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$(L,\odot)$-quasi-uniform Spaces and $(L,\odot)$-neighborhood Systems

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Abstract

In this paper, we introduced the notion of $(L,\odot)$-quasi-uniform spaces and $(L,\odot)$-neighborhood systems on a strictly two-sided, commutative quantale lattice L. We investigate their properties and give the examples. In particular, we study the relations between $(L,\odot)$-quasi-uniform spaces and $(L,\odot)$-neighborhood systems.

Key Words : Quantale lattice, $(L,\odot)$-topologies, $(L,\odot)$-filters, $(L,\odot)$-quasi-uniform spaces, $(L,\odot)$-neighborhood systems

1. Introduction and preliminaries

Uniformities in fuzzy sets, have the entourage approach [1,2,7,9,11,12] based on powersets of the form $L^{X\times X}$, the uniform covering approach of Kotzé [8], the uniform operator approach of Rodabaugh [11] as generalization of Hutton [5] based on powersets of the form $(L^X)^{(L^X)}$, the unification approach of García et al. [2]. For a fixed basis L, algebraic structures in L (cqm-lattices, quantales, MV-algebras) are extended for a completely distributive lattice L [9] or t-norms [12]. Recently, Kim [7] introduced $(L,\odot)$-fuzzy quasi-uniformities as a view point of stsc biquantales L [10].

In this paper, we introduced the notion of $(L,\odot)$-quasi-uniform spaces and $(L,\odot)$-neighborhood systems on a strictly two-sided, commutative quantale lattice L. We investigate their properties and give the examples. In particular, we study the relations between $(L,\odot)$-quasi-uniform spaces and $(L,\odot)$-neighborhood systems.

Definition 1.1. [10] A triple $(L,\le,\odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following conditions:

(Q1) $L=(L,\le,\vee,\wedge,1,0)$ is a completely distributive lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(Q2) $(L,\odot)$ is a commutative semigroup;

(Q3) $a=a\odot 1$, for each $a\in L$;

(Q4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i)\odot b=\bigvee_{i\in\Gamma}(a_i\odot b).$$

Lemma 1.2. [3,7,10] Let $(L,\le,\odot)$ be a stsc-quantale. For each $x,y,z,x_i,y_i\in L$, we have the following properties.

(1) If $y\le z$, $(x\odot y)\le(x\odot z)$, $x\to y\le x\to z$ and $z\to x\le y\to x$.

(2) $x\odot y\le x\wedge y\le x\vee y$.

(3) $x\to(\bigwedge_{i\in\Gamma} y_i)=\bigwedge_{i\in\Gamma}(x\to y_i)$.

(4) $(\bigvee_{i\in\Gamma} x_i)\to y=\bigwedge_{i\in\Gamma}(x_i\to y)$.

(5) $(x\to y)\odot(z\to w)\le(x\odot z)\to(y\odot w)$.

(6) $(y\to z)\le(x\odot y)\to(x\odot z)$.

(7) $(y\to z)\le(x\to y)\to(x\to z)$ and $(y\to x)\le(x\to z)\to(y\to z)$.

(8) $(x_i\to y_i)\le(\bigwedge_{i\in\Gamma} x_i)\to(\bigwedge_{i\in\Gamma} y_i)$.

(9) $(x_i\to y_i)\le(\bigvee_{i\in\Gamma} x_i)\to(\bigvee_{i\in\Gamma} y_i)$.

Definition 1.3. [6] A mapping $\tau: L^X\to L$ is called an $(L,\odot)$-*topology* on X if it satisfies the following conditions:

(O1) $\tau(\overline{0})=\tau(\overline{1})=1$ where $\alpha\in L$, $\overline{\alpha}(x)=\alpha$ for each $x\in X$.

(O2) $\tau(f_1\odot f_2)\ge\tau(f_1)\odot\tau(f_2)$, for any $f_1,f_2\in L^X$.

(O3) $\tau(\bigvee_{i\in\Gamma} f_i)\ge\bigwedge_{i\in\Gamma}\tau(f_i)$, for any $\{f_i\}_{i\in\Gamma}\subset L^X$.

An $(L,\odot)$-topology is called *enriched* if

(E) $\tau(\alpha\odot f)\ge\tau(f)$ for each $f\in L^X$ and $\alpha\in L$.

The pair (X,τ) is called an (resp. enriched) $(L,\odot)$-*topological space*.

Let (X,τ_1) and (Y,τ_2) be two $(L,\odot)$-topological spaces. A mapping $\psi: X\to Y$ is said to be LF-*continuous* iff $\tau_2(g)\le\tau_1(\psi^{\leftarrow}(g))$ for each $g\in L^Y$.

Definition 1.4. [2,6] A mapping $\mathcal{F}: L^X\to L$ is called an $(L,\odot)$-*filter* on X if it satisfies the following conditions:

(F1) $\mathcal{F}(\overline{0})=0$ and $\mathcal{F}(\overline{1})=1$.

(F2) $\mathcal{F}(f\odot g)\ge\mathcal{F}(f)\odot\mathcal{F}(g)$, for each $f,g\in L^X$.

(F3) If $f\le g$, $\mathcal{F}(f)\le\mathcal{F}(g)$.

An $(L,\odot)$-filter is called *stratified* if

(S) $\mathcal{F}(\alpha\odot f)\ge\alpha\odot\mathcal{F}(f)$ for each $f\in L^X$ and $\alpha\in L$.

The pair $(X,\mathcal{F})$ is called an (resp. stratified) $(L,\odot)$-*filter space*. We denote $F_\odot(X)$ (resp. $F^s_\odot(X)$) as the family of (resp. stratified) $(L,\odot)$-filters on X.

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Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be $(L,\odot)$-filters on X. We say $\mathcal{F}_1$ is *finer* than $\mathcal{F}_2$ (or $\mathcal{F}_2$ is *coarser* than $\mathcal{F}_1$), denoted by $\mathcal{F}_2 \leq \mathcal{F}_1$, iff $\mathcal{F}_2(f) \leq \mathcal{F}_1(f)$ for all $f \in L^X$. Let $(X,\mathcal{F}_1)$ and $(Y,\mathcal{F}_2)$ be $(L,\odot)$-filter spaces. A mapping $\psi : X \to Y$ is said to be an *$(L,\odot)$-filter map* iff $\mathcal{F}_2(g) \leq \mathcal{F}_1(\psi^{\leftarrow}(g))$ for each $g \in L^Y$.

Definition 1.5. [6] A map $\mathcal{N} : X \to L^{L^X}$ is called an *(resp. stratified) $(L,\odot)$-neighborhood system* on X if $\mathcal{N}(x) = \mathcal{N}_x$ is an (resp. stratified) $(L,\odot)$-filter and satisfies the following conditions:

(N1) $\mathcal{N}_x(f) \leq [x](f)$, where $[x](f) = f(x)$ for all $f \in L^X$,

(N2) $\mathcal{N}_x(f) \leq \bigvee\{\mathcal{N}_x(g) \mid g(y) \leq \mathcal{N}_y(f), \forall y \in X\}$, for all $f \in L^X$.

2. The Properties of $(L,\odot)$-filters

Theorem 2.1. Let $\mathcal{U},\mathcal{V},\mathcal{W} \in F_\odot(X\times X)$. We define $\mathcal{U}^{-1}$,$\mathcal{U}\circ\mathcal{V} : L^{X\times X} \to L$ as follows:

$$\mathcal{U}^{-1}(w) = \mathcal{U}(w^{-1}),$$

$$(\mathcal{U}\circ\mathcal{V})(w) = \bigvee\{\mathcal{U}(u)\odot\mathcal{V}(v) \mid u\circ v \leq w\}$$

where $u \circ v(x,z) = \bigvee_{y\in X}(u(x,y)\odot v(y,z))$ and $w^{-1}(x,y) = w(y,x)$.

(1) $u\circ v = \bot$ implies $\mathcal{U}(u)\odot\mathcal{V}(v) = \bot$ iff $(\mathcal{U}\circ\mathcal{V}) \in F_\odot(X\times X)$.

(2) If $\mathcal{U}(1_\triangle) = \top$ where $1_\triangle(x,x) = \top$ and $1_\triangle(x,y) = \bot$ for $x \neq y \in X$, then $\mathcal{U}\circ\mathcal{U} \geq \mathcal{U}$.

(3) Put $[(x,x)](u) = u(x,x)$ for all $u \in L^{X\times X}$. Then $\mathcal{U}\circ[(x,x)] \in F^s_\odot(X\times X)$ and $\mathcal{U}\circ[(x,x)] \geq \mathcal{U}$.

(4) $[(x,x)]\circ[(x,x)] = [(x,x)]$.

(5) Put $[\triangle](u) = \bigwedge_{x\in X}[(x,x)](u) = \bigwedge_{x\in X} u(x,x)$ for all $u \in L^{X\times X}$. Then $[\triangle]\circ[\triangle] = [\triangle]$.

(6) $\mathcal{U}\circ\mathcal{U}^{-1} \in F_\odot(X\times X)$.

(7) $(\mathcal{U}\circ\mathcal{V})^{-1} = \mathcal{V}^{-1}\circ\mathcal{U}^{-1}$.

(8) $(\mathcal{U}\circ\mathcal{V})\circ\mathcal{W} = \mathcal{U}\circ(\mathcal{V}\circ\mathcal{W})$.

(9) If $\mathcal{U}_i,\mathcal{V}_i \in F_\odot(X\times X)$ for $i \in \{1,2\}$, then $(\mathcal{U}_1\circ\mathcal{U}_2)\odot(\mathcal{V}_1\circ\mathcal{V}_2) \leq (\mathcal{U}_1\odot\mathcal{V}_1)\circ(\mathcal{U}_2\odot\mathcal{V}_2)$.

Proof. (1) First, we show that $(u_1\odot u_2)\circ(v_1\odot v_2) \leq (u_1\circ v_1)\odot(v_2\circ u_2)$ from:

$$\begin{aligned}
&((u_1\odot u_2)\circ(v_1\odot v_2))(x,z)\\
&= \bigvee_{y\in X}\Big((u_1\odot u_2)(x,y)\odot(v_1\odot v_2)(y,z)\Big)\\
&\leq \bigvee_{y\in X}\Big(u_1(x,y)\odot v_1(y,z)\Big)\\
&\ \odot\bigvee_{w\in X}\Big(u_2(x,w)\odot v_2(w,z)\Big)\\
&= ((u_1\circ v_1)\odot(u_2\circ v_2))(x,z).
\end{aligned}$$

$$\begin{aligned}
&(\mathcal{U}\circ\mathcal{V})(u)\odot(\mathcal{U}\circ\mathcal{V})(v)\\
&= \bigvee_{u_1\circ v_1\leq u}(\mathcal{U}(u_1)\odot\mathcal{V}(v_1))\odot\bigvee_{u_2\circ v_2\leq v}(\mathcal{U}(u_2)\odot\mathcal{V}(v_2))\\
&\leq \bigvee_{(u_1\circ v_1)\odot(u_2\circ v_2)\leq u\odot v}(\mathcal{U}(u_1)\odot\mathcal{V}(v_1)\odot\mathcal{U}(u_2)\odot\mathcal{V}(v_2))\\
&\leq \bigvee_{(u_1\circ v_1)\odot(u_2\circ v_2)\leq u\odot v}(\mathcal{U}(u_1)\odot\mathcal{U}(u_2))\odot\mathcal{V}(v_1)\odot\mathcal{V}(v_2))\\
&\leq \bigvee_{(u_1\odot u_2)\circ(v_1\odot v_2)\leq u\odot v}(\mathcal{U}(u_1\odot u_2)\odot\mathcal{V}(v_1\odot v_2))\\
&\leq (\mathcal{U}\circ\mathcal{V})(u\odot v).
\end{aligned}$$

Hence $(\mathcal{U}\circ\mathcal{V}) \in F_\odot(X\times X)$. Conversely, it easily proved.

(2) For $u\circ 1_\triangle = u$, we have

$$(\mathcal{U}\circ\mathcal{U})(u) \geq \mathcal{U}(u)\odot\mathcal{U}(1_\triangle) = \mathcal{U}(u).$$

(3) Put $[(x,x)](u) = u(x,x)$ for all $u \in L^{X\times X}$. Since $[(x,x)](\alpha\odot u) = \alpha\odot u(x,x) = \alpha\odot[(x,x)](u)$, $[(x,x)] \in F^s_\odot(X\times X)$.

For $u\circ 1_\triangle = u$, we have

$$(\mathcal{U}\circ[(x,x)])(u) \geq \mathcal{U}(u)\odot[(x,x)](1_\triangle) = \mathcal{U}(u).$$

(4) For $u_1\circ u_2 \leq u$, we have

$$\begin{aligned}
([(x,x)]\circ[(x,x)])(u) &= \bigvee_{x\in X}(([(x,x)](u_1)\odot[(x,x)](u_2))\\
&\leq u(x,x) = [(x,x)](u).
\end{aligned}$$

By (3), the result holds.

(5) For $u\circ 1_\triangle = u$, we have

$$\begin{aligned}
&(\textstyle\bigwedge_{x\in X}[(x,x)]\circ\bigwedge_{x\in X}[(x,x)])(u)\\
&\geq \textstyle\bigwedge_{x\in X}[(x,x)](u)\odot\bigwedge_{x\in X}[(x,x)](1_\triangle)\\
&= \textstyle\bigwedge_{x\in X}[(x,x)](u).
\end{aligned}$$

For $u\circ v \leq w$,

$$\begin{aligned}
&\textstyle\bigwedge_{x\in X}[(x,x)](u)\circ\bigwedge_{x\in X}[(x,x)](v)\\
&= \textstyle\bigwedge_{x\in X} u(x,x)\odot\bigwedge_{x\in X} v(x,x)\\
&\leq \textstyle\bigwedge_{x\in X}[(x,x)](u\circ v) \leq \bigwedge_{x\in X}[(x,x)](w).
\end{aligned}$$

(6) For $u\circ v = \bot$, we have $\mathcal{U}(u)\odot\mathcal{U}^{-1}(v) \leq \mathcal{U}(u\odot v^{-1}) = \bot$ because $(u\odot v^{-1})(x,y) \leq u\circ v(x,x) = \bot$.

(7) Since $(v\circ u)^{-1} = u^{-1}\circ v^{-1}$, we have

$$\begin{aligned}
\mathcal{V}^{-1}\circ\mathcal{U}^{-1}(w) &= \bigvee\{\mathcal{V}^{-1}(v)\odot\mathcal{U}^{-1}(u) \mid v\circ u \leq w\}\\
&= \bigvee\{\mathcal{V}(v^{-1})\odot\mathcal{U}(u^{-1}) \mid u^{-1}\circ v^{-1} \leq w^{-1}\}\\
&= \mathcal{U}\circ\mathcal{V}(w^{-1}) = (\mathcal{U}\circ\mathcal{V})^{-1}(w).
\end{aligned}$$

(8) Suppose there exists $e \in L^{X\times X}$ such that

$$((\mathcal{U}\circ\mathcal{V})\circ\mathcal{W})(e) \not\leq (\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W}))(e).$$

Then there exists $d,w \in L^{X\times X}$ with $d\circ w \leq e$ such that

$$(\mathcal{U}\circ\mathcal{V})(d)\odot\mathcal{W}(w) \not\leq (\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W}))(e).$$

Also, there exists $u,v \in L^{X\times X}$ with $u\circ v \leq d$ such that

$$(\mathcal{U}(u)\odot\mathcal{V}(v))\odot\mathcal{W}(w) \not\leq (\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W}))(e).$$

Since $(u\circ v)\circ w = u\circ(v\circ w) \leq e$,

$$(\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W}))(e) \geq \mathcal{U}(u)\odot(\mathcal{V}(v)\odot\mathcal{W}(w)).$$

It is a contradiction. Hence $(\mathcal{U} \circ \mathcal{V}) \circ \mathcal{W} \le \mathcal{U} \circ (\mathcal{V} \circ \mathcal{W})$. Similarly, $(\mathcal{U} \circ \mathcal{V}) \circ \mathcal{W} \ge \mathcal{U} \circ (\mathcal{V} \circ \mathcal{W})$.

(9)

$$\begin{aligned}
&(\mathcal{U}_1 \circ \mathcal{U}_2)(u \circ v) \odot (\mathcal{V}_1 \circ \mathcal{V}_2)(u \circ v) \\
&\le (\mathcal{U}_1(u) \odot \mathcal{U}_2(v)) \odot (\mathcal{V}_1(u) \odot \mathcal{V}_2(v)) \\
&\le (\mathcal{U}_1(u) \odot \mathcal{V}_1(u)) \odot (\mathcal{U}_2(v) \odot \mathcal{V}_2(v)) \\
&\le ((\mathcal{U}_1 \odot \mathcal{V}_1) \circ (\mathcal{U}_2 \odot \mathcal{V}_2))(u \circ v)
\end{aligned}$$

Hence $(\mathcal{U}_1 \circ \mathcal{U}_2) \odot (\mathcal{V}_1 \circ \mathcal{V}_2) \le (\mathcal{U}_1 \odot \mathcal{V}_1) \circ (\mathcal{U}_2 \odot \mathcal{V}_2)$. □

Example 2.2. Let $X = \{a, b, c\}$ be a set, $L = [0, 1]$ the stsc-quantale with $a \odot b = (a + b - 1) \vee 0$ and $u, v \in [0, 1]^{X \times X}$ defined as follows:

$$u(a, a) = u(b, b) = u(c, c) = 1, u(a, b) = u(a, c) = 0.6,$$

$$u(b, a) = u(c, a) = 0.5, u(b, c) = u(c, b) = 0.4.$$

$$v(a, a) = v(b, b) = 1, v(c, c) = 0.7, v(a, b) = v(b, a) = 0.6,$$

$$v(a, c) = v(c, a) = 0.5, v(b, c) = v(c, b) = 0.4.$$

Define $[0, 1]$-filters as $\mathcal{U}, \mathcal{V} : [0, 1]^{X \times X} \to [0, 1]$ as follows:

$$\mathcal{U}(w) = \begin{cases} 1, & \text{if } w = 1_{X \times X}, \\ 0.6, & \text{if } u \le w \ne 1_{X \times X}, \\ 0.3, & \text{if } u \odot u \le w \not\ge u, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{V}(w) = \begin{cases} 1, & \text{if } w \ge 1_{\triangle}, \\ 0.6, & \text{if } v \le w \not\ge 1_{\triangle}, \\ 0.3, & \text{if } v \odot v \le w \not\ge v, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Since $u \circ u = u$, we obtain

$$(\mathcal{U} \circ \mathcal{U})(w) = \begin{cases} 1, & \text{if } w = \overline{1}, \\ 0.2, & \text{if } u \le w \ne \overline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$(\mathcal{U} \odot \mathcal{U})(w) = \begin{cases} 1, & \text{if } w = \overline{1}, \\ 0.2, & \text{if } u \le w \ne \overline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Since $v \circ 1_{\triangle} = v$, we obtain $\mathcal{V} \circ \mathcal{V} = \mathcal{V}$ and

$$(\mathcal{V} \odot \mathcal{V})(w) = \begin{cases} 1, & \text{if } w \ge 1_{\triangle}, \\ 0.2, & \text{if } v \le w \not\ge 1_{\triangle}, \\ 0, & \text{otherwise.} \end{cases}$$

(3) We obtain $[0, 1]$-filter as $\mathcal{U} \circ \mathcal{V} : [0, 1]^{X \times X} \to [0, 1]$ as follows:

$$\mathcal{U} \circ \mathcal{V}(w) = \begin{cases} 1, & \text{if } w = \overline{1}, \\ 0.6, & \text{if } u \le w \ne \overline{1}, \\ 0.3, & \text{if } u \odot u \le w \not\ge u, \\ 0.2, & \text{if } u \circ v \le w \not\ge u \odot u, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{V} \circ \mathcal{U}(w) = \begin{cases} 1, & \text{if } w = \overline{1}, \\ 0.6, & \text{if } v \le w \ne \overline{1}, \\ 0.2, & \text{if } v \circ u \le w \not\ge u, \\ 0, & \text{otherwise.} \end{cases}$$

$$(\mathcal{U} \odot \mathcal{V})(w) = \begin{cases} 1, & \text{if } w = \overline{1}, \\ 0.6, & \text{if } u \le w \ne \overline{1}, \\ 0.3, & \text{if } u \odot u \le w \not\ge u, \\ 0, & \text{otherwise.} \end{cases}$$

(4) $(\mathcal{U} \odot \mathcal{U}) \circ (\mathcal{V} \odot \mathcal{V}) = (\mathcal{U} \circ \mathcal{V}) \odot (\mathcal{U} \circ \mathcal{V})$ as follows:

$$((\mathcal{U} \odot \mathcal{U}) \circ (\mathcal{V} \odot \mathcal{V}))(w) = \begin{cases} 1, & \text{if } w = \overline{1}, \\ 0.2, & \text{if } u \le w \ne \overline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

$(\mathcal{U} \odot \mathcal{V}) \circ (\mathcal{U} \odot \mathcal{V}) = (\mathcal{U} \circ \mathcal{U}) \odot (\mathcal{V} \circ \mathcal{V})$ as follows:

$$(\mathcal{U} \circ \mathcal{U}) \odot (\mathcal{V} \circ \mathcal{V})(w) = \begin{cases} 1, & \text{if } w = \overline{1}, \\ 0.2, & \text{if } u \le w \ne \overline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

3. The Properties of $(L, \odot)$-quasi-uniform Structures

Definition 3.1. [6] An $(L, \odot)$-filter $\mathcal{U}$ on $X \times X$ is called an $(L, \odot)$-*quasi-uniform structure* on X if it satisfies the following conditions:

(QU1) $\mathcal{U} \le [\Delta]$,

(QU2) $\mathcal{U} \le \mathcal{U} \circ \mathcal{U}$ where $\mathcal{U} \circ \mathcal{U} \in F_{\odot}(X \times X)$.

The pair $(X, \mathcal{U})$ is called an $(L, \odot)$-*quasi-uniform space*. An $(L, \odot)$-quasi-uniform structure on X is called an $(L, \odot)$-*uniform structure* if $\mathcal{U} = \mathcal{U}^{-1}$.

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be $(L, \odot)$ quasi-uniform spaces. A map $\psi : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is called *quasi-uniformly continuous* if for $v \in L^{Y \times Y}$, $\mathcal{V}(v) \le \mathcal{U}((\psi \times \psi)^{\leftarrow}(v))$.

Example 3.2. (1) Let X be a set. Define $[\Delta](u) = \bigwedge[(x, x)](u)$ for all $u \in L^{X \times X}$. By Theorem 2.1(5),$[\Delta]$ is an $(L, \odot)$-uniformity on X.

(2) Let $X, ([0, 1], \odot), \mathcal{U}$ and $\mathcal{V}$ be given as in Example 2.2. Since $\mathcal{U} \not\le \mathcal{U} \circ \mathcal{U}$, $\mathcal{U}$ is not an $(L, \odot)$-quasi-uniformity on X. Since $\mathcal{V} = \mathcal{V} \circ \mathcal{V}$ and $\mathcal{V} \le [\Delta]$, $\mathcal{V}$ is an $(L, \odot)$-quasi-uniformity on X.

Theorem 3.3. Let $(X, \mathcal{U})$ be an $(L, \odot)$-quasi-uniform space. We define a map $N^{\mathcal{U}} : X \to L^{L^X}$ as follows:

$$N^{\mathcal{U}}(x)(f) = N^{\mathcal{U}}_x(f) = \bigvee\{\alpha \odot \mathcal{U}(u) \mid \alpha \odot u(x, -) \le f\}.$$

Then $N^{\mathcal{U}}$ is an $(L, \odot)$-neighborhood system on X.

Proof. (F1) Since $\mathcal{U} \leq [\triangle]$, for $\alpha \odot u(x,-) \leq \bar{0}$, we have $\alpha \odot \mathcal{U}(u) \leq \alpha \odot [\triangle] \leq \alpha \odot [(x,x)](u) = \bar{0}(x) = \perp$. Thus $N_x^{\mathcal{U}}(\bar{0}) = \perp$. Moreover, $N_x^{\mathcal{U}}(\bar{1}) \geq \mathcal{U}(\bar{1}) = \top$.

(F2)

$$\begin{array}{l} N_x^{\mathcal{U}}(f) \odot N_x^{\mathcal{U}}(g) \\ = \bigvee\{\alpha \odot \mathcal{U}(u_1) \mid \alpha \odot u_1(x,-) \leq f\} \odot \\ \bigvee\{\beta \odot \mathcal{U}(u_2) \mid \beta \odot u_2(x,-) \leq g\} \\ \leq \bigvee\{\alpha \odot \beta \odot \mathcal{U}(u_1 \odot u_2) \mid \alpha \odot \beta \\ \odot u_1(x,-) \odot u_2(x,-) \leq f \odot g\} \\ = N_x^{\mathcal{U}}(f \odot g). \end{array}$$

(F3) is trivial.

(N1)

$$\begin{array}{rl} N_x^{\mathcal{U}}(f) & = \bigvee\{\alpha \odot \mathcal{U}(u) \mid \alpha \odot u(x,-) \leq f\} \\ & \leq \bigvee\{\alpha \odot [\triangle](u) \mid \alpha \odot u(x,-) \leq f\} \\ & \leq f(x). \end{array}$$

(N2)

$$\begin{array}{l} N_x^{\mathcal{U}}(f) \\ = \bigvee\{\alpha \odot \mathcal{U}(u) \mid \alpha \odot u(x,-) \leq f\}\} \\ \leq \bigvee\{\alpha \odot \mathcal{U}(u_1) \odot \mathcal{U}(u_2) \mid \\ \alpha \odot (u_2 \circ u_1(x,-)) \leq \alpha \odot u(x,-) \leq f\}. \end{array}$$

For $\alpha \odot u_2(y,x) \odot u_1(x,-) \leq \alpha \odot u(y,-) \leq f$, $g(y) = \alpha \odot u_2(y,x) \odot \mathcal{U}(u_1) \leq \alpha \odot \mathcal{U}(u) \leq N_y^{\mathcal{U}}(f)$

$$\begin{array}{l} N_x^{\mathcal{U}}(f) \\ \leq \bigvee\{\alpha \odot \mathcal{U}(u_1) \odot \mathcal{U}(u_2) \mid \\ \alpha \odot (u_2 \circ u_1(x,-)) \leq \alpha \odot u(x,-) \leq f\}. \\ \leq \bigvee\{\alpha \odot \mathcal{U}(u_1) \odot \mathcal{U}(u_2) \mid g(y) \leq N_y^{\mathcal{U}}(f)\} \\ = \bigvee\{N_x^{\mathcal{U}}(g) \mid g(y) \leq N_y^{\mathcal{U}}(f)\}. \end{array}$$

□

Theorem 3.4. Let $(X, \mathcal{U})$ be an $(L,\odot)$-quasi-uniform space and $N^{\mathcal{U}} = \{N_x^{\mathcal{U}} \mid x \in X\}$ be an $(L,\odot)$-neighborhood system on X. We define a map $\tau_U : L^X \to L$ as follows:

$$\tau_U(f) = \bigwedge_{x \in X} (f(x) \to N_x^{\mathcal{U}}(f)).$$

Then (1) τ_U is an $(L,\odot)$-topology.

(2) If $N_x^{\mathcal{U}}$ is a stratified $(L,\odot)$-filter, then τ_U is an enriched $(L,\odot)$-topology.

Proof. (1) (O1)

$$\begin{array}{rl} \tau_U(0) & = \bigwedge_{x \in X}(\bar{0}(x) \to N_x^{\mathcal{U}}(\bar{0})) = 1 \\ \tau_U(1) & = \bigwedge_{x \in X}(\bar{1}(x) \to N_x^{\mathcal{U}}(\bar{1})) = 1 \end{array}$$

(O2)

$$\begin{array}{l} \tau_U(f \odot g) \\ = \bigwedge_{x \in X}((f \odot g)(x) \to N_x^{\mathcal{U}}(f \odot g)) \\ \geq \bigwedge_{x \in X}((f(x) \odot g(x)) \to N_x^{\mathcal{U}}(f) \odot N_x^{\mathcal{U}}(g)) \\ \quad \text{(by Lemma 1.2.(5))} \\ \geq \bigwedge_{x \in X}\Big((f(x) \to N_x^{\mathcal{U}}(f)) \odot (g(x) \to N_x^{\mathcal{U}}(g))\Big) \\ \geq \bigwedge_{x \in X}(f(x) \to N_x^{\mathcal{U}}(f)) \odot \bigwedge_{x \in X}(g(x) \to N_x^{\mathcal{U}}(g)) \\ \geq \tau_U(f) \odot \tau_U(g). \end{array}$$

(O3)

$$\begin{array}{rl} \tau_U(\bigvee_i f_i) & = \bigwedge_{x \in X}((\bigvee_i f_i(x) \to N_x^{\mathcal{U}}(\bigvee_i f_i)) \\ & \geq \bigwedge_{x \in X}((\bigvee_i f_i(x) \to \bigvee_i N_x^{\mathcal{U}}(f_i)) \\ & \quad \text{(by Lemma 1.2.(9))} \\ & \geq \bigwedge_{x \in X} \bigwedge_i (f_i(x) \to N_x^{\mathcal{U}}(f_i)) \\ & \geq \bigwedge_i \bigwedge_{x \in X}(f_i(x) \to N_x^{\mathcal{U}}(f_i)) \\ & = \bigwedge_i \tau_U(f_i) \end{array}$$

(2)

$$\begin{array}{rl} \tau_U(\alpha \odot f) & = \bigwedge_{x \in X}(\alpha \odot f(x) \to N_x^{\mathcal{U}}(\alpha \odot f)) \\ & \geq \bigwedge_{x \in X}((\alpha \odot f(x)) \to (\alpha \odot N_x^{\mathcal{U}}(f))) \\ & \geq \bigwedge_{x \in X}(f(x) \to N_x^{\mathcal{U}}(f)) \text{ (by Lemma 1.2.(6))} \\ & \geq \tau_U(f). \end{array}$$

□

Example 3.5. Let $X = \{x,y,z\}$ be a set, $(L = [0,1], \odot)$ the stsc-quantale with $a \odot b = (a+b-1) \vee 0$ and let $e \in [0,1]^{X \times X}$ defined as

$$v(x,x) = 1, v(x,y) = 0.6, v(x,z) = 0.5,$$

$$v(y,x) = 0.5, v(y,y) = 1, v(y,z) = 0.6,$$

$$v(z,x) = 0.6, v(z,y) = 0.4, v(z,z) = 0.4.$$

We define a $([0,1],\odot)$-quasi-uniformity $\mathcal{U} : [0,1]^{X \times X} \to [0,1]$ as follows:

$$\mathcal{U}(w) = \begin{cases} 1, & \text{if } w \geq 1_\triangle, \\ 0.6, & \text{if } v \leq w \not\geq 1_\triangle, \\ 0.3, & \text{if } v \odot v \leq w \not\geq v, \\ 0, & \text{otherwise.} \end{cases}$$

For $x \in \{x,y,z\}$, we obtain $([0,1],\odot)$-neighborhood filters $N_x^{\mathcal{U}} : [0,1]^X \to [0,1]$ as follows:

$$N_x^{\mathcal{U}}(f) = \begin{cases} \alpha, & \text{if } f \geq \alpha \cdot g_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$N_y^{\mathcal{U}}(f) = \begin{cases} \alpha, & \text{if } f \geq \alpha \cdot g_2, \\ 0, & \text{otherwise.} \end{cases}$$

$$N_z^{\mathcal{U}}(f) = \begin{cases} \alpha, & \text{if } f \geq \alpha \cdot g_3, \\ 0.6 \cdot \beta, & \text{if } \beta \cdot g_4 \leq f \not\geq \alpha \cdot g_3, \\ 0.3 \cdot \gamma, & \text{if } \gamma \cdot g_5 \leq f \not\geq \beta \cdot g_4, \\ 0, & \text{otherwise} \end{cases}$$

$$g_1(x) = 1, g_1(y) = 0, g_1(z) = 0,$$

$$g_2(x) = 0, g_2(y) = 1, g_1(z) = 0,$$

$$g_3(x) = 0, g_3(y) = 0, g_3(z) = 0.4,$$

$$g_4(x) = 0.6, g_4(y) = 0.4, g_4(z) = 0.4,$$

$$g_5(x) = 0.2, g_5(y) = 0, g_5(z) = 0.$$

Theorem 3.6. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be $(L,\odot)$ quasi-uniform spaces. If a map $\psi : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is quasi-uniformly continuous, then a map $\psi : (X, N_x^{\mathcal{U}}) \to (Y, N_{\psi(x)}^{\mathcal{V}})$ is an $(L,\odot)$-filter map and a map $\psi : (X,\tau_U) \to (Y,\tau_V)$ is LF-continuous.

Proof.

$$\begin{aligned} N_{\psi(x)}^{\mathcal{V}}(f) &= \bigvee\{\alpha \odot \mathcal{V}(v) \mid \alpha \odot v(\psi(x),\psi(y)) \leq f(\psi(y))\} \\ &\leq \bigvee\{\alpha \odot \mathcal{U}((\psi\times\psi)^{\leftarrow}(v)) \\ &\mid \alpha \odot (\psi\times\psi)^{\leftarrow}(v)(x,y) \leq \psi^{\leftarrow}(f)(y)\} \\ &\leq N_x^{\mathcal{U}}(\psi^{\leftarrow}(f)). \end{aligned}$$

$$\begin{aligned} &\tau_V(g) \to \tau_U(\psi^{\leftarrow}(g)) \\ &\geq \bigwedge_{y\in Y}(g(y) \to N_y^{\mathcal{V}}(g)) \\ &\to \bigwedge_{x\in X}(\psi^{\leftarrow}(g)(x) \to N_x^{\mathcal{U}}(\psi^{\leftarrow}(g)) \\ &\geq \bigwedge_{x\in X}(\psi^{\leftarrow}(g)(x) \to N_{\psi(x)}^{\mathcal{V}}(g)) \to \\ &\quad \bigwedge_{x\in X}(\psi^{\leftarrow}(g)(x) \to N_x^{\mathcal{U}}(\psi^{\leftarrow}(g))) \\ &\geq \Big((\psi^{\leftarrow}(g)(x) \to N_{\psi(x)}^{\mathcal{V}}(g)) \to \\ &\quad (\psi^{\leftarrow}(g)(x) \to N_x^{\mathcal{U}}(\psi^{\leftarrow}(g))\Big) \text{ (by Lemma 1.2.(8))} \\ &\geq N_{\psi(x)}^{\mathcal{V}}(g) \to N_x^{\mathcal{U}}(\psi^{\leftarrow}(g)). \text{ (by Lemma 1.2.(7))} \end{aligned}$$

□

Theorem 3.7. Let $\mathcal{U}_i$ and $\mathcal{V}_i$ be families of $(L,\odot)$-quasi-uniformities satisfying the condition $\mathcal{U}_1(u) \odot \mathcal{U}_2(v) = \bot$ for each $u \odot v = \bot$. We define $\mathcal{U}_1 \oplus \mathcal{U}_2 \in F_{\odot}(X\times X)$ as follows:

$$(\mathcal{U}_1 \oplus \mathcal{U}_2)(w) = \bigvee\{\mathcal{U}_1(u) \odot \mathcal{U}_2(v) \mid u \odot v \leq w\}.$$

(1) $\mathcal{U}_1^{-1}$ is an $(L,\odot)$-uniformity on X.

(2) $(\mathcal{U}_1 \circ \mathcal{U}_2) \oplus (\mathcal{V}_1 \circ \mathcal{V}_2) \leq (\mathcal{U}_1 \oplus \mathcal{V}_1) \circ (\mathcal{U}_2 \oplus \mathcal{V}_2)$

(3) $\mathcal{U}_1 \oplus \mathcal{U}_2$ is the coarsest $(L,\odot)$-uniformities on X which is finer than $\mathcal{U}_1$ and $\mathcal{U}_2$. Moreover, if $\mathcal{U}_1 = \mathcal{U}_2$, then $\mathcal{U}_1 \oplus \mathcal{U}_1 = \mathcal{U}_1$.

(4) $(\mathcal{U}_1 \oplus \mathcal{U}_2)^{-1} = \mathcal{U}_1^{-1} \oplus \mathcal{U}_2^{-1}$.

(5) $\mathcal{U}_1 \oplus \mathcal{U}_1^{-1}$ is the coarsest $(L,\odot)$-uniformities on X which is finer than $\mathcal{U}_1$ and $\mathcal{U}_1^{-1}$.

(6) $\mathcal{N}_x^{\mathcal{U}_1} \oplus \mathcal{N}_x^{\mathcal{U}_2} \leq \mathcal{N}_x^{\mathcal{U}_1\oplus\mathcal{U}_2}$.

Proof. (1) Since $\mathcal{U}_1 \leq \mathcal{U}_1 \circ \mathcal{U}_1$, we have $\mathcal{U}_1^{-1} \leq \mathcal{U}_1^{-1} \circ \mathcal{U}_1^{-1}$. Other cases are easily proved.

(2) Since $(u_1 \odot v_1) \circ (u_2 \odot v_2) \leq (u_1 \circ u_2) \odot (v_1 \circ v_2)$, for all $u \odot v \leq w$, we have

$$\begin{aligned} &(\mathcal{U}_1 \circ \mathcal{U}_2)(u) \odot (\mathcal{V}_1 \circ \mathcal{V}_2)(v) \\ &= \bigvee\{\mathcal{U}_1(u_1) \odot \mathcal{U}_2(u_2) \mid u_1 \circ u_2 \leq u\} \\ &\odot \bigvee\{\mathcal{V}_1(v_1) \odot \mathcal{V}_2(v_2) \mid v_1 \circ v_2 \leq v\} \\ &= \bigvee\{(\mathcal{U}_1(u_1) \odot \mathcal{U}_2(u_2)) \odot (\mathcal{V}_1(v_1) \odot \mathcal{V}_2(v_2)) \\ &\mid u_1 \circ u_2 \leq u, v_1 \circ v_2 \leq v\} \\ &\leq \bigvee\{(\mathcal{U}_1(u_1) \odot \mathcal{V}_1(v_1)) \odot (\mathcal{U}_2(u_2) \odot \mathcal{V}_2(v_2)) \\ &\mid (u_1 \odot v_1) \circ (u_2 \odot v_2) \leq u \odot v\} \\ &\leq \bigvee\{(\mathcal{U}_1 \oplus \mathcal{V}_1)(u_1 \odot v_1) \odot (\mathcal{U}_2 \oplus \mathcal{V}_2)(u_2 \odot v_2) \\ &\mid (u_1 \odot v_1) \circ (u_2 \odot v_2) \leq u \odot v\} \\ &\leq ((\mathcal{U}_1 \oplus \mathcal{V}_1) \circ (\mathcal{U}_2 \oplus \mathcal{V}_2))(u \odot v). \end{aligned}$$

It follows $(\mathcal{U}_1 \circ \mathcal{U}_2) \oplus (\mathcal{V}_1 \circ \mathcal{V}_2)(w) \leq (\mathcal{U}_1 \oplus \mathcal{V}_1) \circ (\mathcal{U}_2 \oplus \mathcal{V}_2)(w)$ for all $w \in L^{X\times X}$.

(3)

$$\begin{aligned} &(\mathcal{U}_1 \oplus \mathcal{U}_2)(u) \odot (\mathcal{U}_1 \oplus \mathcal{U}_2)(v) \\ &= \bigvee\{\mathcal{U}_1(u_1) \odot \mathcal{U}_2(u_2) \mid u_1 \odot u_2 \leq u\} \\ &\odot \bigvee\{\mathcal{U}_1(v_1) \odot \mathcal{U}_2(v_2) \mid v_1 \odot v_2 \leq v\} \\ &= \bigvee\{(\mathcal{U}_1(u_1) \odot \mathcal{U}_2(u_2)) \odot (\mathcal{U}_1(v_1) \odot \mathcal{U}_2(v_2)) \\ &\mid u_1 \odot u_2 \leq u,\ v_1 \odot v_2 \leq v\} \\ &\leq \bigvee\{\mathcal{U}_1(u_1) \odot \mathcal{U}_1(v_1)) \odot (\mathcal{U}_2(u_2) \odot \mathcal{U}_2(v_2)) \\ &\mid u_1 \odot u_2 \leq u,\ v_1 \odot v_2 \leq v\} \\ &\leq \bigvee\{\mathcal{U}_1(u_1 \odot v_1) \odot \mathcal{U}_2(u_2 \odot v_2) \\ &\mid u_1 \odot u_2 \odot v_1 \odot v_2 \leq u \odot v\} \\ &\leq (\mathcal{U}_1 \oplus \mathcal{U}_2)(u \odot v). \end{aligned}$$

Since $(\mathcal{U}_1 \oplus \mathcal{U}_2) \leq (\mathcal{U}_1 \circ \mathcal{U}_1) \oplus (\mathcal{U}_2 \circ \mathcal{U}_2) \leq (\mathcal{U}_1 \oplus \mathcal{U}_2) \circ (\mathcal{U}_1 \oplus \mathcal{U}_2)$, the results hold.

(4) and (5) are easily proved.

(6)

$$\begin{aligned} &(\mathcal{N}_x^{\mathcal{U}_1} \oplus \mathcal{N}_x^{\mathcal{U}_2})(h) \\ &= \bigvee_{f\odot g\leq h}\Big(\mathcal{N}_x^{\mathcal{U}_1}(f) \odot \mathcal{N}_x^{\mathcal{U}_2}(g)\Big) \\ &= \bigvee_{f\odot g\leq h}\Big(\bigvee\{a_1 \odot \mathcal{U}_1(u_1) \mid a_1 \odot u_1(x,-) \leq f\} \\ &\odot \bigvee\{a_2 \odot \mathcal{U}_2(u_2) \mid a_2 \odot u_2(x,-) \leq g\}\Big) \\ &\leq \bigvee_{f\odot g\leq h}\Big(\bigvee\{a_1 \odot a_2 \odot \mathcal{U}_1(u_1) \odot \mathcal{U}_2(u_2) \\ &\mid a_1 \odot a_2 \odot u_1(x,-) \odot u_2(x,-) \leq f \odot g\} \\ &\leq \mathcal{N}_x^{\mathcal{U}_1\oplus\mathcal{U}_2}(h). \end{aligned}$$

□

Example 3.8. Let $X = \{a,b,c\}$ be a set, $L = [0,1]$ the stsc-quantale with $a \odot b = (a+b-1) \vee 0$ and $u,v \in [0,1]^{X\times X}$ defined as follows:

$$u(a,a) = u(b,b) = 0.6, u(c,c) = 1, u(a,b) = u(a,c) = 0.6,$$

$$u(b,a) = u(c,a) = 0.5, u(b,c) = u(c,b) = 0.4.$$

$$v(a,a) = v(b,b) = 1, v(c,c) = 0.7, v(a,b) = 0.7, v(a,c) = 0.4$$

$$v(b,a) = v(c,a) = v(b,c) = 0.6, v(c,b) = 0.5.$$

Define $[0,1]$-filters as $\mathcal{U},\mathcal{V} : [0,1]^{X\times X} \to [0,1]$ as follows:

$$\mathcal{U}(w) = \begin{cases} 1, & \text{if } w \geq 1_{\triangle}, \\ 0.5, & \text{if } u \leq w \not\geq 1_{\triangle}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{V}(w) = \begin{cases} 1, & \text{if } w \geq 1_{\triangle}, \\ 0.6, & \text{if } v \leq w \not\geq 1_{\triangle}, \\ 0.3, & \text{if } v \odot v \leq w \not\geq v, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{U}$ and $\mathcal{V}$ are $(L,\odot)$-quasi-uniformities on X.

We obtain $[0,1]$-filter $\mathcal{U} \oplus \mathcal{V} : [0,1]^{X\times X} \to [0,1]$ as follows:

$$\mathcal{U}\oplus\mathcal{V}(w) = \begin{cases} 1, & \text{if } w \geq 1_\triangle, \\ 0.6, & \text{if } v \leq w \not\geq 1_\triangle, \\ 0.5, & \text{if } u \leq w \not\geq 1_\triangle, w \not\geq v \\ 0.3, & \text{if } v\odot v \leq w \not\geq v, w \not\geq 1_\triangle, w \not\geq u \\ 0.1, & \text{if } v \odot w \leq w \not\geq v \odot v, \\ & w \not\geq 1_\triangle, w \not\geq u \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{V}\oplus\mathcal{V}^{-1}(w) = \begin{cases} 1, & \text{if } w \geq 1_\triangle, \\ 0.6, & \text{if } v \leq w \not\geq 1_\triangle \text{ or } v \leq w \not\geq 1_\triangle \\ 0.3, & \text{if } v\odot v \leq w \not\geq v, w \not\geq v^{-1} \\ & \text{or } v^{-1}\odot v^{-1} \leq w \not\geq v, w \not\geq v^{-1} \\ 0.2, & \text{if } v \odot v^{-1} \leq w \not\geq v \odot v, \\ & w \not\geq v^{-1} \odot v^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

REFERENCES

[1] A. Craig, G. Jäger, " A common framework for lattice-valued uniform spaces and probabilistic uniform limit spaces", *Fuzzy Sets and Systems*, vol. 158, pp. 424-435, 2007.

[2] J. Gutiérrez García, M. A. de Prade Vicente, A.P. Šostak, *A unified approach to the concept of fuzzy L-uniform spaces*, Chapter 3 in [11], pp. 81-114.

[3] U. Höhle, E. P. Klement, *Non-classical logic and their applications to fuzzy subsets*, Kluwer Academic Publisher, Boston, 1995.

[4] U. Höhle, S. E. Rodabaugh, *Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory*, The Handbooks of Fuzzy Sets Series, Volume 3, Kluwer Academic Publishers, Dordrecht, 1999.

[5] B. Hutton, "Uniformities on fuzzy topological spaces," *J. Math. Anal. Appl.*, vol. 58, pp. 559-571, 1977.

[6] U.Höhle, A.P.Sostak, *Axiomatic foundation of fixed-basis fuzzy topology*, Chapter 3 in [4], 123-272.

[7] Y.C. Kim, Y.S. Kim, "$(L,\odot)$-approximation spaces and $(L,\odot)$-fuzzy quasi-uniform spaces," *Information Sciences*, vol. 179, pp.2028-2048, 2009.

[8] W. Kotzé, *Uniform spaces*, Chapter 8 in [4], pp. 553-580.

[9] Liu Ying-Ming, Luo Mao-Kang, *Fuzzy topology*, World Scientific Publishing Co., Singapore, 1997.

[10] C.J. Mulvey, Quantales, Suppl. Rend. Cric. Mat. Palermo Ser.II 12, pp. 99-104, 1986.

[11] S. E. Rodabaugh, E. P. Klement, Topological And Algebraic Structures In Fuzzy Sets, *The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic 20*, Kluwer Academic Publishers, (Boston/Dordrecht/London), 2003.

[12] D. Zhang, "A comparison of various uniformities in fuzzy topology," *Fuzzy Sets and Systems*, vol. 159, pp. 2503-2519, 2008.

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