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A note on Jensen type inequality for Choquet integrals

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Abstract

The purpose of this paper is to prove a Jensen type inequality for Choquet integrals with respect to a non-additive measure which was introduced by Choquet [1] and Sugeno [20];

$$\Phi((C)\int f d\mu) \leq (C)\int \Phi(f)d\mu,$$

where f is Choquet integrable, $\Phi : [0,\infty) \longrightarrow [0,\infty)$ is convex, $\Phi(\alpha) \leq \alpha$ for all $\alpha \in [0,\infty)$ and $\mu_f(\alpha) \leq \mu_{\Phi(f)}(\alpha)$ for all $\alpha \in [0,\infty)$. Furthermore, we give some examples assuring both satisfaction and dissatisfaction of Jensen type inequality for the Choquet integral.

Key Words : Choquet integral, non-additive measure, Jensen type inequality

1. Introduction

Many researchers have studied characterizations of Choquet integrals (see [1,3,4,5,7,13,20,23,26]), risk analysis (see [15,16,25]), non-additive measures and integration (see [6,9,10]), mathematical economics (see [8,11,21,22]), and expected utility theory (see [2,12,18,24]). Based on these, to study some charecterizations of Choquet integrals in the sense of real analysis is very important, for examples, Jensen inequality, Hölder inequality and Radon Nikodym theorem, etc.

Non-additive measures and their corresponding Choquet integrals are useful tools which are used in mathematical economics, information theory, and risk analysis. It is well-known that Jensen inequality which is also a common generalization of inequalities of Hölder and Minkowski. My present work aims at studying a Jensen type ineqaulity for the Choquet integral with respect to a non-additive measure under some sufficient conditions " $\Phi : [0,\infty) \longrightarrow [0,\infty)$ is convex, $\Phi(\alpha) \leq \alpha$ for all $\alpha \in [0,\infty)$ and $\mu_f(\alpha) \leq \mu_{\Phi(f)}(\alpha)$ for all $\alpha \in [0,\infty)$" :

$$\Phi((C)\int f d\mu) \leq (C)\int \Phi(f)d\mu,$$

where f is Choquet integrable. We also give some examples assuring both satisfaction and dissatisfaction of Jensen type inequality for the Choquet integral.

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2. Preliminaries and Definitions

In this section, we list some definitions and basic properties of the Choquet integral which is used in the next section.

Definition 2.1. Let X be a set and Ω a σ-algebra of subsets of X. A set function $\mu : X \longrightarrow \Omega$ is said to be non-additive measure (or fuzzy measure) if

(i) $\mu(\emptyset) = 0$ and

(ii) $\mu(A) \leq \mu(B)$, where $A, B \in \Omega$ and $A \subset B$.

We recall that the concept of the Choquet integral with respect to a classical measure was first introduced in capacity theory by Choquet ([1]).

Definition 2.2. (1) The Choquet integral of a measurable function f with respect to a non-additive measure μ is defined by

$$(C)\int f d\mu = \int_0^\infty \mu(\{x \in X | f(x) > r\})dr$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is said to be Choquet integrable if the Choquet integral of f can be defined and its value is finite.

We remark that the Choquet integral with respect to a non-additive measure is often used in information fusion and data mining as an aggregation tool. Now, we introduce the concept of comonotonic between two functions.

Definition 2.3. Let $f, g : X \longrightarrow [0, \infty)$ be measurable functions. f and g are said to be comonotonic, the symbol $f \sim g$ if

$$f(x_1) < f(x_2) \Longrightarrow g(x_1) \leq g(x_2)$$

for all $x_1, x_2 \in X$.

Theorem 2.4. Let $f, g, h : X \longrightarrow [0, \infty)$ be measurable functions. Then we have the followings.

(1) $f \sim f$.

(2) $f \sim g \Longrightarrow g \sim f$.

(3) $f \sim a$ for all $a \in [0, \infty)$.

(4) $f \sim g$ and $f \sim h \Longrightarrow f \sim g + h$.

Theorem 2.5. Let $f, g : X \longrightarrow [0, \infty)$ be measurable functions.

(1) If $f \leq g$, then $(C)\int f d\mu \leq (C)\int g d\mu$.

(2) If $f \sim g$ and $a, b \in [0, \infty)$, then

$$(C)\int (af + bg)d\mu = a(C)\int f d\mu + b(C)\int g d\mu.$$

3. Jensen type inequality for Choquet integrals

In this section we will prove the Jesen type inequality for Choquet integral with respect to a non-additive measure. First, we consider a convex function that will be used in the main theorem as follows.

Definition 3.1. A function Φ defined on $[0, \infty)$ is said to be convex if for each $x, y \in [0, \infty)$ and each λ, $0 \leq \lambda \leq 1$ we have

$$\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y).$$

Now, we introduce the classical inequality theorem in [5].

Theorem 3.2. ([5]) Let Φ be a convex function on $[0, \infty)$ and f an integrable function on $[0, \infty)$. Then

$$\Phi\left(\int f d\mu\right) \leq \int \Phi(f) d\mu,$$

We note that if μ is a non-additive measure, then f is Choquet integrable if and only if $\mu_f(\alpha) = \mu\{x|f(x) > \alpha\}$ is integrable on $[0, \infty)$, that is, $\int_0^\infty \mu_f(\alpha)d\alpha \in [0, \infty)$. Let $\mathbf{F}^\mu(X)$ be the class of all Choquet integrable functions $f : X \longrightarrow [0, \infty)$.

In order to obtain a new Jensen type inequality for the Choquet integral, it is clear that the classical condition "Φ is convex" must be more strong sufficient condition as in the following theorem.

Theorem 3.3. Let (X, Ω, μ) be a non-additive measure space and let $f \in \mathbf{F}^\mu(X)$. If Φ is a convex function on $[0, \infty)$ and $\Phi(\alpha) \leq \alpha$ and $\mu_f(\alpha) \leq \mu_{\Phi f}(\alpha)$ for all $\alpha \in [0, \infty)$, then

$$\Phi\left((C)\int f d\mu\right) \leq (C)\int \Phi(f) d\mu,$$

Proof. By Theorem 3.2 and the hypothesis $\Phi(\alpha) \leq (\alpha)$ and $\mu_f(\alpha) \leq \mu_{\Phi f}(\alpha)$ for all $\alpha \in [0, \infty)$,

$$\Phi\left((C)\int f d\mu\right) d\mu = \Phi\left(\int_0^\infty \mu_f(\alpha)d\alpha\right)$$

$$\leq \int_0^\infty \Phi(\mu_f(\alpha))d\alpha \leq \int_0^\infty \mu_f(\alpha)d\alpha$$

$$\leq \int_0^\infty \mu_{\Phi f}(\alpha)d\alpha = (C)\int \Phi(f)d\mu.$$

Remark 3.4. We can compare the conditions " Φ is a strictly increasing and $\Phi(\alpha) \leq \alpha$ for all $\alpha \in [0, \infty)$" of Jensen type inequality for the Sugeno integral(see [20]) with the conditions " Φ is convex and $\Phi(\alpha) \leq \alpha$ and $\mu_f(\alpha) \leq \mu_{\Phi f}(\alpha)$ for all $\alpha \in [0, \infty)$" of Jensen type inequality for the Choquet integral.

Now, we give an example assuring the satisfaction of Jensen type inequality for the Choquet integral.

Example 3.5. Let m be the Lebesgue measure and $\mu = m^2$. Then clearly, μ is a non-additive measure. If

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } 1 \leq x \leq 5 \\ 0 & \text{if else} \end{cases}$$

and

$$\Phi(r) = \begin{cases} r^2 & \text{if } 0 \leq r \leq 1 \\ 0 & \text{if else} \end{cases}$$

then Φ is convex and $\Phi(r) = r^2 \leq r$ for all $r \in [0, \infty)$. We also can calculate the following integrals: At first, one has

$$(C)\int f d\mu$$
$$= \int_0^\infty \mu\{x|f(x) > \alpha\}d\alpha$$
$$= \int_{\frac{1}{25}}^1 \left(1 - \frac{1}{\sqrt{\alpha}}\right)^2 d\alpha$$
$$= \frac{4}{25} + ln25.$$

Then one has

$$\Phi\left((C)\int f d\mu\right) = 0.$$

Secondly, one has

$$(C)\int \Phi(f) d\mu$$
$$= \int_0^\infty \mu\{x|\Phi(f(x)) > \alpha\}d\alpha$$
$$= \int_{\frac{1}{5^4}}^1 \left(1 - \frac{1}{\sqrt[4]{\alpha}}\right)^2 d\alpha$$
$$= \frac{4932}{2175}.$$

Therefore we obtain the following inequality:

$$\Phi\left((C)\int f d\mu\right) = 0$$
$$< \frac{4932}{2175} = (C)\int \Phi(f) d\mu.$$

That is, the Jensen type inequality in the theorem 3.3 holds.

We give the following example to see that the conditions "$\Phi(r) \leq r$ for all $r \in [0, \infty]$ and $\mu_f(\alpha) \leq \mu_{\Phi(f)}(\alpha)$ for all $\alpha \in [0, \infty)$" are not satisfied, but the conclusion of the theorem 3.3 holds. Thus we can see that these conditions in the theorem 3.3 are sufficient conditions.

Example 3.6. Let m be the Lebesgue measure and $\mu = m^2$. Then clearly, μ is a non-additive measure. Let

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } 1 \leq x \leq 5 \\ 0 & \text{if else} \end{cases}$$

and

$$\Phi(r) = \begin{cases} \sqrt{r} & \text{if } 0 \leq r \leq 1 \\ 0 & \text{if else} \end{cases}$$

Then $\Phi(r) > r$ for all $r \in [0, \infty)$, that is, Φ does not satisfy the condition $\Phi(r) \leq r$ for all $r \in [0, \infty)$. And one also has

$$\mu_f(\alpha) = \begin{cases} \left(\frac{1}{\sqrt{\alpha}} - 1\right)^2 & \text{if } \frac{1}{25} \leq \alpha \leq 1 \\ 0 & \text{if else} \end{cases}$$
$$>$$
$$\mu_{\Phi(f)}(\alpha) = \begin{cases} \left(\frac{1}{\alpha} - 1\right)^2 & \text{if } \frac{1}{5} \leq \alpha \leq 1 \\ 0 & \text{if else} \end{cases}$$

Thus, we can see that the condition " $\mu_f(\alpha) \leq \mu_{\Phi(f)}(\alpha)$ for all $\alpha \in [0, \infty)$" does not hold. But we obtain the followings: At first, one has

$$(C)\int f d\mu$$
$$= \int_0^\infty \mu\{x|f(x) > \alpha\}d\alpha$$
$$= \int_{\frac{1}{25}}^1 \left(1 - \frac{1}{\sqrt{\alpha}}\right)^2 d\alpha$$
$$= \frac{4}{25} + ln25.$$

Then one has

$$\Phi\left((C)\int f d\mu\right) = 0.$$

Secondly, one has

$$(C)\int \Phi(f) d\mu$$
$$= \int_0^\infty \mu\{x|\Phi(f(x)) > \alpha\}d\alpha$$
$$= \int_{\frac{1}{5}}^1 \left(1 - \frac{1}{\alpha}\right)^2 d\alpha$$
$$= \frac{113}{25} - \frac{1}{2}ln5.$$

Therefore we obtain the following inequality:

$$\Phi\left((C)\int f d\mu\right) = 0$$
$$< \frac{113}{25} - \frac{1}{2}ln5 = (C)\int \Phi(f) d\mu.$$

That is, the conclusion of the theorem 3.4 holds.

Finally, we give the following two examples assuring both satisfaction and dissatisfaction of the condition " $\mu_f(\alpha) \leq \mu_{\Phi(f)}(\alpha)$ for all $\alpha \in [0, \infty)$"

Example 3.7. Let m be the Lebesgue measure and $\mu = m^2$. Then clearly, μ is a non-additive measure. Let

$$f(x) = \begin{cases} x^2 & \text{if } 1 \leq x \leq 5 \\ 0 & \text{if else} \end{cases}$$

and

$$\Phi(r) = \begin{cases} r^2 & \text{if } 0 \leq r \leq 1 \\ 0 & \text{if else} \end{cases}$$

Then Φ is convex and satisfy the condition "$\Phi(r) \leq r$ for all $r \in [0, \infty]$" , and one also has

$$\mu_f(\alpha) = \mu(\{x|f(x) > \alpha\} \cap [1, 5])$$
$$= \begin{cases} (5 - \sqrt{\alpha})^2 & \text{if } 1 \leq \alpha \leq 5 \\ 0 & \text{if else} \end{cases}$$
$$>$$
$$\mu_{\Phi(f)}(\alpha) = \mu(\{x|\Phi(f)(x) > \alpha\} \cap [1, 5])$$
$$= \begin{cases} (5 - \sqrt[4]{\alpha})^2 & \text{if } 1 \leq \alpha \leq 5 \\ 0 & \text{if else} \end{cases}$$

Thus, we can see that the condition " $\mu_f(\alpha) \leq \mu_{\Phi(f)}(\alpha)$ for all $\alpha \in [0,\infty)$" does not hold.

Example 3.8. Let m be the Lebesgue measure and $\mu = m^2$. Then clearly, μ is a non-additive measure. Let

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } 1 \leq x \leq 5 \\ 0 & \text{if else} \end{cases}$$

and

$$\Phi(r) = \begin{cases} r & \text{if } 0 \leq r \leq 1 \\ 0 & \text{if else} \end{cases}$$

Then Φ is both convex and the condition "$\Phi(r) \leq r$ for all $r \in [0,\infty]$" , and one also has

$$\mu_f(\alpha) = \mu(\{x|f(x) > \alpha\} \cap [1,5])$$

$$= \begin{cases} \left(\frac{1}{\sqrt{\alpha}} - 1\right)^2 & \text{if } \frac{1}{25} \leq \alpha \leq 1 \\ 0 & \text{if else} \end{cases}$$

$$= \mu_{\Phi(f)}(\alpha)\mu(\{x|\Phi(f)(x) > \alpha\} \cap [1,5])$$

$$= \begin{cases} (\frac{1}{\sqrt{\alpha}} - 1)^2 & \text{if } \frac{1}{25} \leq \alpha \leq 1 \\ 0 & \text{if else} \end{cases}$$

Thus, we can see that the condition " $\mu_f(\alpha) \leq \mu_{\Phi(f)}(\alpha)$ for all $\alpha \in [0,\infty)$" holds.

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