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Some Properties and Theorems on Intuitionistic Fuzzy Metric Space

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Abstract

In this paper, we introduce and formulate the definitions of Banach operator type k and Banach operator pair on intuitionistic fuzzy metric space. Thereafter we prove some properties and theorems on intuitionistic fuzzy metric space.

Key words : Intuitionistic fuzzy metric space, f-contraction, Banach operator type k, Banach operator pair, fixed point.

1. Introduction

Park et.al.[4] defined the intuitionistic fuzzy metric space, and we studied many contents on intuitionistic fuzzy metric space. Also, many authors([1],[3],[4] etc) studied some definitions and theories on intuitionistic fuzzy metric space.

In this paper, we first introduce and formulate the definitions of Banach operator type k and Banach operator pair on intuitionistic fuzzy metric space. Thereafter we prove some properties on intuitionistic fuzzy metric space. These results partially improve and generalize [6].

2. Preliminaries and Properties

Throughout this paper, **N** denote the set of all positive integers. Now, we begin with some definitions, properties in intuitionistic fuzzy metric space as following:

Let us recall(see [5]) that a continuous $t-$norm is an operation $* : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (a)$*$ is commutative and associative, (b)$*$ is continuous, (c)$a * 1 = a$ for all $a \in [0,1]$, (d)$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a,b,c,d \in [0,1])$. Also, a continuous $t-$conorm is an operation $\diamond : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (a)$\diamond$ is commutative and associative, (b)$\diamond$ is continuous, (c)$a \diamond 0 = a$ for all $a \in [0,1]$, (d)$a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ $(a,b,c,d \in [0,1])$.

Also, let us recall (see [2]) that the following conditions are satisfied: (a)For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_4 \diamond r_2 \leq r_1$; (b)For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

Definition 2.1. ([1])The $5-$tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions; for all $x, y, z \in X$,

(a)$M(x,y,t) > 0$,
(b)$M(x,y,t) = 1$ if and only if $x = y$,
(c)$M(x,y,t) = M(y,x,t)$,
(d)$M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
(e)$M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous,
(f)$N(x,y,t) > 0$,
(g)$N(x,y,t) = 0$ if and only if $x = y$,
(h)$N(x,y,t) = N(y,x,t)$,
(i)$N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$,
(j)$N(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Note that (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Lemma 2.2. ([3])Let X be an intuitionistic fuzzy metric space. If there exists a number $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,

$$M(x,y,kt) \geq M(x,y,t), \quad N(x,y,kt) \leq N(x,y,t),$$

then $x = y$.

Definition 2.3. Let X be an intuitionistic fuzzy metric space.

(a)A self mapping T on X is said to be f-contraction if there exists a real number $0 < k \leq 1$ such that

$$M(Tx,Ty,kt) \geq M(fx,fy,t),$$
$$N(Tx,Ty,kt) \leq N(fx,fy,t)$$

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for all $x, y \in X$. If $k = 1$, then T is said to be f-nonexpansive.

(b)A mapping T on X is said to be asymptotically f-nonexpansive if there exists a sequence $\{\mu_n\}$ of real numbers with $\mu_n \geq 1$ and $\lim_{n\to\infty} \mu_n = 1$ such that

$$M(T^n x, T^n y, \mu_n t) \geq M(fx, fy, t),$$
$$N(T^n x, T^n y, \mu_n t) \leq N(fx, fy, t)$$

for all $x, y \in X$ and $n = 1, 2, 3, \cdots, \infty$.

(c)T is said to be uniformly asymptotically regular on X if for each $r > 0$, there exists $\mathbf{N}(\epsilon) = \mathbf{N}$ such that

$$M(T^n x, T^{n+1} y, \mu_n t) > 1 - r,$$
$$N(T^n x, T^{n+1} y, \mu_n t) < r$$

for all $n \geq \mathbf{N}$ and $x \in X$.

(d)Two self mappings T and f on X are said to be commuting if $Tfx = fTx$ for all $x \in X$.

Definition 2.4. Let T be a self mapping of an intuitionistic fuzzy metric space X. Then T is called a Banach operator of type k if

$$M(T^2 x, Tx, kt) \geq M(Tx, x, t),$$
$$N(T^2 x, Tx, kt) \leq N(Tx, x, t)$$

for some $k \geq 0$ and for all $x \in X$.

Definition 2.5. Let T and f be two self mappings on intuitionistic fuzzy metric space X. Then (T, f) is a Banach operator pair if any one of the following conditions is satisfied

(a)$T(F(f)) \subseteq F(f)$(the set of fixed points of f),

(b)$fTx = Tx$ for each $x \in F(f)$,

(c)$fTx = Tfx$ for $x \in F(f)$,

(d)$M(Tfx, fx, kt) \geq M(fx, x, t)$, $N(Tfx, fx, kt) \leq N(fx, x, t)$ for some $k \geq 0$.

Proposition 2.6. If T and f are two self mappings of an intuitionistic fuzzy metric space X, then (T, f) is a Banach operator pair if and only if T and f commute on $F(f)$.

Proof. This proposition is satisfied from the Definition 2.5. □

Proposition 2.7. If T and f are two continuous self mappings of an intuitionistic fuzzy metric space X, then (T, f) is a Banach operator pair if and only if for each $\{x_n\} \subset X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = x$, it follows that

$$\lim_{n\to\infty} M(Tfx_n, Tx_n, kt) = 1$$
$$\text{and } \lim_{n\to\infty} N(Tfx_n, Tx_n, kt) = 0$$

or

$$\lim_{n\to\infty} M(Tfx_n, fTx_n, kt) = 1$$
$$\text{and } \lim_{n\to\infty} N(Tfx_n, fTx_n, kt) = 0.$$

Proof. Let T and f be two continuous self mappings of an intuitionistic fuzzy metric space X. If (T, f) is a Banach operator pair, and for each $\{x_n\} \subset X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = x$, then $fTx_n = Tx_n$ from Definition 2.5. Also, By continuity of T, f, Proposition 2.6 and $Tfx_n = fTx_n$,

$$\begin{aligned}
&\lim_{n\to\infty} M(Tfx_n, Tx_n, kt) \\
&= \lim_{n\to\infty} M(Tfx_n, Tfx_n, kt) \\
&\geq \lim_{n\to\infty} M(fx_n, fx_n, t) = 1, \\
&\lim_{n\to\infty} N(Tfx_n, Tx_n, kt) \\
&= \lim_{n\to\infty} N(Tfx_n, Tfx_n, kt) \\
&\leq \lim_{n\to\infty} N(fx_n, fx_n, t) = 0.
\end{aligned}$$

or

$$\begin{aligned}
&\lim_{n\to\infty} M(Tfx_n, fTx_n, kt) \\
&= \lim_{n\to\infty} M(Tfx_n, Tfx_n, kt) \\
&\geq \lim_{n\to\infty} M(fx_n, fx_n, t) = 1, \\
&\lim_{n\to\infty} N(Tfx_n, fTx_n, kt) \\
&= \lim_{n\to\infty} N(Tfx_n, Tfx_n, kt) \\
&\leq \lim_{n\to\infty} N(fx_n, fx_n, t) = 0.
\end{aligned}$$

Conversely, for each $\{x_n\} \subset X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = x$, if

$$\lim_{n\to\infty} M(Tfx_n, fTx_n, kt) = 1$$
$$\text{and } \lim_{n\to\infty} N(Tfx_n, fTx_n, kt) = 0,$$

then from Definition 2.5, (T, f) is a Banach operator pair. □

3. Some Results

Now, we prove some fixed point theorems satisfying some conditions on intuitionistic fuzzy metric space.

Theorem 3.1. Let Y be a nonempty closed subset of an intuitionistic fuzzy metric space X with $t * t \geq t$, $t \diamond t \leq t$ for all $t \in [0, 1]$ and let $f, T : Y \to Y$ be commuting self mappings on $Y - \{q\}$ for some $q \in X$ such that

$T(Y-\{q\}) \subset f(Y)-\{q\}$. Suppose that there exists $k \in (0,1)$ such that

$$\begin{aligned} &M(Tx,Ty,kt)\\ &\geq \min\{M(fx,fy,t),M(fx,Tx,t),M(fy,Ty,t),\\ &\qquad M(fx,Ty,t)*M(fy,Tx,t)\}, \qquad (1)\\ &N(Tx,Ty,kt)\\ &\leq \max\{N(fx,fy,t),N(fx,Tx,t),N(fy,Ty,t),\\ &\qquad N(fx,Ty,t)\diamond N(fy,Tx,t)\} \end{aligned}$$

for all $x,y \in Y$. Further, if T is continuous and $\overline{(T(Y-\{q\})}$ is complete, then $F(f)\cap F(T)$ has a unique point in Y.

Proof. Let $x_0 \in Y$. Since $T(Y-\{q\}) \subset f(Y)-\{q\}$, define a sequence $\{x_n\} \subset Y$ as $fx_n = Tx_{n-1}$ for each $n \geq 1$. Then we have

$$\begin{aligned} &M(fx_{n+1},fx_n,kt)\\ &= M(Tx_n,Tx_{n+1},kt)\\ &\geq \min\{M(fx_n,fx_{n-1},t),M(fx_n,Tx_n,t),\\ &\qquad M(fx_n,Tx_{n-1},t)*M(fx_{n-1},Tx_n,t),\\ &\qquad M(fx_{n-1},Tx_{n-1},t)\}\\ &\geq \min\{M(fx_n,fx_{n-1},t),M(fx_n,fx_{n+1},t),\\ &\qquad M(fx_{n-1},fx_n,t)*M(fx_n,fx_{n+1},t)\}\\ &\geq M(fx_n,fx_{n-1},t),\\ &N(fx_{n+1},fx_n,kt)\\ &= N(Tx_n,Tx_{n+1},kt)\\ &\leq \max\{N(fx_n,fx_{n-1},t),N(fx_n,Tx_n,t),\\ &\qquad N(fx_n,Tx_{n-1},t)\diamond N(fx_{n-1},Tx_n,t),\\ &\qquad N(fx_{n-1},Tx_{n-1},t)\}\\ &\leq \max\{N(fx_n,fx_{n-1},t),N(fx_n,fx_{n+1},t),\\ &\qquad N(fx_{n-1},fx_n,t)\diamond N(fx_n,fx_{n+1},t)\}\\ &\leq N(fx_n,fx_{n-1},t) \end{aligned}$$

for all $n \in \mathbf{N}$. Therefore $\{x_n\}$ is a Cauchy sequence in Y. So, $\{Tx_n\}$ is a Cauchy sequence in Y and since $\overline{T(Y-\{q\})}$ is complete, $\lim_{n\to\infty} Tx_n = y \in Y$ and consequently, $\lim_{n\to\infty} fx_n = y$. Since T and f are commuting on $Y-\{q\}$, $Tfx_n = fTx_n$. As T is continuous, $\lim_{n\to\infty} fTx_n = \lim_{n\to\infty} Tfx_n = Ty$. Now

$$\begin{aligned} &M(Tx_n,TTx_n,kt)\\ &\geq \min\{M(fx_n,fTx_n,t),M(fx_n,Tx_n,t),\\ &\qquad M(fx_n,TTx_n,t)*M(fTx_n,Tx_n,t),\\ &\qquad M(fTx_n,TTx_n,t)\},\\ &N(Tx_n,TTx_n,kt)\\ &\leq \max\{N(fx_n,fTx_n,t),N(fx_n,Tx_n,t),\\ &\qquad N(fx_n,TTx_n,t)\diamond N(fTx_n,Tx_n,t),\\ &\qquad N(fTx_n,TTx_n,t)\}. \end{aligned}$$

Taking the limit as $n\to\infty$ in above equation, we obtain

$$\begin{aligned} &M(y,Ty,kt)\\ &\geq \min\{M(y,Ty,t),M(y,y,t),M(Ty,Ty,t),\\ &\qquad M(y,Ty,t)*M(Ty,y,t)\},\\ &N(y,Ty,kt)\\ &\leq \max\{N(y,Ty,t),N(y,y,t),N(Ty,Ty,t),\\ &\qquad N(y,Ty,t)\diamond N(Ty,y,t)\}. \end{aligned}$$

Thus since $a*a \geq a$ and $a\diamond a \leq a$ for all $a \in [0,1]$, $M(y,Ty,kt) \geq M(y,Ty,t)$, $N(y,Ty,kt) \leq N(y,Ty,t)$. Thus by Lemma 2.2, $y = Ty \in T(Y)$ and $T(Y) \subset f(Y)$, there exists $z \in Y$ such that $y = Ty = fz$. Now, we prove that $Tz = fz$. Since

$$\begin{aligned} &M(TTx_n,Tz,kt)\\ &\geq \min\{M(fTx_n,fz,t),M(fTx_n,TTx_n,t),\\ &\qquad M(fTx_n,Tz,t)*M(TTx_n,fz,t),\\ &\qquad M(fz,Tz,t)\},\\ &N(TTx_n,Tz,kt)\\ &\leq \max\{N(fTx_n,fz,t),N(fTx_n,TTx_n,t),\\ &\qquad N(fTx_n,Tz,t)\diamond N(TTx_n,fz,t),\\ &\qquad N(fz,Tz,t)\}. \end{aligned}$$

Taking the limit as $n\to\infty$ in above equation, we obtain

$$\begin{aligned} &M(Ty,Tz,kt)\\ &\geq \min\{M(Ty,fz,t),M(Ty,Tz,t),M(fz,Tz,t),\\ &\qquad M(Ty,Tz,t)*M(fz,Ty,t)\},\\ &N(Ty,Tz,kt)\\ &\leq \max\{N(Ty,fz,t),N(Ty,Tz,t),N(fz,Tz,t),\\ &\qquad N(Ty,Tz,t)\diamond N(fz,Ty,t)\}. \end{aligned}$$

Since $y = Ty$ and $a*a \geq a$ and $a\diamond a \leq a$ for all $a \in [0,1]$, therefore

$$M(y,Tz,kt) \geq M(y,Tz,t), \quad N(y,Tz,kt) \leq N(y,Tz,t).$$

From Lemma 2.2, $y = Tz$. Hence $y = Tz = Ty = fz$. Also, since $Tfz = fTz$, $y = Ty = fy$. That is, y is a unique point of $F(f)\cap F(T)$. ☐

Corollary 3.2. Let T and f be two self mappings of a nonempty closed subset Y of an intuitionistic fuzzy metric space X with $t*t \geq t$, $t\diamond t \leq t$ for all $t \in [0,1]$ such that $\overline{T(Y-\{q\})}$ is complete for some $q \in X$. Suppose that (T,f) is a Banach operator pair on $Y-\{q\}$ satisfying inequality (1) for all $x,y \in Y$ and $k \in [0,1)$. If f is continuous and $F(f)$ is nonempty, then there is a unique common fixed point of T and f.

Proof. Since $F(f)$ is the fixed point set of f, $f(F(f)) = F(f)$. Also, since (T, f) is a Banach operator pair on $Y - \{q\}$, $T(F(f) - \{q\}) \subseteq F(f) - \{q\}$ and $T(F(f)) \subseteq f(F(f))$. Also, $\overline{T(F(f) - \{q\})}$ is complete. Furthermore, since (T, f) satisfies inequality (1) for all $x, y \in Y$ and by Theorem 3.1, T and f have a unique common fixed point z in $F(f)$. □

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