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On Fuzzy α-Weakly r-Continuous Mappings

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Abstract

In this paper, we introduce the concept of fuzzy α-weakly r-continuous mapping on a fuzzy topological space and investigate some properties of such a mapping and the relationships among fuzzy α-weakly r-continuity, fuzzy r-continuity and fuzzy weakly r-continuity.

Key words : fuzzy α-weakly r-continuous, fuzzy weakly r-semicontinuous, fuzzy S-weakly r-continuous, fuzzy weakly r-continuous

1. Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Chang [1] defined a fuzzy topological space using fuzzy sets. In [3, 4], Chattopadhyay, Hazra and Samanta introduced the concept of smooth fuzzy topological space which is a generalization of the fuzzy topological space. Lee and Lee [8] introduced the concepts of fuzzy strongly r-semiopen sets and fuzzy strongly r-semicontinuous mappings in fuzzy topological spaces defined by Chattopadhyay. These concepts are generalizations of fuzzy strongly semiopen seta and fuzzy strongly semicontinuous mappings. In this paper, we introduce the concept of fuzzy α-weakly r-continuous mapping as a generalization of the fuzzy strongly r-semicontinuous mapping and study some properties of the mapping and the relationships among fuzzy α-weakly r-continuity, fuzzy r-continuity and fuzzy weakly r-continuity.

2. Preliminaries

Let I be the unit interval $[0,1]$ of the real line. A member μ of I^X is called a fuzzy set of X. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. For any $\mu \in I^X$, μ^c denotes the complement $\tilde{1} - \mu$. All other notations are standard notations of fuzzy set theory.

An *fuzzy point* x_α in X is a fuzzy set x_α defined by

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point x_α is said to belong to an fuzzy set A in X, denoted by $x_\alpha \in A$, if $\alpha \leq A$ for $x \in X$.

A fuzzy set A in X is the union of all fuzzy points which belong to A.

Let $f : X \to Y$ be a mapping and $\alpha \in I^X$ and $\beta \in I^Y$. Then $f(\alpha)$ is a fuzzy set in Y, defined by

$$f(\alpha)(y) = \begin{cases} \sup_{z\in f^{-1}(y)} \alpha(z), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for $y \in Y$.

$f^{-1}(\beta)$ is a fuzzy set in X, defined by $f^{-1}(\beta)(x) = \beta(f(x)), x \in X$.

A *fuzzy topology* [3, 4] on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$ for $\mu_1, \mu_2 \in I^X$.

(3) $\mathcal{T}(\vee \mu_i) \geq \wedge \mathcal{T}(\mu_i)$ for $\mu_i \in I^X$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*. And $\mu \in I^X$ is said to be *fuzzy r-open* (resp., *fuzzy r-closed*) if $\mathcal{T}(\mu) \geq r$ (resp., $\mathcal{T}(\mu^c) \geq r$).

Let A be a fuzzy set in an FTS $(X, \mathcal{T})$ and $r \in (0,1] = I_0$.

The *r-closure* of A, denoted by $cl(A,r)$, is defined as $cl(A,r) = \cap\{B \in I^X : A \subseteq B \text{ and } B \text{ is fuzzy } r\text{-closed}\}$.

The *r-interior* of A, denoted by $int(A,r)$, is defined as $int(A,r) = \cup\{B \in I^X : B \subseteq A \text{ and } B \text{ is fuzzy } r\text{-open}\}$.

Definition 2.1 ([5, 6, 7]). Let A be a fuzzy set in an FTS $(X, \mathcal{T})$ and $r \in (0,1] = I_0$. Then A is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set B in X such that $B \subseteq A \subseteq cl(B,r)$,

(2) *fuzzy r-preopen* if $A \subseteq int(cl(A,r),r)$,

(3) *fuzzy r-regular open* if $A = int(cl(A,r),r)$,

(4) *fuzzy r-strong semiopen* if

$$A \subseteq int(cl(int(A,r),r),r).$$

Let A be a fuzzy set in an FTS $(X, \mathcal{T})$ and $r \in I_0$. The *fuzzy r-strong semi-closure* and the *fuzzy r-strong semi-interior* of A, denoted by $sscl(A,r)$ and $ssint(A,r)$, respectively, are defined as

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$sscl(A,r) = \cap\{B \in I^X : A \subseteq B$ and B is fuzzy r-strong semiclosed$\}$,
$ssint(A,r) = \cup\{B \in I^X : B \subseteq A$ and B is fuzzy r-strong semiopen$\}$.

We have the following relationships.
$int(A,r) \subseteq ssint(A,r) \subseteq A \subseteq sscl(A,r) \subseteq cl(A,r)$

Definition 2.2 ([6, 7, 8]). Let $f : X \to Y$ be a mapping from FTS's $(X,\mathcal{T})$ and $(Y,\mathcal{U})$. Then f is said to be

(1) *fuzzy r-continuous* if for each fuzzy r-open set B of Y, $f^{-1}(B)$ is a fuzzy r-open set in X,

(2) *fuzzy almost r-continuous* if for each fuzzy r-open set B of Y, $f^{-1}(B)$ is a fuzzy r-regular open set in X,

(3) *fuzzy r-semicontinuous* if for each fuzzy r-open set B of Y, $f^{-1}(B)$ is a fuzzy r-semiopen set in X,

(4) *fuzzy strongly r-semicontinuous* if for each fuzzy r-open set B of Y, $f^{-1}(B)$ is a fuzzy strongly r-semiopen set in X,

(5) *fuzzy weakly r-continuous* if for each fuzzy r-open set B of Y, $f^{-1}(B) \subseteq int(f^{-1}(cl(B,r)),r)$,

3. Main Results

Definition 3.1. Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a mapping between FTS's $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ ($r \in I_0$). Then f is said to be *fuzzy α-weakly r-continuous* if for each fuzzy r-open set μ of Y, $f^{-1}(\mu) \subseteq ssint(f^{-1}(cl(\mu,r)),r)$.

Remark 3.2. Every fuzzy strongly r-semicontinuous is fuzzy α-weakly r-continuous mapping but the converse is not always true.

Example 3.3. Let $X = I$ and let β and μ be fuzzy sets of X defined as

$$\beta(x) = -\frac{1}{3}x + \frac{2}{3}, \text{ for } x \in I,$$

$$\mu(x) = \frac{1}{2}x, \text{ for } x \in I.$$

Define a fuzzy topology $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\sigma) = \begin{cases} 1, & \text{if } \sigma = \tilde{0}, \tilde{1}, \\ \frac{1}{2}, & \text{if } \sigma = \beta, \\ 0, & \text{otherwise ;} \end{cases}$$

and a fuzzy topology $\mathcal{U} : I^X \to I$ by

$$\mathcal{U}(\sigma) = \begin{cases} 1, & \text{if } \sigma = \tilde{0}, \tilde{1}, \\ \frac{2}{3}, & \text{if } \sigma = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Note that:

$$int(cl(int(f^{-1}(\mu), \frac{1}{2}), \frac{1}{2})) = \tilde{0};$$

$$ssint(f^{-1}(cl(\mu, \frac{1}{2})), \frac{1}{2}) = \mu^c.$$

Hence the identity mapping $f : (X,\mathcal{T}) \to (X,\mathcal{U})$ is a fuzzy α-weakly $\frac{1}{2}$-continuous mapping but it is not fuzzy strongly $\frac{1}{2}$-semicontinuous.

Theorem 3.4. Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a mapping on FTS's $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ ($r \in I_0$). Then f is a fuzzy α-weakly r-continuous mapping if and only if for every fuzzy point x_α and each fuzzy r-open set V containing $f(x_\alpha)$, there exists a fuzzy r-strong semiopen set U containing x_α such that $f(U) \subseteq cl(V,r)$.

Proof. Suppose f is a fuzzy α-weakly r-continuous mapping. Let x_α be a fuzzy point in X and V a fuzzy r-strong semiopen set containing $f(x_\alpha)$. Then there exists a fuzzy r-open set B such that $f(x_\alpha) \in B \subseteq V$. Since f is a fuzzy α-weakly r-continuous mapping,

$$\begin{aligned} f^{-1}(B) &\subseteq ssint(f^{-1}(cl(B,r)),r) \\ &\subseteq ssint(f^{-1}(cl(V,r)),r). \end{aligned}$$

Set $U = ssint(f^{-1}(cl(V,r)),r)$. Then U is a fuzzy r-strong semiopen set such that $f^{-1}(B) \subseteq U$. So $f(U) \subseteq cl(V,r)$.

For the converse, let V be a fuzzy r-open set in Y. For each $x_\alpha \in f^{-1}(V)$, by hypothesis, there exists a fuzzy r-strong semiopen set U_{x_α} containing x_α such that $f(U_{x_\alpha}) \subseteq cl(V,r)$. Now we can say for each $x_\alpha \in f^{-1}(V)$, $x_\alpha \in U_{x_\alpha} \subseteq f^{-1}(cl(V,r))$.

Thus $\cup\{U_{x_\alpha} : x_\alpha \in f^{-1}(V)\} \subseteq f^{-1}(cl(V,r))$. Since $\cup\{U_{x_\alpha} : x_\alpha \in f^{-1}(V)\}$ is fuzzy r-strong semiopen, we have $f^{-1}(V) \subseteq ssint(f^{-1}(cl(V,r)),r)$. □

Theorem 3.5. Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a mapping on FTS's $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ ($r \in I_0$). Then the following statements are equivalent:

(1) f is fuzzy α-weakly r-continuous.

(2) $sscl(f^{-1}(int(F,r)),r) \subseteq f^{-1}(F)$ for each fuzzy r-closed set F in Y.

(3) $sscl(f^{-1}(int(cl(B,r),r)),r) \subseteq f^{-1}(cl(B,r))$ for each fuzzy set B in Y.

(4) $f^{-1}(int(B,r)) \subseteq ssint(f^{-1}(cl(int(B,r),r)),r)$ for each fuzzy set B in Y.

(5) $sscl(f^{-1}(V),r) \subseteq f^{-1}(cl(V,r))$ for a fuzzy r-open set V in Y.

Proof. (1) ⇒ (2) Let F be any fuzzy r-closed set of Y. Then since $\tilde{1} - F$ is a fuzzy r-open set in Y, from (1), it follows

$$\begin{aligned} f^{-1}(\tilde{1} - F) &\subseteq ssint(f^{-1}(cl(\tilde{1} - F,r)),r) \\ &= ssint(f^{-1}(\tilde{1} - int(F,r)),r) \\ &= ssint(\tilde{1} - f^{-1}(int(F,r)),r) \\ &= \tilde{1} - sscl(f^{-1}(int(F,r)),r). \end{aligned}$$

Hence we have $sscl(f^{-1}(int(F,r)),r) \subseteq f^{-1}(F)$.

(2) ⇒ (3) Let $B \in I^Y$. Since $cl(B,r)$ is a fuzzy r-closed set in Y, by (2), $sscl(f^{-1}(int(cl(B,r),r)) \subseteq f^{-1}(cl(B,r))$.

(3) ⇒ (4) For $B \in I^Y$, from (3), it follows

$$\begin{aligned} f^{-1}(int(B,r) &= \tilde{1} - (f^{-1}(cl(\tilde{1} - B,r))) \\ &\subseteq \tilde{1} - sscl(f^{-1}(int(cl(\tilde{1} - B,r),r)),r) \\ &= ssint(f^{-1}(cl(int(B,r),r)),r). \end{aligned}$$

Hence we have $f^{-1}(int(B,r)) \subseteq ssint(f^{-1}(cl(int(B,r),r)),r)$.

(4) ⇒ (5) Let V be any fuzzy r-open set of Y. Then from (4) and $(V,r) \subseteq int(cl(V,r),r))$, it follows

$$\begin{aligned} &\tilde{1} - f^{-1}(cl(V,r)) \\ &= f^{-1}(int(\tilde{1} - V,r)) \\ &\subseteq ssint(f^{-1}(cl(int(\tilde{1} - V,r),r)),r) \\ &= ssint(\tilde{1} - (f^{-1}(int(cl(V,r),r))),r) \\ &= \tilde{1} - sscl(f^{-1}(int(cl(V,r),r)),r) \\ &\subseteq \tilde{1} - sscl(f^{-1}(V),r). \end{aligned}$$

Hence $sscl(f^{-1}(V),r) \subseteq f^{-1}(cl(V,r))$.

(5) ⇒ (1) Let V be a fuzzy r-open set in Y. From $(V,r) \subseteq int(cl(V,r),r)$ and (5), we have

$$\begin{aligned} f^{-1}(V) &\subseteq f^{-1}(int(cl(V,r),r)) \\ &= \tilde{1} - f^{-1}(cl(\tilde{1} - cl(V,r),r)) \\ &\subseteq \tilde{1} - sscl(f^{-1}(\tilde{1} - cl(V,r)),r) \\ &= ssint(f^{-1}(cl(V,r)),r). \end{aligned}$$

Hence f is a fuzzy α-weakly r-continuous mapping. □

Theorem 3.6. Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a mapping on FTS's $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ ($r \in I_0$). Then f is fuzzy α-weakly r-continuous if and only if $sscl(f^{-1}(int(cl(V,r),r)),r) \subseteq f^{-1}(cl(V,r))$ for each fuzzy r-preopen set V in Y.

Proof. Assume f is fuzzy α-weakly r-continuous. Let V be a fuzzy r-preopen of Y. Then $V \subseteq int(cl(V,r),r)$. Set $A = int(cl(V,r),r)$. Since A is a fuzzy r-open set, by Theorem 3.5 (5), we have

$$sscl(f^{-1}(int(cl(A,r),r)),r,s) \subseteq f^{-1}(cl(A,r)).$$

From $cl(A,r) = cl(V,r)$, it follows

$$sscl(f^{-1}(int(cl(V,r),r)),r) \subseteq f^{-1}(cl(V,r)).$$

For the converse, let G be a fuzzy r-open set of Y. Then since G is a fuzzy r-preopen set, we have $sscl(f^{-1}(G),r) \subseteq sscl(f^{-1}(int(cl(G,r),r)),r) \subseteq f^{-1}(cl(G,r))$.

Hence, by Theorem 3.5 (5), f is a fuzzy α-weakly r-continuous mapping. □

Theorem 3.7. Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a mapping on FTS's $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ ($r \in I_0$). Then f is fuzzy α-weakly r-continuous if and only if $sscl(f^{-1}(int(cl(G,r),r)),r) \subseteq f^{-1}(cl(G,r))$ for each fuzzy r-semiopen set G in Y.

Proof. Assume f is fuzzy α-weakly r-continuous. Let V be a fuzzy r-semiopen in Y. Then $V \subseteq cl(int(V,r),r)$. Set $F = cl(int(V,r),r)$. Since F is a fuzzy r-closed set, by Theorem 3.5 (2), we have

$$sscl(f^{-1}(int(F,r)),r) \subseteq f^{-1}(F).$$

From $cl(V,r) = cl(int(V,r),r) = F$, it follows

$$sscl(f^{-1}(int(cl(V,r),r)),r) \subseteq f^{-1}(cl(V,r)).$$

For the converse, let G be a fuzzy r-open set of Y. Then since G is a fuzzy r-semiopen set, by hypothesis and Theorem 3.5 (5), f is a fuzzy α-weakly r-continuous mapping. □

Theorem 3.8 ([8]). Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a mapping on FTS's $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ ($r \in I_0$). Then f is fuzzy strongly r-semicontinuous if and only if $cl(int(cl(f^{-1}(cl(G,r)),r),r),r) \subseteq f^{-1}(cl(G,r))$ for each fuzzy set G in Y.

Theorem 3.9. Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a mapping on FTS's $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ ($r \in I_0$). Then if f is fuzzy strongly r-semicontinuous, then it is fuzzy weakly r-continuous.

Proof. Let B be fuzzy r-open in Y. Since f is fuzzy strongly r-semicontinuous, $f^{-1}(B)$ is fuzzy strongly r-semiopen and $f^{-1}(cl(B,r))$ is fuzzy strongly r-semiclosed in X. Thus from Theorem 3.8, it follows

$$\begin{aligned} f^{-1}(B) &\subseteq int(cl(int(f^{-1}(B),r),r),r) \\ &\subseteq cl(int(cl(f^{-1}(cl(B,r)),r),r),r) \\ &\subseteq f^{-1}(cl(B,r)). \end{aligned}$$

This implies $f^{-1}(B) \subseteq int(f^{-1}(cl(B,r)),r)$. Hence f is fuzzy weakly r-continuous. □

Example 3.10. Let $X = I$ and let A and B be fuzzy sets defined as follows

$$A(x) = \frac{1}{4}x + \frac{3}{4}, \text{ for all } \ x \in I;$$

$$B(x) = \frac{1}{4}x + \frac{1}{4}, \text{ for all } \ x \in I.$$

Define fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X as follows.

$$\mathcal{T}_1(\sigma) = \begin{cases} 1, & \text{if } \sigma = \tilde{0}, \tilde{1}, \\ \frac{1}{2}, & \text{if } \sigma = B, \\ 0, & \text{otherwise ;} \end{cases}$$

$$\mathcal{T}_2(\sigma) = \begin{cases} 1, & \text{if } \sigma = \tilde{0}, \tilde{1}, \\ \frac{2}{3}, & \text{if } \sigma = A, \\ 0, & \text{otherwise.} \end{cases}$$

Then the identity mapping $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a fuzzy weakly $\frac{1}{2}$-continuous mapping but it is not fuzzy strongly $\frac{1}{2}$-semicontinuous.

Lemma 3.11. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a mapping on FTS's $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ ($r \in I_0$). Then if f is fuzzy weakly r-continuous mapping, then it is fuzzy α-weakly r-continuous.

Proof. Obvious. □

Example 3.12. Let $X = I$ and let A,B, C and D be fuzzy sets defined as follows.

$$A(x) = \frac{1}{3};$$

$$B(x) = \frac{8}{9};$$

$$C(x) = \frac{2}{9};$$

$$D(x) = -\frac{1}{18}(5x - 14), \text{ for all } \ x \in I.$$

Consider two fuzzy topologies $\mathcal{T}_1$, $\mathcal{T}_2$:

$$\mathcal{T}_1(\sigma) = \begin{cases} 1, & \text{if } \sigma = \tilde{0}, \tilde{1}, \\ \frac{2}{3}, & \text{if } \sigma = A, \\ \frac{1}{2}, & \text{if } \sigma = B, \\ 0, & \text{otherwise ;} \end{cases}$$

$$\mathcal{T}_2(\sigma) = \begin{cases} 1, & \text{if } \sigma = \tilde{0}, \tilde{1}, \\ \frac{1}{2}, & \text{if } \sigma = C, \\ \frac{2}{3}, & \text{if } \sigma = D, \\ 0, & \text{otherwise.} \end{cases}$$

Then the identity mapping $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a fuzzy α-weakly $\frac{1}{2}$-continuous mapping but it is not fuzzy weakly $\frac{1}{2}$-continuous.

Finally we get the following implications:

fuzzy r-continuous $\Rightarrow$ fuzzy strongly r-semicontinuous $\Rightarrow$ fuzzy weakly r-continuous $\Rightarrow$ fuzzy α-weakly r-continuous

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