)}$. Therefore, the proof is complete. □

Theorem 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix and $A = M - N$ be an M-splitting of A. Let $M = D - B = D - L_k - U_k$ $(1 \le k \le \ell)$, where $D = diag(M)$, $L_k \ge 0$ is a strictly lower triangular matrix, and $U_k \ge 0$ is a general matrix, and let $\left(B_k(\omega), C_k(\omega), E_k\right)$, $k = 1, 2, \ldots, \ell$, be the SOR multisplitting of M.*

Let $\delta = \rho(D^{-1}(B+N))$ and $\alpha = \max\left\{\delta, \max\{\rho\left((B_k(\omega))^{-1}|C_k(\omega)|\right) \mid 1 \le k \le \ell\}\right\}$. Let $H_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \left((R_{\beta,\omega,k})^s + \beta \sum_{j=0}^{s-1} (R_{\beta,\omega,k})^j (B_k(\omega))^{-1} N\right)$ be an iteration matrix of the relaxed two-stage multisplitting method using an outer splitting $A = M - N$ and inner splittings $M = B_k(\omega) - C_k(\omega)$, where $R_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}C_k(\omega) + (1-\beta)I$. Then the following hold.

(a) *If $1 < \omega < \dfrac{2}{1+\delta}$ and $0 < \beta \le 1$, then $\rho(H_{\beta,\omega}) < 1$.*

(b) *If $\omega > 1$ is chosen so that $\omega(1+\alpha) < 2$ and if $0 < \beta < \dfrac{2}{\omega(1+\alpha)}$, then $\rho(H_{\beta,\omega}) < 1$.*

Proof. Let $\hat{M} = B_k(\omega) - |C_k(\omega)|$ for $1 < \omega < \dfrac{2}{1+\delta}$. Then $\hat{M} = B_k(\omega) - |C_k(\omega)| = \dfrac{2-\omega}{\omega} D - B$ are regular splittings of $\hat{M}$. Since $\frac{\omega\delta}{2-\omega} < 1$, $\hat{M}$ is an M-matrix and thus $\alpha < 1$. Let

$$\hat{R}_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}|C_k(\omega)| + (1-\beta)I,$$

$$\tilde{R}_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}|C_k(\omega)| + (\beta-1)I,$$

$$\hat{H}_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \hat{H}_{\beta,\omega,k}, \quad \hat{H}_{\beta,\omega,k} = (\hat{R}_{\beta,\omega,k})^s + \beta \sum_{j=0}^{s-1} (\hat{R}_{\beta,\omega,k})^j (B_k(\omega))^{-1} N,$$

$$\tilde{H}_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,\omega,k}, \quad \tilde{H}_{\beta,\omega,k} = (\tilde{R}_{\beta,\omega,k})^s + \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,\omega,k})^j (B_k(\omega))^{-1} N.$$

We first prove part (a). Let $\hat{A} = \hat{M} - N$. Then $\hat{A} = \hat{M} - N = \dfrac{2-\omega}{\omega} D - (B+N)$ are regular splittings of $\hat{A}$. Since $\dfrac{\omega\delta}{2-\omega} < 1$, $\hat{A}$ is also an M-matrix. Since $\hat{H}_{\beta,\omega}$ can be viewed as an iteration matrix of the relaxed two-stage multisplitting method using an outer splitting $\hat{A} = \hat{M} - N$ and inner splittings $\hat{M} = B_k(\omega) - |C_k(\omega)|$, $\rho(\hat{H}_{\beta,\omega}) < 1$ is obtained from Theorem 2.1. Since it can be easily shown that $|H_{\beta,\omega}| \le \hat{H}_{\beta,\omega}$, $\rho(H_{\beta,\omega}) < 1$. Next we prove part (b). Assume that $\omega > 1$ is chosen so that $\omega(1+\alpha) < 2$. Then $\omega(1+\delta) < 2$. For $0 < \beta \le 1$, $\rho(H_{\beta,\omega}) < 1$ is directly obtained from part (a). Now we consider the case of $1 < \beta < \dfrac{2}{\omega(1+\alpha)}$.

Let

$$\tilde{A}_{\beta,\omega,k} = \frac{2-\beta}{\beta}B_k(\omega) - |C_k(\omega)| - N \text{ and } \tilde{A}_{\beta,\omega} = \frac{2-\beta\omega}{\beta\omega}D - B - N.$$

Since $|R_{\beta,\omega,k}| \leq \tilde{R}_{\beta,\omega,k}$, $|H_{\beta,\omega}| \leq \tilde{H}_{\beta,\omega}$. Since $\omega > 1$, $\beta < \dfrac{2}{\omega(1+\alpha)} < \dfrac{2}{(1+\alpha)}$ and so $\dfrac{\beta\alpha}{2-\beta} < 1$. It follows that $I - \tilde{R}_{\beta,\omega,k} = (2-\beta)I - \beta(B_k(\omega))^{-1}|C_k(\omega)|$ is nonsingular. Hence, one obtains

$$\tilde{H}_{\beta,\omega,k} = I - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j(B_k(\omega))^{-1}\tilde{A}_{\beta,\omega,k}. \tag{16}$$

Since $\beta > 1$, $B_k(\omega) = \dfrac{1}{\omega}(D - \omega L_k)$ and $|C_k(\omega)| = \dfrac{1}{\omega}((\omega-1)D + \omega U_k)$, one obtains

$$\begin{aligned}\tilde{A}_{\beta,\omega,k} &= \frac{2-\beta\omega}{\beta\omega}D - \frac{2-\beta}{\beta}L_k - U_k - N\\ &\geq \frac{2-\beta\omega}{\beta\omega}D - L_k - U_k - N\\ &= \frac{2-\beta\omega}{\beta\omega}D - B - N\\ &= \tilde{A}_{\beta,\omega}\end{aligned} \tag{17}$$

for each k. Since $\beta < \dfrac{2}{\omega(1+\alpha)}$, $2-\beta\omega > 0$ and $\dfrac{\beta\omega\alpha}{2-\beta\omega} < 1$. It follows that $\tilde{A}_{\beta,\omega} = \dfrac{2-\beta\omega}{\beta\omega}D - (B+N)$ is a regular splitting of $\tilde{A}_{\beta,\omega}$ and thus $\tilde{A}_{\beta,\omega}^{-1} \geq 0$. Since $\tilde{R}_{\beta,\omega,k}$ and $(B_k(\omega))^{-1}$ are nonnegative, from (16) and (17) one obtains

$$\tilde{H}_{\beta,\omega,k} \leq I - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j(B_k(\omega))^{-1}\tilde{A}_{\beta,\omega}. \tag{18}$$

Let $e = (1,1,\ldots,1)^T$ and $v = \tilde{A}_{\beta,\omega}^{-1}e$. Then $v > 0$ and $(B_k(\omega))^{-1}e > 0$. The remaining part of the proof can be done in a similar way as was done in that of Theorem 3.2. Hence, $\rho(H_{\beta,\omega}) < 1$ is obtained for $1 < \beta < \dfrac{2}{\omega(1+\alpha)}$. Therefore, the proof is complete. □

Notice that $1 < \dfrac{2}{1+\alpha} \leq \dfrac{2}{1+\delta}$ in Theorem 3.3. The following theorem is directly obtained by combining Theorems 3.2 and 3.3.

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix and $A = M - N$ be an M-splitting of A. Let $M = D - B = D - L_k - U_k$ $(1 \leq k \leq \ell)$, where $D = diag(M)$, $L_k \geq 0$ is a strictly lower triangular matrix, and $U_k \geq 0$ is a general matrix, and let $\Big(B_k(\omega), C_k(\omega), E_k\Big)$, $k = 1,2,\ldots,\ell$, be the SOR multisplitting of M.*

Let $\delta = \rho(D^{-1}(B+N))$ *and* $\alpha = \max\Big\{\delta,\ \max\{\rho\left((B_k(\omega))^{-1}|C_k(\omega)|\right) \mid 1 \leq k \leq \ell\}\Big\}$. *Let* $H_{\beta,\omega} = \sum_{k=1}^{\ell} E_k\left((R_{\beta,\omega,k})^s + \beta\sum_{j=0}^{s-1}(R_{\beta,\omega,k})^j(B_k(\omega))^{-1}N\right)$ *be an iteration matrix of the relaxed two-stage multisplitting method using an outer splitting* $A = M - N$ *and inner splittings* $M = B_k(\omega) - C_k(\omega)$, *where* $R_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}C_k(\omega) + (1-\beta)I$. *Then the following hold.*

(*a*) *If* $1 < \omega < \dfrac{2}{1+\delta}$ *and* $0 < \beta \leq 1$, *then* $\rho(H_{\beta,\omega}) < 1$.

(*b*) *If* $\omega > 0$ *is chosen so that* $\omega(1+\alpha) < 2$ *and if* $0 < \beta < \dfrac{2}{1+\omega\alpha+|1-\omega|}$, *then* $\rho(H_{\beta,\omega}) < 1$.

In Theorem 3.4, notice that if $0 < \omega \leq 1$, then the condition $\omega(1+\alpha) < 2$ in part (b) is automatically satisfied from the fact that $\alpha < 1$.

4. Numerical experiments

In this section, we provide numerical experiments for the convergence of the relaxed two-stage multisplitting method using SOR multisplitting as inner splittings. All numerical values are computed using MATLAB.

Example 4.1. *Suppose that* $\ell = 3$. *Consider an M-matrix* A *of the form*

$$A = \begin{pmatrix} F & -I & 0 \\ -I & F & -I \\ 0 & -I & F \end{pmatrix}, \quad F = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $A = M - N$, *where*

$$M = \begin{pmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}, \quad N = \begin{pmatrix} 0 & I & 0 \\ I & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

Let $D = diag(M)$, $B = D - M$,

$$L_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$L_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} L_{11} & 0 & 0 \\ 0 & L_{12} & 0 \\ 0 & 0 & L_{12} \end{pmatrix}, \quad L_2 = \begin{pmatrix} L_{12} & 0 & 0 \\ 0 & L_{11} & 0 \\ 0 & 0 & L_{12} \end{pmatrix}, \quad L_3 = \begin{pmatrix} L_{12} & 0 & 0 \\ 0 & L_{12} & 0 \\ 0 & 0 & L_{11} \end{pmatrix},$$

$$U_1 = \begin{pmatrix} U_{11} & 0 & 0 \\ 0 & U_{12} & 0 \\ 0 & 0 & U_{12} \end{pmatrix}, \quad U_2 = \begin{pmatrix} U_{12} & 0 & 0 \\ 0 & U_{11} & 0 \\ 0 & 0 & U_{12} \end{pmatrix}, \quad U_3 = \begin{pmatrix} U_{12} & 0 & 0 \\ 0 & U_{12} & 0 \\ 0 & 0 & U_{11} \end{pmatrix},$$

$$E_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Then, $A = M - N$ is an M-splitting of A and $M = D - L_k - U_k$ for $k = 1,2,3$. Let $(B_k(\omega), C_k(\omega), E_k)$, $k = 1,2,3$, be the SOR multisplitting of M. That is, $B_k(\omega) = \frac{1}{\omega}(D - \omega L_k)$ and $C_k(\omega) = \frac{1}{\omega}((1-\omega)D + \omega U_k)$ for $k = 1,2,3$. Then δ, α and $H_{\beta,\omega}$ are defined as in Theorem 3.4.

Note that $\delta = \rho(D^{-1}(B+N)) \approx 0.7071$ and $\frac{2}{1+\delta} \approx 1.1716$. For various values of ω, the numerical values of α, $\omega(1+\alpha)$ and $\frac{2}{1+\omega\alpha+|1-\omega|}$ are listed in Table 1. Note that the upper bound of β, which is $\frac{2}{1+\omega\alpha+|1-\omega|}$, becomes maximum when $\omega = 1$. When $\omega > 1$ and $\omega(1+\alpha) < 2$, the upper bound of β is greater than 1, but it is close to 1. Numerical values of $\rho(H_{\beta,\omega})$ for various values of ω, β and s are listed in Table 2. For this test problem, $\rho(H_{\beta,\omega})$ is optimal when $\omega(1+\alpha)$ is close to 2 and β is close to its upper bound $\frac{2}{1+\omega\alpha+|1-\omega|}$.

TABLE 1. Numerical values of α, $\omega(1+\alpha)$ and $\frac{2}{1+\omega\alpha+|1-\omega|}$ for Example 4.1.

ω	α	$\omega(1+\alpha)$	$\frac{2}{1+\omega\alpha+\|1-\omega\|}$	ω	α	$\omega(1+\alpha)$	$\frac{2}{1+\omega\alpha+\|1-\omega\|}$
0.2	0.8683	0.3737	1.0133	1.0	0.7071	1.7071	1.1716
0.3	0.8006	0.5402	1.0308	1.1	0.7071	1.8778	1.0651
0.5	0.7071	0.8536	1.0790	1.15	0.7071	1.9632	1.0188
0.8	0.7071	1.3657	1.1327	1.17	0.7071	1.9973	1.0013
0.9	0.7071	1.5364	1.1518	1.18	0.7071	2.0144	0.9929

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TABLE 2. Numerical values of $\rho(H_{\beta,\omega})$ for Example 4.1.

s	ω	β	$\rho(H_{\beta,\omega})$	s	ω	β	$\rho(H_{\beta,\omega})$	s	ω	β	$\rho(H_{\beta,\omega})$
1	0.2	0.8	0.9523	2	0.2	0.8	0.9090	3	0.2	0.8	0.8714
		1.0	0.9404			1.0	0.8886			1.0	0.8436
		1.01	0.9398			1.01	0.8875			1.01	0.8423
	0.5	0.8	0.8773		0.5	0.8	0.7879		0.5	0.8	0.7226
		1.0	0.8466			1.0	0.7452			1.0	0.6781
		1.07	0.8359			1.07	0.7313			1.07	0.6645
	0.8	0.8	0.7978		0.8	0.8	0.6858		0.8	0.8	0.6238
		1.0	0.7472			1.0	0.6354			1.0	0.5860
		1.13	0.7143			1.13	0.6086			1.13	0.5696
	1.0	0.8	0.7419		1.0	0.8	0.6307		1.0	0.8	0.5828
		1.0	0.6773			1.0	0.5836			1.0	0.5576
		1.17	0.7851			1.17	0.5554			1.17	0.5496
	1.1	0.8	0.7130		1.1	0.8	0.6075		1.1	0.8	0.5689
		1.0	0.6513			1.0	0.5640			1.0	0.5512
		1.06	0.7504			1.06	0.5540			1.06	0.5496
	1.15	0.8	0.6983		1.15	0.8	0.5969		1.15	0.8	0.5635
		1.0	0.7111			1.0	0.5553			1.0	0.5497
		1.01	0.7282			1.01	0.5536			1.01	0.5495

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