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THE ITERATION METHOD OF SOLVING A TYPE OF THREE-POINT BOUNDARY VALUE PROBLEM

XIPING LIU* AND MEI JIA

ABSTRACT. This paper studies the iteration method of solving a type of second-order three-point boundary value problem with non-linear term f, which depends on the first order derivative. By using the upper and lower method, we obtain the sufficient conditions of the existence and uniqueness of solutions. Furthermore, the monotone iterative sequences generated by the method contribute to the minimum solution and the maximum solution. And the error estimate formula is also given under the condition of unique solution. We apply the solving process to a special boundary value problem, and the result is interesting.

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1. Introduction

In this paper, we study the iterative method of solving a second-order three-point boundary value problem

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & t \in (0,1), \\ x(0) = kx(\xi), \quad x'(1) = 0, \end{cases} \tag{1}$$

where $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $0 < \xi \leq 1$, $0 \leq k \leq 1$ are constants.

In recent years, there are many literature on the existence of solutions for the second-order boundary value problem (see [1,2,6-9]). However, the literature which study the method for approximate functional solutions on such a problem are fewer. In this paper, the method is investigated. Those literature in this area mostly study the existence and uniqueness of solutions of the boundary value problem with nonlinear term $f(t, u)$. However, the nonlinear term f usually

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*Corresponding author.
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satisfies Nagumo condition when f depends on the first order derivative (see, for example [1,2,4,5]), which weaken the role of the first order derivative term.

Special mention is given here that in our consideration the problem with the nonlinear term f depending on the first order derivative need not satisfy Nagumo condition. By constructing a special cone and using the upper and lower method, we obtain the sufficient conditions of the existence and uniqueness of solutions. Furthermore, we obtain the monotone iterative sequence generated by the method. The error estimate formula is also given under the condition of unique solution. Finally, we apply the solving process to a special boundary value problem, and the result is satisfying.

2. Preliminaries

Throughout this paper, the following hypothesis is satisfied:

(H_0) The constant $0 < \theta < \arccos k$. Namely $0 \le k < \cos\theta$.

For convenience, we denote $\rho = \theta(\cos\theta - k\cos((1-\xi)\theta)$. Then $\rho > 0$ when the hypothesis (H_0) holds. To investigate the boundary value problem (1), first we consider the Green's function of the boundary value problem

$$\begin{cases} -x''(t) - \theta^2 x(t) = 0, \quad t \in (0,1), \\ x(0) = kx(\xi), \quad x'(1) = 0. \end{cases} \tag{2}$$

The following lemmas (Lemma 1-3) are easy to prove.

Lemma 1. *The Green's function of the boundary value problem* (2) *is*

$$G(t,s) = \frac{1}{\rho}\begin{cases} \sin(\theta t)\cos((1-s)\theta) + k\sin((s-t)\theta)\cos((1-\xi)\theta), & t \le s \le \xi, \\ \cos((1-t)\theta)\sin(s\theta), & s \le \min\{t,\xi\}, \\ \cos((1-s)\theta)(\sin(t\theta) - k\sin((t-\xi)\theta), & \max\{t,\xi\} \le s, \\ \cos((1-t)\theta)(\sin(s\theta) - k\sin((s-\xi)\theta), & \xi \le s \le t. \end{cases} \tag{3}$$

Lemma 2. *If* (H_0) *holds, then*

(1) $$\frac{\partial G(t,s)}{\partial t} = \frac{\theta}{\rho}\begin{cases} \cos(\theta t)\cos((1-s)\theta) - k\cos((s-t)\theta)\cos((1-\xi)\theta), & 0 \le t < s \le \xi \le 1, \\ \sin((1-t)\theta)\sin(s\theta), & 0 \le s < \min\{t,\xi\} \le 1, \\ \cos((1-s)\theta)(\cos(t\theta) - k\cos((t-\xi)\theta), & 0 \le \max\{t,\xi\} < s \le 1, \\ \sin((1-t)\theta)(\sin(s\theta) - k\sin((s-\xi)\theta), & 0 \le \xi < s \le t \le 1; \end{cases}$$

(2) *for any* $t \in [0,1]$,

$$\int_0^1 G(t,s)\mathrm{d}s = \frac{1}{\rho\theta}(-\cos(\theta) + (1-k)\cos((1-t)\theta) + k\cos((1-\xi)\theta))$$

and

$$\max_{t\in[0,1]}\int_0^1 G(t,s)\mathrm{d}s = \frac{1}{\rho\theta}(-\cos(\theta) + (1-k) + k\cos((1-\xi)\theta))$$

(3) *for any* $t \in [0,1]$,

$$\int_0^1 \frac{\partial G(t,s)}{\partial t}\mathrm{d}s = \frac{1}{\rho}((1-k)\sin((1-t)\theta))$$

and

$$\max_{t\in[0,1]}\int_0^1 \frac{\partial G(t,s)}{\partial t}\mathrm{d}s = \frac{(1-k)\sin\theta}{\rho}$$

Lemma 3. *If* (H_0) *holds, then for any* $t, s \in [0,1]$, $G(t,s) \geq 0$, $\dfrac{\partial G(t,s)}{\partial t} \geq 0$.

In the following, we establish a comparison principle.

Lemma 4. *Suppose* (H_0) *holds and* $x \in C^2[0,1]$ *satisfies*

$$\begin{cases} -x''(t) - \theta^2 x(t) \geq 0, & t \in (0,1), \\ x(0) \geq kx(\xi), \\ x'(1) \geq 0. \end{cases} \tag{4}$$

Then $x(t) \geq 0, t \in [0,1]$.

Proof. By the definition of Green's function, the unique solution of the boundary value problem

$$\begin{cases} -x''(t) - \theta^2 x(t) = h(t), \ \ t \in (0,1), \\ x(0) = kx(\xi), \ \ x'(1) = 0. \end{cases} \tag{5}$$

is

$$x_1(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s \tag{6}$$

Suppose $x_0(t)$ satisfies

$$\begin{cases} -x''(t) - \theta^2 x(t) = 0, \ \ t \in (0,1), \\ x(0) = kx(\xi) + a, \ \ x'(1) = b \end{cases}$$

where $a, b \in \mathbb{R}$. Then $x_0(t) = r_1\cos(\theta t) + r_2\sin(\theta t)$ satisfies the boundary conditions, and we have

$$\begin{cases} -r_1\theta\sin\theta + r_2\theta\cos\theta = b \\ r_1 - k(r_1\cos(\theta\xi) + r_2\sin(\theta\xi) = a \end{cases}$$

Solving the equations we obtain

$$\begin{cases} r_1 = \dfrac{1}{\rho}(bk\sin(\theta\xi) + a\theta\cos\theta) \\ r_2 = \dfrac{1}{\rho}(b - bk\cos(\theta\xi) + a\theta\sin\theta). \end{cases}$$

Then

$$x_0(t) = \frac{1}{\rho}\Big(b(\sin(\theta t) - k\sin((t-\xi)\theta) + a\cos((1-t)\theta)\Big) \tag{7}$$

and $x_0(t) \geq 0, t \in [0,1]$ when (H_0) holds.

Therefore, the solution of the boundary value problem

$$\begin{cases} -x''(t) - \theta^2 x(t) = h(t), \ \ t \in (0,1), \\ x(0) = kx(\xi) + a, \ \ x'(1) = b \end{cases}$$

is

$$x(t) = x_0(t) + \int_0^1 G(t,s)h(s)\mathrm{d}s \tag{8}$$

Since $h(t) \geq 0$, $t \in (0,1)$, $a, b \geq 0$, we obtain $x(t) \geq 0$ for $t \in [0,1]$ by Lemma 3, (7) and (8).

For convenience, we give out the following two lemmas which are used in the later proof.

Lemma 5. (*see Lemma 1.1.2 of* [3]) *Let E be partially ordered Banach space, $\{x_n\} \subset E$ is monotone sequence and relatively compact set, then $\{x_n\}$ is convergent.*

Lemma 6. (*see Lemma 1.1.1 of* [3]) *Let E be partially ordered Banach space, $x_n \preceq y_n$, $(n = 1, 2, 3 \cdots)$, if $x_n \to x^*$, $y_n \to y^*$, we have $x^* \preceq y^*$.*

Remark 1. " $\preceq$ " denotes the partially ordering of partially ordered Banach space E.

3. Existence of solutions of the boundary value problem

Let $E = C^1[0,1]$ with $||x|| = \max\{|x|_\infty, |x'|_\infty\}$, where $|x|_\infty = \max\limits_{t\in[0,1]} |x(t)|$,

$$P = \{x | x \in E, x(t) \geq 0, x'(t) \geq 0, t \in [0,1]\}$$

Then P is a cone of E and E is a partially ordered Banach space.

Obviously, for any $x \preceq y \in E$ if and only if $y - x \in P$, namely

$$x \preceq y \in E \;\;\Leftrightarrow\;\; x(t) \leq y(t) \text{ and } x'(t) \leq y'(t), \text{ for any } t \in [0,1]. \tag{9}$$

For any $\alpha \preceq \beta \in E$, denote $D_0 = [\alpha, \beta] = \{x \in E \mid \alpha \preceq x \preceq \beta\}$, then D_0 is a bounded set.

Definition 1. Let $\varphi_0 \in C^2([0,1])$. Then φ_0 is a lower solution of boundary value problem (1), if

$$\begin{cases} -\varphi_0''(t) \leq f(t, \varphi_0(t), \varphi_0'(t)), \\ \varphi_0(0) \leq k\varphi_0(\xi), \\ \varphi_0'(1) \leq 0. \end{cases}$$

Let $\psi_0 \in C^2([0,1])$. Then ψ_0 is an upper solution of the boundary value problem (1), if

$$\begin{cases} -\psi_0''(t) \geq f(t, \psi_0(t), \psi_0'(t)), \\ \psi_0(0) \geq k\psi_0(\xi), \\ \psi_0'(1) \geq 0. \end{cases}$$

Theorem 1. *Suppose (H_0) holds, and there exist a upper solution ψ_0 and a lower solution φ_0 of boundary value problem* (1) *such that $\varphi_0 \preceq \psi_0$ on $[0,1]$. And if f satisfies*

(H_1) $f(t, u_2, v) - f(t, u_1, v) \geq \theta^2(u_2 - u_1)$, *for any* $t \in [0,1]$, $\varphi_0'(t) \leq v \leq \psi_0'(t)$, $\varphi_0(t) \leq u_1 \leq u_2 \leq \psi_0(t)$;

$f(t,u,v_2)-f(t,u,v_1)\geq 0$, *for any* $t\in[0,1]$, $\varphi_0(t)\leq u\leq\psi_0(t)$, $\varphi_0'(t)\leq v_1\leq v_2\leq\psi_0'(t)$,
then the boundary value problem (1) *has the minimal solution* φ^* *and the maximal solution* ψ^* *on the ordered interval* $[\varphi_0,\psi_0]$. *Moreover, the iterative sequences defined by*

$$\varphi_n(t)=\int_0^1 G(t,s)(f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s))-\theta^2\varphi_{n-1}(s))\mathrm{d}s$$

$$\psi_n(t)=\int_0^1 G(t,s)(f(s,\psi_{n-1}(s),\psi_{n-1}'(s))-\theta^2\psi_{n-1}(s))\mathrm{d}s.$$

converge uniformly on $[0,1]$ *to* φ^*, ψ^* *respectively.*

Proof. It is easy to see that $x=\varphi_0=\psi_0$ is the solution of the boundary value problem (1) if $\varphi_0\equiv\psi_0$. Now suppose $\varphi_0\not\equiv\psi_0$.

Denote $D=[\varphi_0,\psi_0]$. For any $h\in D$, we consider the following boundary value problem

$$\begin{cases} -x''(t)-\theta^2x(t)=f(t,h(t),h'(t))-\theta^2h(t), & t\in(0,1),\\ x(0)=kx(\xi), \quad x'(1)=0. \end{cases} \tag{10}$$

By the definition of Green's function the unique solution of the boundary value problem (10) is

$$x(t)=\int_0^1 G(t,s)(f(s,h(s),h'(s))-\theta^2h(s))\mathrm{d}s:=(Qh)(t)$$

It is clear that x is a solution of the boundary value problem (1) if and only if x is a fixed point of Q.

Let $F:D\to C([0,1])$, $(Fh)(t)=f(t,h(t),h'(t))-\theta^2h(t)$. Then F is a continuous and bounded operator.

Let $T:C([0,1])\to C^1([0,1])$, $(Th)(t)=\displaystyle\int_0^1 G(t,s)h(s)\mathrm{d}s$. Then T is continuous , increasing and linear compact operator.

Denote $Q=T\circ F$, so $Q:D\to C([0,1])$ is continuous and relatively compact. That is, $Q(D)$ is a relatively compact set. We have three claims as follows.

Claim 1: Q is an increasing operator.

For any h_1, $h_2\in D$ and $h_1\preceq h_2$, by (9), we have

$$\varphi_0(t)\leq h_1(t)\leq h_2(t)\leq\psi_0(t),\ \varphi_0'(t)\leq h_1'(t)\leq h_2'(t)\leq\psi_0'(t),\ \text{for any } t\in[0,1]$$

By (H_1), we have

$$\begin{aligned}
&(Fh_2-Fh_1)(t)\\
&=[f(t,h_2(t),h_2'(t))-\theta^2h_2(t)]-[f(t,h_1(t),h_1'(t))-\theta^2h_1(t)]\\
&=[f(t,h_2(t),h_2'(t))-f(t,h_1(t),h_1'(t)))]-\theta^2(h_2(t)-h_1(t))\\
&=[f(t,h_2(t),h_2'(t))-f(t,h_1(t),h_2'(t)))]+[f(t,h_1(t),h_2'(t))\\
&\quad -f(t,h_1(t),h_1'(t)))]-\theta^2(h_2(t)-h_1(t))\\
&\geq\theta^2(h_2(t)-h_1(t))-\theta^2(h_2(t)-h_1(t))\\
&=0
\end{aligned}$$

Therefore $(Qh_2)(t) - (Qh_1)(t) = (T \circ F)h_2(t) - (T \circ F)h_1(t)$. By Lemma 3, we get $(Qh_1)(t) \leq (Qh_2)(t)$, for any $t \in [0, 1]$.

And for any $t \in [0, 1]$, by Lemma 3,

$$(Qh_2)'(t) - (Qh_1)'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t}((Fh_2)(s) - (Fh_1)(s))\mathrm{d}s \geq 0.$$

Hence we have

$$(Qh_1)'(t) \leq (Qh_2)'(t).$$

So $(Qh_2) - (Qh_1) \in P$, namely $Qh_1 \preceq Qh_2$. Therefore, $Q : D \to C^1([0, 1])$ is an increasing operator.

Claim 2: $\varphi_0 \preceq Q\varphi_0$, $Q\psi_0 \preceq \psi_0$.

Denote $\varphi_1 = Q\varphi_0$. Since φ_0 is the lower solution of the boundary value problem (1), then

$$\begin{cases} -\varphi_0''(t) \leq f(t, \varphi_0(t), \varphi_0'(t)), \\ \varphi_0(0) \leq k\varphi_0(\xi), \\ \varphi_0'(1) \leq 0. \end{cases}$$

Let $\varphi = \varphi_1 - \varphi_0$, by the definition of Q, for any $t \in [0, 1]$, we have

$$\begin{aligned} &-\varphi''(t) - \theta^2\varphi(t) \\ &= -(\varphi_1(t) - \varphi_0(t))'' - \theta^2(\varphi_1(t) - \varphi_0(t)) \\ &= (-\varphi_1''(t) - \theta^2\varphi_1(t)) - (-\varphi_0''(t) - \theta^2\varphi_0(t)) \\ &\geq (f(t, \varphi_0(t), \varphi_0'(t)) - \theta^2\varphi_0(t)) - (f(t, \varphi_0(t), \varphi_0'(t)) - \theta^2\varphi_0(t)) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &\varphi(0) = \varphi_1(0) - \varphi_0(0) = k\varphi_1(\xi) - k\varphi_0(\xi) \geq k\varphi(\xi), \\ &\varphi'(1) = \varphi_1'(1) - \varphi_0'(1) \geq 0. \end{aligned}$$

Then φ satisfies (4), by Lemma 4 we have $\varphi(t) \geq 0$ on $[0, 1]$. That is $\varphi_0(t) \leq Q\varphi_0(t) = \varphi_1(t)$, for any $t \in [0, 1]$. Moreover for any $t \in [0, 1]$

$$\begin{aligned} \varphi''(t) &= \varphi_1''(t) - \varphi_0''(t) \\ &\leq -f(t, \varphi_0(t), \varphi_0'(t)) - \theta^2\varphi_1(t) + \theta^2\varphi_0(t) + f(t, \varphi_0(t), \varphi_0'(t)) \\ &= \theta^2[\varphi_0(t) - \varphi_1(t)] \\ &\leq 0. \end{aligned}$$

Hence $\varphi'(t)$ is monotone decreasing for any $t \in [0, 1]$. We have $\varphi'(t) \geq \varphi'(1) \geq 0$, i.e., $\varphi_1'(t) - \varphi_0'(t) \leq 0$, $t \in [0, 1]$.

Therefore, $\varphi_0 \preceq Q\varphi_0$. Similarly, we can prove that $Q\psi_0 \preceq \psi_0$.

Claim 3: There exist of the minimal solution and the maximal solution of boundary value problem (1).

By repeating the steps of Claim 2, we can construct iterative sequences. For $n = 1, 2, \cdots$

$$\varphi_n = Q\varphi_{n-1} = \int_0^1 G(t, s)(f(s, \varphi_{n-1}(s), \varphi_{n-1}'(s)) - \theta^2\varphi_{n-1}(s))\mathrm{d}s$$

$$\psi_n = Q\psi_{n-1} = \int_0^1 G(t,s)(f(s,\psi_{n-1}(s),\psi'_{n-1}(s)) - \theta^2\psi_{n-1}(s))\mathrm{d}s$$

And we get

$$\varphi_0 \preceq \varphi_1 \preceq \varphi_2 \preceq \cdots \leq \varphi_n \preceq \cdots \preceq \psi_n \preceq \cdots \preceq \psi_2 \preceq \psi_1 \preceq \psi_0.$$

From $\{\psi_n\}$, $\{\varphi_n\} \subset Q(D)$ and Lemma 5 we can show that there exist φ^*, $\psi^* \in D$ such that $\psi_n \to \psi^*$, $\varphi_n \to \varphi^*$ $(n \to \infty)$. By the continuity of Q, we have $\varphi^* = Q\varphi^*$ and $\psi^* = Q\psi^*$ when $n \to \infty$. So φ^* and ψ^* are the fixed points of Q.

In the following, we prove that φ^*, ψ^* are the minimal solution and the maximal solution of boundary value problem (1) respectively.

Assume $z \in D = [\varphi_0, \psi_0]$ is a fixed point of Q, then $\varphi_0 \preceq z \preceq \psi_0$. As Q is an increasing operator, we get $Q\varphi_0 \preceq Qz \preceq Q\psi_0$, i.e., $\varphi_1 \preceq z \preceq \psi_1$. In a similar way we get $Q\varphi_1 \preceq Qz \preceq Q\psi_1$, i.e., $\varphi_2 \preceq z \preceq \psi_2$. Do it repeatedly. Then we have $\varphi_n \preceq z \preceq \psi_n$. By Lemma 6, $\varphi^* \preceq z \preceq \psi^*$ holds. That is φ^*, ψ^* are the minimal fixed point and the maximal fixed point of Q respectively.

Therefore, φ^*, ψ^* are the minimal solution and maximal solution of boundary value problem (1) respectively. □

4. Uniqueness of solutions of the boundary value problem

Denote $M = \max_{t\in[0,1]} \int_0^1 G(t,s)\mathrm{d}s = \frac{1}{\rho\theta}(-\cos(\theta) + (1-k) + k\cos((1-\xi)\theta))$,

$$m = \max_{t\in[0,1]} \int_0^1 \frac{\partial G(t,s)}{\partial t}\mathrm{d}s = \frac{(1-k)\sin\theta}{\rho}, \quad \overline{M} = \max\{M, m\}.$$

Then $M, m, \overline{M} > 0$.

Theorem 2. *Suppose that all the hypotheses of Theorem 1 hold, and*

(H_2) There exists constant M_1 with $0 < M_1 < \frac{1}{\overline{M}}$, such that

$$f(t,u,v_2) - f(t,u,v_1) \leq M_1(v_2 - v_1)$$

for any $t \in [0,1]$, $\varphi_0(t) \leq u \leq \psi_0(t)$, $\varphi'_0(t) \leq v_1 \leq v_2 \leq \psi'_0(t)$;
and there exists constant M_2 with $\theta^2 < M_2 < \theta^2 + \frac{1}{\overline{M}} - M_1$, such that

$$f(t,u_2,v) - f(t,u_1,v) \leq M_2(u_2 - u_1)$$

for any $t \in [0,1]$, $\varphi'_0(t) \leq v \leq \psi'_0(t)$, $\varphi_0(t) \leq u_1 \leq u_2 \leq \psi_0(t)$.

Then boundary value problem (1) *has a unique solution x^* on $[\varphi_0, \psi_0]$ and for any $x_0 \in [\varphi_0, \psi_0]$, iterative sequence*

$$x_n(t) = \int_0^1 G(t,s)(f(s,x_{n-1}(s),x'_{n-1}(s)) - \theta^2 x_{n-1}(s))\mathrm{d}s, \quad n = 1, 2, \cdots$$

converge uniformly to x^* on [0,1], and its error estimate is

$$||x_n - x^*|| \leq 2\Big(\overline{M}(M_1 + M_2 - \theta^2)\Big)^n ||\psi_0 - \varphi_0||, \quad n = 1, 2, \cdots. \tag{11}$$

Or for $n = 1, 2, \cdots$.

$$|x_n - x^*|_\infty \le 2M\Big(M_1 m + M(M_2-\theta^2)\Big)^{n-1}\Big(M_1|\psi_0' - \varphi_0'|_\infty + (M_2-\theta^2)|\psi_0 - \varphi_0|_\infty\Big) \tag{12}$$

$$|x_n' - (x^*)'|_\infty \le 2m\Big(M_1 m + M(M_2-\theta^2)\Big)^{n-1}\Big(M_1|\psi_0' - \varphi_0'|_\infty + (M_2-\theta^2)|\psi_0 - \varphi_0|_\infty\Big) \tag{13}$$

Proof. According to the assumptions of Theorem 2 we have

$$\begin{aligned}
0 &\le \psi_n(t) - \varphi_n(t) = Q\psi_{n-1}(t) - Q\varphi_{n-1}(t) \\
&= \int_0^1 G(t,s)\Big(f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - \theta^2\psi_{n-1}(s)\Big)\mathrm{d}s \\
&\quad - \int_0^1 G(t,s)\Big(f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)) - \theta^2\varphi_{n-1}(s)\Big)\mathrm{d}s \\
&= \int_0^1 G(t,s)\Big(f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)) + \theta^2(\varphi_{n-1}(s) \\
&\quad -\psi_{n-1}(s))\Big)\mathrm{d}s \\
&= \int_0^1 G(t,s)\Big((f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - f(s,\psi_{n-1}(s),\varphi_{n-1}'(s))) + (f(s,\psi_{n-1}(s), \\
&\quad \varphi_{n-1}'(s)) - f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s))) + \theta^2(\varphi_{n-1}(s) - \psi_{n-1}(s))\Big)\mathrm{d}s \\
&\le \int_0^1 G(t,s)\Big(M_1(\varphi_{n-1}'(s)) - \psi_{n-1}'(s)) + M_2(\psi_{n-1}(s)) - \varphi_{n-1}(s)) \\
&\qquad + \theta^2(\varphi_{n-1}(s) - \psi_{n-1}(s)\Big)\mathrm{d}s \\
&= \int_0^1 G(t,s)\Big(M_1(\varphi_{n-1}'(s)) - \psi_{n-1}'(s)) + (M_2-\theta^2)(\psi_{n-1}(s)) - \varphi_{n-1}(s))\Big)\mathrm{d}s \\
&\le (M_1 + M_2 - \theta^2)\int_0^1 G(t,s) \parallel \psi_{n-1} - \varphi_{n-1} \parallel \mathrm{d}s \\
&\le M(M_1 + M_2 - \theta^2) \parallel \psi_{n-1} - \varphi_{n-1} \parallel
\end{aligned}$$

And similarly

$$\begin{aligned}
0 &\le \psi_n'(t) - \varphi_n'(t) = (Q_{n-1})\psi'(t) - (Q\varphi_{n-1}(t))' \\
&= \int_0^1 \frac{\partial G(t,s)}{\partial t}(f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - \theta^2\psi_{n-1}(s))\mathrm{d}s \\
&\quad - \int_0^1 \frac{\partial G(t,s)}{\partial t}(f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)) - \theta^2\varphi_{n-1}(s))\mathrm{d}s \\
&= \int_0^1 \frac{\partial G(t,s)}{\partial t}\Big(f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s))
\end{aligned}$$

$$
\begin{aligned}
&+\theta^2(\varphi_{n-1}(s)-\psi_{n-1}(s))\Big)\mathrm{d}s\\
&\leq (M_1+M_2-\theta^2)\int_0^1\frac{\partial G(t,s)}{\partial t}\parallel \psi_{n-1}-\varphi_{n-1}\parallel \mathrm{d}s\\
&\leq m(M_1+M_2-\theta^2)\parallel \psi_{n-1}-\varphi_{n-1}\parallel
\end{aligned}
$$

Hence

$$
\begin{aligned}
\parallel \psi_n-\varphi_n\parallel &\leq \max\{M,m\}(M_1+M_2-\theta^2)\parallel \psi_{n-1}-\varphi_{n-1}\parallel\\
&=\overline{M}(M_1+M_2-\theta^2)\parallel \psi_{n-1}-\varphi_{n-1}\parallel
\end{aligned}
$$

Using the inequality repeatedly, we have

$$\parallel \psi_n-\varphi_n\parallel\leq \Big(\overline{M}(M_1+M_2-\theta^2)\Big)^n \parallel \psi_0-\varphi_0\parallel$$

Noticing that $0<\overline{M}(M_1+M_2-\theta^2)<1$, we have $||\psi_n-\varphi_n||\to 0,\quad (n\to\infty)$.

Since $\psi_n\to\psi^*$, $\varphi_n\to\varphi^*$, there exists a unique $x^*\in\bigcap_{n=1}^{\infty}[\varphi_n,\psi_n]$ such that $\psi_n\to x^*$, $\varphi_n\to x^*$, $(n\to\infty)$. So from Lemma 6, we get

$$\varphi_n\preceq x^*\preceq\psi_n,\quad x^*\in D. \tag{14}$$

By the monotonicity of Q we have

$$\varphi_{n+1}=Q\varphi_n\preceq Qx^*\preceq Q\psi_n=\psi_{n+1}.$$

Let $n\to\infty$, then $x^*\preceq Qx^*\preceq x^*$. So $x^*=Qx^*$.

Consequently, x^* is the unique solution of boundary value problem (1).

For any $x_0\in[\varphi_0,\psi_0]$, by the monotonicity of Q we get

$$\varphi_n\preceq x_n\preceq\psi_n,\quad n=1,2,\cdots. \tag{15}$$

From (14) and (15), we obtain that for any $t\in[0,1]$

$$
\begin{aligned}
&0\leq x_n(t)-\varphi_n(t)\leq\psi_n(t)-\varphi_n(t)\\
&0\leq x^*(t)-\varphi_n(t)\leq\psi_n(t)-\varphi_n(t)\\
&0\leq x_n'(t)-\varphi_n'(t)\leq\psi_n'(t)-\varphi_n'(t)\\
&0\leq (x^*)'(t)-\varphi_n(t)\leq\psi_n'(t)-\varphi_n'(t)
\end{aligned}
$$

Therefore $||x_n-\varphi_n||\leq||\psi_n-\varphi_n||$ and $||\varphi_n-x^*||\leq||\psi_n-\varphi_n||$.
Then

$$
\begin{aligned}
||x_n-x^*|| &\leq ||x_n-\varphi_n||+||\varphi_n-x^*||\leq 2||\psi_n-\varphi_n||\\
&\leq 2\Big(\overline{M}(M_1+M_2-\theta^2)\Big)^n||\psi_0-\varphi_0||.
\end{aligned}
$$

So (11) holds. In the following, we prove (12) and (13) hold. Since

$$
\begin{aligned}
0 &\leq\psi_n(t)-\varphi_n(t)=Q\psi_{n-1}(t)-Q\varphi_{n-1}(t)\\
&=\int_0^1 G(t,s)\Big((f(s,\psi_{n-1}(s),\psi_{n-1}'(s))-f(s,\psi_{n-1}(s),\varphi_{n-1}'(s)))\\
&\quad+(f(s,\psi_{n-1}(s),\varphi_{n-1}'(s))-f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)))
\end{aligned}
$$

$$+\theta^2(\varphi_{n-1}(s)-\psi_{n-1}(s))\Big)\mathrm{d}s$$

$$\leq \int_0^1 G(t,s)\Big(M_1(\varphi'_{n-1}(s))-\psi'_{n-1}(s))+(M_2-\theta^2)(\psi_{n-1}(s))-\varphi_{n-1}(s))\Big)\mathrm{d}s$$

$$\leq M\Big(M_1 \mid \varphi'_{n-1}-\psi'_{n-1}\mid_\infty +(M_2-\theta^2)\mid \psi_{n-1}-\varphi_{n-1}\mid_\infty\Big)$$

That is

$$\mid \psi_n-\varphi_n\mid_\infty \leq M\Big(M_1 \mid \varphi'_{n-1}-\psi'_{n-1}\mid_\infty +(M_2-\theta^2)\mid \psi_{n-1}-\varphi_{n-1}\mid_\infty\Big) \quad (16)$$

And similarly

$$\mid \psi'_n-\varphi'_n\mid_\infty \leq m\Big(M_1 \mid \varphi'_{n-1}-\psi'_{n-1}\mid_\infty +(M_2-\theta^2)\mid \psi_{n-1}-\varphi_{n-1}\mid_\infty\Big) \quad (17)$$

Using the inequalities (16) and (17) repeatedly, (12) and (13) also hold. □

5. Illustration

In this section, we apply the iterative method to an example.

Let $\theta=1$, $\xi=\frac{1}{3}$, $k=\frac{1}{3}<\frac{1}{2}=\cos\frac{\pi}{3}<\cos 1=\cos\theta$, $f(t,u,v)=1+(1+\frac{1}{10}t^2)u+\frac{1}{10}v$. Then $\rho=\cos 1-\frac{1}{3}\cos\frac{2}{3}$ and θ, k , ξ, f satisfy the hypothesis (H_0).

Consider the second-order non-homogeneous linear ordinary differential equation with variable coefficients boundary value problem

$$\begin{cases} -x''(t)=\frac{1}{10}x'(t)+(1+\frac{1}{10}t^2)x(t)+1, & t\in(0,1) \\ x(0)=\frac{1}{3}x(\frac{1}{3}), \quad x'(1)=0 \end{cases} \quad (18)$$

It is difficult to solve the boundary value problem by means of conventional method. In the following, we solve the boundary value problem (18) by means of the method presented in this paper. And the error estimate formula is given.

Let $\varphi_0(t)=\int_0^1 G(t,s)\mathrm{d}s$, $\psi_0(t)=2\int_0^1 G(t,s)\mathrm{d}s$ for any $t\in[0,1]$ where

$$G(t,s)=\frac{1}{\cos 1-\frac{1}{3}\cos\frac{2}{3}}\begin{cases} \sin t\cos(1-s)+\frac{1}{3}\sin(s-t)\cos\frac{2}{3}, & t\leq s\leq\frac{1}{3}, \\ \cos(1-t)\sin s, & s\leq\min\{t,\frac{1}{3}\}, \\ \cos(1-s)(\sin t-\frac{1}{3}\sin(t-\frac{1}{3})), & \max\{t,\frac{1}{3}\}\leq s, \\ \cos(1-t)(\sin s-\frac{1}{3}\sin(s-\frac{1}{3})), & \frac{1}{3}\leq s\leq t. \end{cases}$$

Then by (5), (6), Lemma 2 and Lemma 3 we have

$$\begin{cases} -\varphi''_0(t)=\varphi_0(t)+1\leq 1+\frac{1}{10}\varphi'_0(t)+(1+\frac{1}{10}t^2)\varphi_0(t), & t\in(0,1) \\ \varphi_0(0)=\frac{1}{3}\varphi_0(\frac{1}{3}), \quad \varphi'_0(1)=0 \end{cases}$$

and

$$\begin{cases} -\psi''_0(t)=\psi_0(t)+2\geq 1+\frac{1}{10}\psi'_0(t)+(1+\frac{1}{10}t^2)\psi_0(t), & t\in(0,1) \\ \psi_0(0)=\frac{1}{3}\psi_0(\frac{1}{3}), \quad \psi'_0(1)=0 \end{cases}$$

Hence φ_0, ψ_0 are the lower solution and the upper solution of the boundary value problem (18), respectively, and $\varphi_0\preceq\psi_0$.

It is easy to verify that $M_1 = \frac{1}{10}$, $M_2 = \frac{11}{10}$, $M = \max_{t\in[0,1]} \int_0^1 G(t,s)\mathrm{d}s \approx 1.39515$, $m = \max_{t\in[0,1]} \int_0^1 \frac{\partial G(t,s)}{\partial t}\mathrm{d}s \approx 2.1545$.

Therefore, the boundary value problem (18) satisfies the conditions of Theorem 2. Then the boundary value problem (18) has a unique solution x^* on $[\varphi_0, \psi_0]$ and for any $x_0 \in [\varphi_0, \psi_0]$, iterative sequence

$$x_n(t) = \int_0^1 G(t,s)(f(s, x_{n-1}(s), x'_{n-1}(s)) - x_{n-1}(s))\mathrm{d}s, \quad n = 1, 2, \cdots$$

converge uniformly to x^* on [0,1], and its error estimate is

$$||x_n - x^*|| \le 2\overline{M}\Big(\frac{\overline{M}}{5}\Big)^n, \quad n = 1, 2, \cdots. \tag{19}$$

Or for $n = 1, 2, \cdots$.

$$|x_n - x^*|_\infty \le 2M\Big(\frac{m + M}{10}\Big)^n \tag{20}$$

$$|x'_n - (x^*)'|_\infty \le 2m\Big(\frac{m + M}{10}\Big)^n \tag{21}$$

Take $x_0(t) = \frac{3}{2}\int_0^1 G(t,s)\mathrm{d}s$, then $\varphi_0 \preceq x_0 \preceq \psi_0$, and

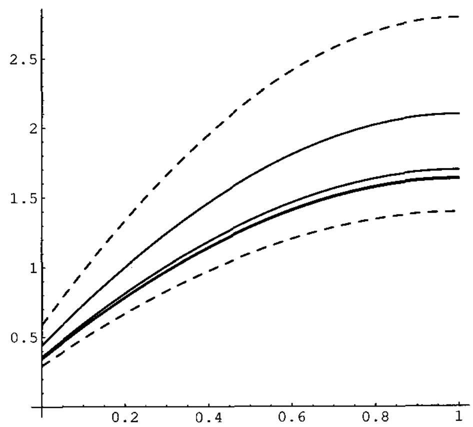
$$\begin{aligned} x_n(t) &= \int_0^1 G(t,s)\mathrm{d}s + \frac{1}{10}\int_0^1 G(t,s)(s^2 x_{n-1}(s) + x'_{n-1}(s))\mathrm{d}s \\ &= \varphi_0(t) + \frac{1}{10}\int_0^1 G(t,s)(s^2 x_{n-1}(s) + x'_{n-1}(s))\mathrm{d}s, \quad n = 1, 2, \cdots \end{aligned}$$

By using computers, we get

$$\begin{aligned} x_1(t) = {} & \tfrac{1}{2160(\cos(\frac{2}{3}) - 3\cos(1))^2}\big[-14040 + 1620t^2 + (8424 - 972t^2)\cos(\tfrac{1}{3}) \\ & + (-1404 + 162t^2)\cos(\tfrac{4}{3}) + (8424 - 972t^2)\cos(\tfrac{5}{3}) + (-12636 + 1458t^2)\cos(2)) \\ & + 27(-107 + 6t + 3t^2)\cos(\tfrac{1}{3} - t) + 27(-107 + 6t + 3t^2)\cos(\tfrac{5}{3} - t) \\ & + (8478 - 486t - 243t^2)\cos(2 - t) + (8478 - 486t - 243t^2)\cos(t) \\ & + (619 + 81t - 54t^3)\sin(\tfrac{1}{3} - t) - 972\sin(\tfrac{2}{3} - t) + 3240\sin(1 - t) - 972\sin(\tfrac{4}{3} - t) \\ & + (-25 + 81t - 54t^3)\sin(\tfrac{5}{3} - t) + (-243t + 162t^3)\sin(2 - t) \\ & + (1782 + 243t - 162t^3)\sin(t) - 324\sin(\tfrac{1}{3} + t) + 1944\sin(\tfrac{2}{3} + t) - 2916\sin(1 + t)\big] \end{aligned}$$

$$\begin{aligned} x_2(t) = {} & \tfrac{1}{139968000(\cos(\frac{2}{3}) - 3\cos(1))^3}\Big((510183360 + 9447840t - 97627680t^2 + 4723920t^4)\cos(\tfrac{1}{3}) \\ & + (-3231161280 - 59836320t + 618308640t^2 - 29918160t^4)\cos(\tfrac{2}{3}) \\ & + (5612016960 + 103926240t - 1073904480t^2 + 51963120t^4)\cos(1) \\ & + (-1530550080 - 28343520t + 292883040t^2 - 14171760t^4)\cos(\tfrac{4}{3}) \\ & + (-56687040 - 1049760t + 10847520t^2 - 524880t^4)\cos(2) \\ & + (510183360 + 9447840t - 97627680t^2 + 4723920t^4)\cos(\tfrac{7}{3}) \\ & + (-1530550080 - 28343520t + 292883040t^2 - 14171760t^4)\cos(\tfrac{8}{3}) \\ & + (1530550080 + 28343520t - 292883040t^2 + 14171760t^4)\cos(3) \\ & - 3240(275 - 3078t + 2052t^3)\cos(\tfrac{1}{3} - t) \\ & -30(-23241575 + 1024407t + 561330t^2 - 91962t^3 - 41553t^4 + 2916t^6)\cos(\tfrac{2}{3} - t) \\ & +(-2306293680 + 96577920t + 55724760t^2 - 5832000t^3 - 4155300t^4 + 291600t^6)\cos(1 - t) \end{aligned}$$

$$+(699688590-30066390t-16839900t^2+2314980t^3+1246590t^4-87480t^6)\cos(\tfrac{4}{3}-t)$$
$$+(-19174320+4986360t-3324240t^3)\cos(\tfrac{5}{3}-t)$$
$$+(32484240-8660520t+5773680t^3)\cos(2-t)$$
$$+(-125473570+7021890t+2832165t^2-1692360t^3-207765t^4+14580t^6)\cos(\tfrac{7}{3}-t)$$
$$+(690820710-27714150t-16839900t^2+746820t^3+1246590t^4-87480t^6)\cos(\tfrac{8}{3}-t)$$
$$+(-1028006640+41203080t+25030215t^2-1180980t^3-1869885t^4+131220t^6)\cos(3-t)$$
$$+(729000-17321040t+11547360t^3)\cos(t)$$
$$+(-114900370+9905490t+2832165t^2-3614760t^3-207765t^4+14580t^6)\cos(\tfrac{1}{3}+t)$$
$$+(679790130-30722490t-16839900t^2+2752380t^3+1246590t^4-87480t^6)\cos(\tfrac{2}{3}+t)$$
$$+(-1011166740+45795780t+25030215t^2-4242780t^3-1869885t^4+131220t^6)\cos(1+t)$$
$$+(-8660520-2361960t+1574640t^3)\cos(\tfrac{4}{3}+t)$$
$$+(25981560+7085880t-4723920t^3)\cos(\tfrac{5}{3}+t)$$
$$+(-25981560-7085880t+4723920t^3)\cos(2+t)$$
$$+(722847240-19945440t-9972720t^2)\sin(\tfrac{1}{3}-t)$$
$$+(-163404837-12499920t+3744630t^2+11751480t^3-524880t^4-419904t^5)\sin(\tfrac{2}{3}-t)$$
$$+(167495040+51350760t-7435800t^2-38899440t^3+1749600t^4+1399680t^5)\sin(1-t)$$
$$+(-109626075-13831560t+3078810t^2+11751480t^3-524880t^4-419904t^5)\sin(\tfrac{4}{3}-t)$$
$$+(364179240-9972720t-4986360t^2)\sin(\tfrac{5}{3}-t)$$
$$+(-628718760+17321040t+8660520t^2)\sin(2-t)$$
$$+(168837777-1588815t-2472930t^2-1975590t^3+87480t^4+69984t^5)\sin(\tfrac{7}{3}-t)$$
$$+(2788209-18536040t+726570t^2+11751480t^3-524880t^4-419904t^5)\sin(\tfrac{8}{3}-t)$$
$$+(27392175t-1180980t^2-17474130t^3+787320t^4+629856t^5)\sin(3-t)$$
$$+(1259274600-34642080t-17321040t^2)\sin(t)$$
$$+(-363155805+7356015t+5356530t^2+1975590t^3-87480t^4-69984t^5)\sin(\tfrac{1}{3}+t)$$
$$+(104183361+12519360t-3734910t^2-11751480t^3+524880t^4+419904t^5)\sin(\tfrac{2}{3}+t)$$
$$+(-169479378-18206775t+5773680t^2+17474130t^3-787320t^4-629856t^5)\sin(1+t)$$
$$+(171635760-4723920t-2361960t^2)\sin(\tfrac{4}{3}+t)$$
$$+(-514907280+14171760t+7085880t^2)\sin(\tfrac{5}{3}+t)$$
$$+(514907280-14171760t-7085880t^2)\sin(2+t)\Big)$$



Since $x_3(t)$ is too long, we do not give $x_3(t)$.

If $n = 3$, by (19) we have absolute value error $||x_3 - x^*|| \leq 0.264004$, and by (20) and(21) we have absolute value error $|x_3 - x^*|_\infty \leq 0.1107$, $|x_3' - (x^*)'|_\infty \leq 0.159918$.

If $n = 10$, then absolute value error $||x_{10} - x^*|| \leq 0.000456485$, $|x_{10} - x^*|_\infty \leq 0.0000594246$, $|x_{10}' - (x^*)'|_\infty \leq 0.0000858454$.

Practically, we adopt (19) when n is bigger. In Fig. 1, we give the solving process. The two broken lines denote the upper solution ψ_0 (above) and the lower solution φ_0 (below). From up to down, four real lines denote x_0, x_1, x_2, and x_3 respectively, where x_3 and x_2 greatly approach to each other. The figure shows the approximative process of $\{x_n\}$, which is satisfying.

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Xiping Liu is a professor in University of Shanghai for Science and Technology. His research interests focus on the theory and application of ordingary differential equations and functional differential equations.

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, P.R.China

e-mail:xipingliu@163.com

Mei Jia received her Master of Science at Beijing Institute of Technology under the direction of Prof. Weigao Ge. She is an associate professor in University of Shanghai for Science and Technology. Her research interests focus on the theory and application of ordingary differential equations and functional differential equations.

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, P.R.China

e-mail: jiamei-usst@163.com