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APPROXIMATION BY QUASI-INTERPOLATORY COMPACTLY SUPPORTED BIORTHOGONAL WAVELET SYSTEMS

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ABSTRACT. A family of quasi-interpolatory wavelet system was introduced in [10], extending and unifing the biorthogonal Coiffman wavelet system. The corresponding refinable functions and wavelets have vanishing moment of a certain order (say, L), which is a key property for data representation and approximation . One of main advantages of this wavelet systems is that we can get optimal smoothness in the sense of smoothing factors in the scaling filters. In this paper, we first discuss the biorthogonality condition of the quisi-interpolatory wavelet system. Then, we study the properties of the scaling and wavelet filters, related to the polynomial reproduction and the vanishing moment respectively, which in fact determines the approximation orders of biorthogonal projections. In addition, we discuss the approximation orders of the wavelet projections.

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1. Introduction

The discrete wavelet transform is a power tool for signal processing applications of many types of signals like seismic shaking, human speech, and music. There are two main reasons for the success of this wavelet transform. First, it is decomposed and reconstructed by the so-called Mallat's Fast Wavelet Transform [11]. Second, a good part of its success is due to the vanishing moment property which leads to sparse representations for signals. The construction of classical wavelets is now well-understood due to the pioneer works such as [1, 5, 6]. Many

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properties, such as symmetry (or antisymmetry), vanishing moments, regularity and short support, are required in a practical use for application areas. It has been well-known that orthogonality and symmetry are conflicting properties for the design of compactly supported wavelets [6]. After the multiresolution analysis had been developed [11], most of useful wavelet functions are derived from refinable functions; we say that ϕ is a refinable (or scaling) function if it satisfies the *refinement equation*

$$\phi = \sum_{k\in\mathbb{Z}} a_k \phi(2\cdot -k), \tag{1.1}$$

where $\mathbf{a} := \{a_k : k \in \mathbb{Z}\}$ is usually called *the mask* for ϕ. On the other hand, in the context subdivsion, the sequence $\mathbf{a}$ in (1.1) is used as a rule in refinement process of given control point to make a smooth curve. It can be written in Fourier domain as

$$\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2) \tag{1.2}$$

with

$$m_0(\xi) := \sum_{n\in\mathbb{Z}} a_n e^{-in\xi}/2. \tag{1.3}$$

In signal and image processing, m_0 is called low-pass filter or scaling filter. The function ϕ can also be reinterpreted as the basic limit function of a subdivision scheme, which is a powerful tool for curve and surface design in the computer aided geometric design. Recently, there has been introduced a biorthogonal wavelet systems based on quasi-interpolatory refinable function [10], which means that it reproduces a certain polynomial space Π_L where N is an even positive integer. The associated Laurent polynomial $a(z)$ is of the form: putting $y = \sin^2 \xi/2$,

$$m_0(\xi) = \frac{1}{2}(1-y)^N \left[2\sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n + \omega 2^{4N}(-1)^N y^N \right]. \tag{1.4}$$

The primary goal of this paper is to characterize this symbol $a(z)$. We study the properties of this Laurent polynomial and the corresponding wavelet filters, related to the polynomial reproducing property and the vanishing moment property respectively. Letting $\tilde{a}(z)$ be the dual Laurent polynomial of $a(z)$, we especially relates the vanishing moment of the pair of the dual masks $(\mathbf{a}, \tilde{\mathbf{a}})$, the pair of refinable functions $(\phi, \tilde{\phi})$ and the corresponding biorthogonal wavelet functions (say $(\psi, \tilde{\psi})$).

2. Quasi-interpolatory subdivision schemes

2.1. Subdivision scheme. In this section, we briefly introduce the relation between subdivision and refinable function. Let $f^0 = \left\{ f_n^0 \in \mathbb{R} : n \in \mathbb{Z} \right\}$ be the initially given values. A subdivision scheme defines recursively new discrete values $f^k = \left\{ f_n^k \in \mathbb{R} : n \in \mathbb{Z} \right\}$ on finer levels by linear sums of existing values as follows:

$$f_j^{k+1} = \sum_{n\in\mathbb{Z}} a_{j-2n} f_n^k, \quad k \in \mathbb{Z}_+, \tag{2.1}$$

where the sequence $\mathbf{a} = \{a_n : n \in \mathbb{Z}\}$ is termed the *mask* of the given subdivision. We denote the rule at each level by S and have the formal relation

$$f^k = S^k f^0. \tag{2.2}$$

Letting $\delta^0 := \{\delta_{n,0} : n \in \mathbb{Z}\}$, the function

$$\phi := S^\infty \delta^0 \in C^\nu(\mathbb{R})$$

is called the *basic limit function* of S and it satisfies the *refinement equation* in (1.1) [9]. To simplify the following presentation of a refinable function (or (the corresponding subdivision) and its analysis, it is convenient to introduce the Laurent polynomial defined by

$$a(z) := \sum_{n\in\mathbb{Z}} a_n z^n, \quad z \in \mathbb{C} \setminus \{0\}. \tag{2.3}$$

The Laurent polynomial $a(z)$ is also called the symbol of its corresponding refinable function ϕ.

2.2. The mask of quasi-interpolatory subdivision scheme. A family of our refinement masks is obtained by the requirement of reproducing polynomials in $\Pi_{<L}$, $L = 2N$. We briefly review its construction process; for the details see [2]. First, for the construction of the even and odd masks, we use the stencil of $L = 2N$ points to reproduce polynomials of degree $< 2N$. Let β_ℓ, $\ell = 1, \ldots, L$, be a basis of $\Pi_{<L}$. Then, putting $\tau_j = \delta_{j,1}$, the even and odd masks ($j = 0, 1$, respectively)

$$\mathbf{a}_j := \left\{ a_{j-2n} : n = -N + \tau_j, \ldots, N \}, \quad j \in \{0, 1\right\}$$

are obtained by solving the linear system which can be written in the matrix form

$$\mathbf{a}_j = \mathbf{P}^{-1} \mathbf{r}_j, \tag{2.4}$$

where

$$\mathbf{M}(n, \ell) = \beta_\ell(n), \ \mathbf{r}_j(\ell) = \beta_\ell(2^{-1}\tau_j), \quad j \in \{0, 1\}.$$

If $j = 0$, this is an underdetermined system of $L+1$ unknowns in L equations, and hence there is one degree of freedom which will be used as a tension parameter ω. Here and in the sequel, for convenience, we put $\omega := a_{2N}$.

3. Compactly supported biorthogonal wavelets

3.1. Biorthogonal wavelet systems. The set $\left\{\phi(\cdot - k) : n \in \mathbb{Z}\right\}$ forms a Riesz basis if and only if there exist constants $A, B > 0$ such that

$$A \le \sum_{n\in\mathbb{Z}} |\phi(\xi + 2\pi n)|^2 \le B, \tag{3.1}$$

We say that the function $\phi \in L_2(\mathbb{R})$ is *stable* if (3.1) is satisfied. If the integer translates of ϕ are linearly independent, the stability of ϕ_j is immediate. Let $\phi \in L_2(\mathbb{R})$ be a stable refinable function with the symbol $a(z)$ such that $a(-1) = 0$ and $a(1) = 2$. The first step for the construction of biorthogonal wavelet systems is to find a refinable function $\tilde{\phi} \in L_2(\mathbb{R})$ such that

$$\left\langle \phi, \tilde{\phi}(\cdot - \ell) \right\rangle = \delta_{0,\ell}, \quad \ell \in \mathbb{Z}. \tag{3.2}$$

A necessary condition for ϕ and $\tilde{\phi}$ to satisfy the duality condition (3.2) is

$$\overline{a}\tilde{a} + \overline{a(-\cdot)}\tilde{a}(-\cdot) = 4. \tag{3.3}$$

If $\tilde{\phi}$ is stable and satisfies the condition (3.2), we call $\tilde{\phi}$ the dual refinable function of ϕ (or just dual of ϕ). Let $\tilde{a}(z)$ be the symbol of $\tilde{\phi}$ such that $\tilde{a}(0) = 2$ and $\tilde{a}(-1) = 0$. Then, the refinement functions ϕ and $\tilde{\phi}$ are defined respectively by

$$\hat{\tilde{\phi}}(\xi) = \prod_{j=1}^{\infty} \tilde{m}_0(\xi/2^j). \tag{3.4}$$

where

$$\tilde{m}_0(\xi) = \tilde{a}(e^{-i\xi})/2.$$

These infinite product in (3.4) converge absolutely and uniformly on compact sets and is the Fourier transforms of compactly supported $\tilde{\phi}$ with their support widths given by the filter lengths [1, 7]. Given a pair of dual refinable functions ϕ and $\tilde{\phi}$ with their associated filters $m_0(\xi)$ and $\tilde{m}_0(\xi)$, the dual wavelet functions ψ and $\tilde{\psi}$ are defined via the relations

$$\hat{\psi}(\xi) = m_1(\xi/2)\hat{\phi}(\xi/2), \quad \hat{\tilde{\psi}}(\xi) = \tilde{m}_1(\xi/2)\hat{\tilde{\phi}}(\xi/2), \tag{3.5}$$

where

$$m_1(\xi) = e^{-i\xi}\overline{\tilde{m}_0(\xi + \pi)}, \quad \tilde{m}_1(\xi) = e^{-i\xi}\overline{m_0(\xi + \pi)}. \tag{3.6}$$

We will give a sufficient condition for ψ and $\tilde{\psi}$ to be biorthognal wavelets associated with ϕ and $\tilde{\phi}$.

Proposition 3.1. *Let ϕ and $\tilde{\phi}$ be refinable functions whose symbols $a(z)$ and $\tilde{a}(z)$ are respectively of the form*

$$a(z) = \left(\frac{1+z}{2}\right)^{\ell} b(z), \ \tilde{a}(z) = \left(\frac{1+z}{2}\right)^{\tilde{\ell}} \tilde{b}(z) \tag{3.7}$$

for some $\ell, \tilde{\ell} \in \mathbb{N}$, where $b(z)$ and $\tilde{b}(z)$ are Laurent polynomials such that $b(1) = \tilde{b}(1) = 2$. Let

$$B_k = \max_{\xi} \left| \prod_{j=0}^{k-1} F(\xi) \right|^{1/k}, \quad \tilde{B}_{\tilde{k}} = \max_{\xi} \left| \prod_{j=0}^{\tilde{k}-1} \tilde{F}(\xi) \right|^{1/k}. \tag{3.8}$$

where $F(\xi); = \frac{1}{2} b(e^{-i\xi})$ and $\tilde{F}(\xi) := \frac{1}{2}\tilde{b}(e^{-i\xi})$. Suppose that $B_k \tilde{B}_{\tilde{k}} < 2^{\ell+\tilde{\ell}-1}$ for some integers $k, \tilde{k} > 0$. Then if $\phi, \tilde{\phi} \in L_2(\mathbb{R})$, $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ is a biorthognal wavelet system.

Proof. Define u_n and $\tilde{u}_n$ by

$$u_n(w) := \left[\prod_{j=1}^{n} m_0(2^{-j} w) \right] \chi_{[-\pi,\pi]}(2^{-n} w),$$

Similarly, define $\tilde{u}_n$ with $\tilde{m}_0$. Then u_n and $\tilde{u}_n$ converge pointwise to $\hat{\phi}$ and $\hat{\tilde{\phi}}$ respectively. Using the similar technique in Proposition 4.9 in [1] and applying (3.3), it is easy to see the relation

$$\begin{aligned} \int_{-\infty}^{\infty} u_n(w)\overline{\tilde{u}}_n(w) e^{i\ell w} dw &= \int_0^{2^{n+1}\pi} \left(\prod_{j=1}^{n} m_0(2^{-j}w) \overline{\tilde{m}}_0(2^{-j}w) \right) e^{i\ell w} dw \\ &= \int_{-\infty}^{\infty} u_{n-1}(w) \overline{\tilde{u}}_{n-1}(w) e^{i\ell w} dw. \end{aligned}$$

Repeating this process yields the identity

$$\int_{-\infty}^{\infty} u_n(w)\overline{\tilde{u}}_n(w) e^{i\ell w} dw = \int_{-\infty}^{\infty} u_1(w)\overline{\tilde{u}}_1(w) e^{i\ell w} dw = 2\pi \delta_{0,\ell}. \tag{3.9}$$

Hence, it suffices to show that $u_n(\cdot)\overline{\tilde{u}}_n(\cdot)$ converges to $\hat{\phi}(\cdot)\overline{\hat{\tilde{\phi}}}(\cdot)$ in the sense of L^1-norm in order to prove $\int_{-\infty}^{\infty} \phi(x)\overline{\tilde{\phi}}(x-\ell)dx = \delta_{0\ell}$. Since

$$\prod_{j=1}^{n} \left| \frac{1 + e^{-i2^{-j}w}}{2} \right| = \prod_{j=1}^{n} \left| \frac{\sin(2^{-j}w)}{2\sin(2^{-j-1}w)} \right| = \left| \frac{\sin(2^{-1}w)}{2^n \sin(2^{-n-1}w)} \right|,$$

we have

$$|u_n(w)| = \left| \frac{\sin(2^{-1}w)}{2^n \sin(2^{-n-1}w)} \right|^L \prod_{j=1}^{n} \left| F(2^{-j}w) \right| \chi_{[-\pi,\pi]}(2^{-n}w), \tag{3.10}$$

Similar relation we obtain for $|\tilde{u}_n(w)|$ by using $\tilde{F}(2^{-j}w)$. Since $|\sin w| \geq \frac{2}{\pi}|w|$ for $|w| \leq \pi/2$, it is easy to see that

$$\left|\sin(2^{-n-1}w)\right|^{-1} \chi_{[-\pi,\pi]}(2^{-n}w) \leq \frac{\pi}{2} 2^{n+1} |w|^{-1}.$$

Therefore, we arrive at the relation

$$\left|\frac{\sin(2^{-1}w)}{2^n\sin(2^{-n-1}w)}\right|\chi_{[-\pi,\pi]}(2^{-n}w)\le\frac{\pi}{2}\left|\frac{\sin(2^{-1}w)}{2^{-1}w}\right|\le C(1+|w|)^{-1}. \tag{3.11}$$

We now compute an upper bound for $\prod_{j=1}^{n}F(2^{-j}w)$. At $w=0$, we have $m_0(0)=1$; so $F(0)=1$. Since F is a trigonometric polynomial, there exists C_1 such that $|F(w)|\le 1+C_1|w|$ for $|w|\le 1$. Consequently, we have

$$\left|\prod_{j=1}^{n}F(2^{-j}w)\right|\le\prod_{j=1}^{n}(1+C_12^{-j}|w|)\le\prod_{j=1}^{\infty}e^{C_12^{-j}|w|}\le e^{C_1}. \tag{3.12}$$

If $|w|>1$, then, for the given k in (3.8), there exists $\ell_0>0$ such that $2^{k\ell_0}\le|w|<2^{k(\ell_0+1)}$. Write $n=kn'+q$ with $0\le q<k$, and assume without loss of generality that $\ell_0<n'$. Note that $\sup_\zeta|F(\zeta)|\ge 1$, since $F(0)=1$. Then, letting $G(w)=F(2^{-1}w)F(2^{-2}w)\cdots F(2^{-k}w)$, we obtain

$$\left|\prod_{j=1}^{n}F(2^{-j}w)\right|\le\left[\sup_\zeta|F(\zeta)|\right]^{k-1}\prod_{j=0}^{\ell_0}\left|G(2^{-kj}w)\right|\prod_{j'=\ell_0+1}^{n'}\left|G(2^{-k\ell'}w)\right|.$$

Since $|2^{-(\ell_0+1)k}w|\le 1$, applying the same argument in (3.12) yields

$$\prod_{j=\ell_0+1}^{n'}\left|G(2^{-kj}w)\right|=\prod_{j=0}^{n'-\ell_0-1}\left|G(2^{-kj}2^{-(\ell_0+1)k}w)\right|\le e^{C_1}. \tag{3.13}$$

Moreover, invoking the definition of B_k in (3.8), we have

$$\begin{aligned}\prod_{j=0}^{\ell_0}\left|G(2^{-kj}w)\right| &\le B_k^{k(\ell_0+1)}\le B_k^{k+\log|w|/\log 2}\\ &\le C_2(1+|w|)^{\log B_k/\log 2}.\end{aligned} \tag{3.14}$$

Hence, we arrive at the bound

$$\left|\prod_{j=1}^{n}F(2^{-j}w)\right|\le C_3(1+|w|)^{\log B_k/\log 2}, \tag{3.15}$$

where the constant c is independent of n. This together with (3.11) leads to the estimate of u_n in (3.10) as follows:

$$|u_n(w)|\le D(1+|w|)^{-L+\log B_k/\log 2}.$$

Next, with $\tilde{B}_{\tilde{k}}$ in (3.8), the estimate of $|\tilde{u}_n(w)|$ can be done similarly :

$$|\tilde{u}_n(w)|\le\tilde{D}(1+|w|)^{-L+\log\tilde{B}_{\tilde{k}}/\log 2}.$$

Thus, since $B_k\tilde{B}_{\tilde{k}} < 2^{L+\tilde{L}-1}$ by assumption, it is obvious that

$$\sup_n |u_n(w)\tilde{u}_n(w)| \le E(1+|w|)^{-L-\tilde{L}+\log(B_k\tilde{B}_{\tilde{k}})/\log 2} \in L^1(\mathbb{R}).$$

Since $u_n\tilde{u}_n \to \hat{\phi}\hat{\tilde{\phi}}$ pointwise as n tends to ∞, by the Lebesgue dominated convergence theorem, we have the convergence property $\|u_n\tilde{u}_n - \hat{\phi}\hat{\tilde{\phi}}\|_{L_1(\mathbb{R})} \to 0$ as $n \to \infty$. Therefore, from (3.9) we get the required result. □

4. Approximation order and vanishing moment

The following theorem treats the relations between dual (refinable, wavelet) functions and dual symbols. Some of them may be already well-known in the literature. But, we clarify all the relations together. For this, we use the notation.

Theorem 4.1. *Let* $\left\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\right\}$ *be a biorthognal wavelet system as in the previous section and their symbols are of the form*

$$a(z) = (1+z)^L b(z), \quad \tilde{a}(z) = (1+z)^L \tilde{b}(z),$$

where $b(z)$ *and* $\tilde{b}(z)$ *are Laurent polynomials such that* $b(-1) \neq 0$ *and* $\tilde{b}(-1) \neq 0$. *Then, the following conditions are equivalent:*

(1) *For any* $p \in \Pi_{<L}$, $\sum_{n\in\mathbb{Z}} a_{j-2n}p(n) = p(j/2)$ *with* $j = 0, 1$.

(2) *For any* $p \in \Pi_{<L}$, $\sum_{n\in\mathbb{Z}} \tilde{a}_{j-2n}p(n) = p(j/2)$ *with* $j = 0, 1$.

(3) *The refinable function* ϕ *and the wavelet* ψ *have the vanishing moments of order* L:

$$\begin{aligned} &\int_{\mathbb{R}} \phi(t)t^\ell dt = \delta_{0,\ell} \\ &\int_{\mathbb{R}} \psi(t)t^\ell dt = 0, \quad \forall \ell = 0, \ldots, L-1. \end{aligned} \tag{4.1}$$

(4) *The dual functions* $\tilde{\phi}$ *and* $\tilde{\psi}$ *have the vanishing moments of order* L:

$$\begin{aligned} &\int_{\mathbb{R}} \tilde{\phi}(t)t^\ell dt = \delta_{0,\ell} \quad \text{and} \\ &\int_{\mathbb{R}} \tilde{\psi}(t)t^\ell dt = 0, \quad \forall \ell = 0, \ldots, L-1. \end{aligned} \tag{4.2}$$

Proof. The proof of this theorem will be done in the following way: (1) ⇒ (4) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (4): Since ϕ is the basic limit function of the subdivision scheme with the mask $\{a_n\}_{n\in\mathbb{Z}}$, the polynomial reproducing property in (1) is equivalent to the

case of the refinable function ϕ. Hence, by some elementary calculations, it is immediate that for any $\ell = 0, \dots, L-1$,

$$t^\ell = \sum_{n\in\mathbb{Z}} \phi(t-n)n^\ell.$$

Now, in order to prove the vanishing moment property of the refinable function ϕ, we apply the duality condition (3.2), that is., $\langle\tilde{\phi}, \phi(\cdot - n)\rangle = \delta_{n,0}$ for $n \in \mathbb{Z}$. It yields that

$$\begin{aligned}\int_{\mathbb{R}} \tilde{\phi}(t)t^\ell dt &= \int_{\mathbb{R}} \tilde{\phi}(t) \sum_{n\in\mathbb{Z}} \phi(t-n)n^\ell dt \\ &= \sum_{n\in\mathbb{Z}} n^\ell \int_{\mathbb{R}} \tilde{\phi}(t)\phi(t-n)dt = \delta_{\ell,0}.\end{aligned}$$

for all $\ell = 0, \dots, L-1$. In a simlilar way, using the orthogonality condition $\langle\tilde{\psi}, \phi(\cdot - n)\rangle = 0$, we can show that

$$\begin{aligned}\int_{\mathbb{R}} \tilde{\psi}(t)t^\ell dt &= \sum_{n\in\mathbb{Z}} n^\ell \int_{\mathbb{R}} \tilde{\psi}(t)\phi(t-n)dt \\ &= 0,\end{aligned}$$

for all $\ell = 0, \dots, L-1$.

(4) $\Rightarrow$ (2) Let $\tilde{a}(z)$ be the Laurent polynomial associated to the dual refinable function $\tilde{\phi}$, i.e.,

$$\tilde{a}(z) = \sum_{n\in\mathbb{Z}} \tilde{a}_n z^n.$$

By applying the refinement equation $\tilde{\phi} = \sum_{n\in\mathbb{Z}} \tilde{a}_n \tilde{\phi}(\cdot - n)$ to (4.2), we can easily obtain that

$$\sum_{n\in\mathbb{Z}} \tilde{a}_n n^\ell = \sum_{n\in\mathbb{Z}} \tilde{a}_n \int_{\mathbb{R}} \tilde{\phi}(2t-n)t^\ell dt = 2\delta_{\ell,0}$$

with $\ell = 0, \dots, L-1$. It implies the identity

$$\tilde{a}^{(\ell)}(1) = 2\delta_{\ell,0}, \quad \ell = 0, \dots, L-1. \tag{4.3}$$

By construction, $\tilde{a}^{(\ell)}(-1) = 0$ for any $\ell = 0, \dots, L-1$. Thus combining it with (4.3), the statement (2) is proved. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) : The proof can be done similarly as (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2). □

A biorthogonal wavelet system with compact support is called a biorthogonal Coifman wavelet system for degree L if the synthesis refinable function ϕ and the dual wavelets ψ and $\tilde{\psi}$ have the vanishing moments L, that is, for all $n = 0, \dots, L$,

$$\hat{\phi}^{(n)} = \delta_{0,n}, \text{and} \quad \hat{\psi}^{(n)} = \hat{\tilde{\psi}} = 0. \tag{4.4}$$

Remark 4.2. It is easy to see that the vanishing moment of order L of ϕ (and $\tilde{\phi}$) in (4.1) implies the polynomial reproducing property of degree $< L$, i.e., in the sense that

$$\int_{\mathbb{R}} \phi(\cdot - t)p(t)dt = p, \quad p \in \Pi_{<L}.$$

The following proposition estimates the size of $|\langle f, \tilde{\psi}_{j,k}\rangle|$. In particular, our approximaion spaces are chosen from the Sobolev space

$$W_p^L(\Omega) := \|f\|_{W_p^L} := \Big\{ f \in L_p(\mathbb{R}) : \|f^{(L)}\|_{L_p(\Omega)} < \infty \Big\}. \tag{4.5}$$

Proposition 4.3. *Let ϕ be the refinable function with its symbol in* (1.4) *and let $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ be a corresponding biorthgonal wavelet system. Assume that $f \in W_2^L(\mathbb{R})$. Then, for any fixed $j \in \mathbb{N}$ and $k \in \mathbb{Z}$,*

$$|\langle f, \tilde{\psi}_{j,k}\rangle| \le c2^{-jL} \left\| f^{(L)} \right\|_{L_2(B_{2^{-j}r}(2^{-j}k))}.$$

Proof. It is clear that for any $j, k \in \mathbb{Z}$,

$$\begin{aligned} 2^{j/2}\langle f, \tilde{\psi}_{j,k}\rangle &= 2^{j/2} \int_{\mathbb{R}} f(t)\tilde{\psi}_{j,k}(t)dt \\ &= \int_{\mathbb{R}} f(2^{-j}t)\tilde{\psi}(t-k)dt. \end{aligned}$$

Let $T_L f$ be the Taylor polynomial of f of degree $L-1$ around $\vartheta = 2^{-j}k$ and $R_L f$ be its remainder. Since ψ has the vanishing moment of order L,

$$\int_{\mathbb{R}} T_L f(2^{-j}t)\psi_{j,k}(t)dt = 0.$$

Then we get

$$\begin{aligned} 2^{j/2}\langle f, \tilde{\psi}_{j,k}\rangle &= \int_{\mathbb{R}} (f(2^{-j}t) - T_L f(2^{-j}t))\tilde{\psi}(t-k)dt \\ &= \int_{\mathbb{R}} R_L f(t)\tilde{\psi}(t-k)dt. \end{aligned}$$

Note that the remainder $R_L f$ is of the form

$$R_L f(t) = 2^{-jL}\frac{(t-k)^L}{(L-1)!}\int_0^1 (1-y)^L f^{(L)}(2^{-j}k + y(t-2^{-j}k))dy.$$

Thus, it follows that

$$|2^{j/2}\langle f, \tilde{\psi}_{j,k}\rangle| \le c2^{-jL} \left\| f^{(L)} \right\|_{L_2(B_{2^{-j}r}(2^{-j}k))} \int_{\mathbb{R}} |(t-k)^L \tilde{\psi}(t-k)|^2 dt.$$

Since $\tilde{\psi}$ is compactly supported, $\int_{\mathbb{R}} |(t-k)^L\tilde{\psi}(t-k)|^2 dt < \infty$, which completes the proof. ∎

Corollary 4.4. *Let ϕ be the refinable function with its symbol in* (1.4) *and let* $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ *be a corresponding biorthgonal wavelet system. Assume that* $f \in W_2^L(\mathbb{R})$. *Then, for any fixed* $j \in \mathbb{N}$ *and* $k \in \mathbb{Z}$,

$$\left(| \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{j,k} \rangle |^2 \right)^{1/2} = O(2^{-jL}).$$

For a pair of biorthogonal wavelets ψ and $\tilde{\psi}$, we define an approximation to the wavelet projection $Q_j f \in W_j := \overline{\text{span}}\{\psi_{j,k} : k \in \mathbb{Z}\}$ by

$$Q_j^* f = \sum_{k \in \mathbb{Z}} f_{j,k}^* \psi_{j,k}$$

where

$$f_{j,k}^* = \sum_{k \mathbb{Z}} (-1)^k a_{1-2k} f(k 2^{-j}).$$

The next theorem treats the accuracy of this approximation.

Theorem 4.5. *Let ϕ be the refinable function with its symbol in* (1.4) *and let* $\left\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\right\}$ *be a corresponding biorthgonal wavelet system. Assume that* $f \in W_\infty^L(\mathbb{R})$. *Then, for any* $j \in \mathbb{N}$ *and* $k \in \mathbb{Z}$,

$$|f_{j,k}^*| \le c 2^{-jL} \left\| f^{(L)} \right\|_{L_\infty(B_{2^{-j}r}(2^{-j}k))}.$$

Proof. Due to Thereom 4.1, we find the relation

$$\sum_{n \in \mathbb{Z}} (-1)^n a_{1-n} n^\alpha = \sum_{n \in \mathbb{Z}} a_{1-2n} (2n)^\alpha - \sum_{n \in \mathbb{Z}} a_{-2n} (2n)^\alpha = 0$$

because each summation on the righthand side of the above identity has the value $(-1)^\alpha$. It implies that

$$\sum_{n \in \mathbb{Z}} (-1)^n a_{1-n} p(n) = 0$$

for any polynomial of degree$< L$. Now, take $T_L f$ as the Taylor polynomial of f of degree $L-1$ around $\vartheta = 2^{-j}k$. Then,

$$\sum_{n \in \mathbb{Z}} (-1)^n a_{1-n} T_L f(k 2^{-j}) = 0.$$

By using the same argument of Proposition 4.3, we get the required result. □

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