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On Fuzzy S-Weakly r-Continuous Mappings

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Abstract

In this paper, we introduce the concept of fuzzy S-weakly r-continuous mapping on an fuzzy topological space and investigate some properties of such mappings.

Key words : fuzzy S-weakly r-continuous, fuzzy weakly r-semicontinuous, fuzzy weakly r-continuous

1. Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Chang [1] defined fuzzy topological spaces using fuzzy sets. In [3, 4], Chattopadhyay, Hazra and Samanta introduced the concept of smooth fuzzy topological spaces which are a generalization of fuzzy topological spaces.

Lee and Kim [8] introduced the concept of fuzzy weakly r-semicontinuous mappings in fuzzy topological spaces defined by Chattopadhyay. And they showed that there is no any relationship between fuzzy weakly r-semicontinuous mappings and fuzzy weakly r-continuous mappings. In this paper, we introduce fuzzy S-weakly r-continuous mappings on the fuzzy topological space and study some properties. In particular, we show that every fuzzy weakly r-semicontinuous mapping (or fuzzy weakly r-continuous mapping) is fuzzy S-weakly r-continuous but the converse may not be true.

2. Preliminaries

Let I be the unit interval $[0,1]$ of the real line. A member μ of I^X is called a fuzzy set of X. For any $\mu \in I^X$, μ^c denotes the complement $\tilde{1}-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

An *fuzzy point* x_α in X is a fuzzy set x_α defined by

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point x_α is said to belong to an fuzzy set A in X, denoted by $x_\alpha \in A$, if $\alpha \leq A$ for $x \in X$.

A fuzzy set A in X is the union of all fuzzy points which belong to A.

Let $f : X \to Y$ be a mapping and $\alpha \in I^X$ and $\beta \in I^Y$. Then $f(\alpha)$ is a fuzzy set in Y, defined by

$$f(\alpha)(y) = \begin{cases} \sup_{z\in f^{-1}(y)} \alpha(z), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise ,} \end{cases}$$

for $y \in Y$.

$f^{-1}(\beta)$ is a fuzzy set in X, defined by $f^{-1}(\beta)(x) = \beta(f(x)), x \in X$.

A *fuzzy topology* [3, 4] on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$ for $\mu_1, \mu_2 \in I^X$.

(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$ for $\mu_i \in I^X$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*. And $\mu \in I^X$ is said to be *fuzzy r-open* (resp., *fuzzy r-closed*) if $\mathcal{T}(\mu) \geq r$ (resp., $\mathcal{T}(\mu^c) \geq r$).

The *r-closure* of A, denoted by $cl(A,r)$, is defined as $cl(A,r) = \cap\{B \in I^X : A \subseteq B$ and B is fuzzy r-closed$\}$.

The *r-interior* of A, denoted by $int(A,r)$, is defined as $int(A,r) = \cup\{B \in I^X : B \subseteq A$ and B is fuzzy r-open$\}$.

Definition 2.1. ([5, 6, 7]) Let A be a fuzzy set in a FTS $(X, \mathcal{T})$ and $r \in (0,1] = I_0$. Then A is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set B in X such that $B \subseteq A \subseteq cl(B,r)$,

(2) *fuzzy r-preopen* if $A \subseteq int(cl(A,r),r)$,

(3) *fuzzy r-regular open* if $A = int(cl(A,r),r)$.

Let A be a fuzzy set in a FTS $(X, \mathcal{T})$ and $r \in I_0$. The *fuzzy r-semi-closure* of A, denoted by $scl(A,r)$ is defined as

$$scl(A,r) = \cap\{B \in I^X : A \subseteq B \text{ and } B \text{ is fuzzy } r\text{-semiclosed}\}.$$

The *fuzzy r-semi-interior* of A, denoted by $sint(A,r)$, is defined as

$$sint(A,r) = \cup\{B \in I^X : B \subseteq A \text{ and } B \text{ is fuzzy } r\text{-semiopen}\}.$$

Note that:

$$int(A,r) \subseteq sint(A,r) \subseteq A \subseteq scl(A,r) \subseteq cl(A,r).$$

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Definition 2.2. ([6, 7, 8]) Let $f : X \to Y$ be a mapping from FTS's X and Y. Then f is said to be

(1) a *fuzzy r-continuous* if for each fuzzy r-open set B of Y, $f^{-1}(B)$ is a fuzzy r-open set in X,

(2) a *fuzzy r-semicontinuous* if for each fuzzy r-open set B of Y, $f^{-1}(B)$ is a fuzzy r-semiopen set in X,

(3) a *fuzzy almost r-continuous* if for each fuzzy r-open set B of Y, $f^{-1}(B)$ is a fuzzy r-regular open set in X,

(4) a *fuzzy weakly r-continuous* if for each fuzzy r-open set B of Y, $f^{-1}(B) \subseteq int(f^{-1}(cl(B, r)), r)$,

(5) a *fuzzy weakly r-semicontinuous* if for each fuzzy r-open set B of Y, $f^{-1}(B) \subseteq sint(f^{-1}(scl(B, r)), r)$.

3. Main Results

Definition 3.1. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a mapping between FTS's X and Y ($r \in I_0$). Then f is said to be *fuzzy S-weakly r-continuous* if for each fuzzy r-open set μ of Y, $f^{-1}(\mu) \subseteq sint(f^{-1}(cl(\mu, r)), r)$.

Remark 3.2. Every fuzzy weakly r-continuous mapping is fuzzy S-weakly r-continuous but the converse is not always true.

Example 3.3. Let $X = I$ and let β and μ be fuzzy sets of X defined as

$$\beta(x) = -\frac{1}{3}x + \frac{2}{3} \text{ for } x \in I,$$

$$\mu(x) = \frac{1}{2}x \text{ for } x \in I.$$

Define a fuzzy topology $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\sigma) = \begin{cases} 1 & \text{if } \sigma = \tilde{0}, \tilde{1} \\ \frac{1}{2} & \text{if } \sigma = \beta \\ 0 & \text{otherwise ;} \end{cases}$$

and a fuzzy topology $\mathcal{U} : I^X \to I$ by

$$\mathcal{U}(\sigma) = \begin{cases} 1 & \text{if } \sigma = \tilde{0}, \tilde{1}, \\ \frac{2}{3} & \text{if } \sigma = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Consider a mapping $f : (X, \mathcal{T}) \to (X, \mathcal{U})$ defined as follows: $f(x) = x$ for all $x \in X$.

Since

$$int(f^{-1}(cl(\mu, \frac{1}{2})), \frac{1}{2}) = int(f^{-1}(\mu^c), \frac{1}{2}) = \beta$$

$$sint(f^{-1}(cl(\mu, \frac{1}{2})), \frac{1}{2}) = sint(f^{-1}(\mu^c), \frac{1}{2}) = \mu^c,$$

clearly f is a fuzzy S-weakly $\frac{1}{2}$-continuous mapping but it is not fuzzy weakly $\frac{1}{2}$-continuous.

Remark 3.4. Every fuzzy weakly r-semicontinuous mapping is fuzzy S-weakly r-continuous but the converse is not always true.

Example 3.5. In the above Example 3.3, the identity mapping f is fuzzy S-weakly r-continuous. But f is not fuzzy weakly $\frac{1}{2}$-semicontinuous because $cl(\mu, \frac{1}{2}) = scl(\mu, \frac{1}{2})$ in $(X, \mathcal{U})$.

Now get the following implications:

f. r-cont. $\rightarrow$ f. weakly r-cont.

$\searrow$

$\downarrow$ $\quad$ $\not\downarrow$ $\not\uparrow$ $\quad$ f. S-weakly r-cont.

$\nearrow$

f. r-semicont. $\rightarrow$ f. weakly r-semicont.

Theorem 3.6. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a mapping on FTS's $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ ($r \in I_0$). Then f is a fuzzy S-weakly r-continuous mapping if and only if for every fuzzy point x_α and each fuzzy r-open set V containing $f(x_\alpha)$, there exists a fuzzy r-semiopen set U containing x_α such that $f(U) \subseteq cl(V, r)$.

Proof. Suppose f is a fuzzy S-weakly r-continuous mapping. Let x_α be a fuzzy point in X and V a fuzzy r-semiopen set containing $f(x_\alpha)$; then there exists a fuzzy r-open set B such that $f(x_\alpha) \in B \subseteq V$. Since f is a fuzzy S-weakly r-continuous mapping,

$$f^{-1}(B) \subseteq sint(f^{-1}(cl(B, r)), r) \subseteq sint(f^{-1}(cl(V, r)), r).$$

Set $U = sint(f^{-1}(cl(V, r)), r)$; then U is a fuzzy r-semiopen set such that $f^{-1}(B) \subseteq U$. So $f(U) \subseteq cl(V, r)$.

For the converse, let V be a fuzzy r-open set in Y. For each $x_\alpha \in f^{-1}(V)$, by hypothesis, there exists a fuzzy r-semiopen set U_{x_α} containing x_α such that $f(U_{x_\alpha}) \subseteq cl(V, r)$. Now we can say for each $x_\alpha \in f^{-1}(V)$,

$$x_\alpha \in U_{x_\alpha} \subseteq f^{-1}(cl(V, r)).$$

Thus $\cup\{U_{x_\alpha} : x_\alpha \in f^{-1}(V)\} \subseteq f^{-1}(cl(V, r))$.

Since $\cup\{U_{x_\alpha} : x_\alpha \in f^{-1}(V)\}$ is fuzzy r-semiopen, we have $f^{-1}(V) \subseteq sint(f^{-1}(cl(V, r)), r)$. □

Theorem 3.7. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a mapping on FTS's $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ ($r \in I_0$). Then the following statements are equivalent:

(1) f is fuzzy S-weakly r-continuous.

(2) $scl(f^{-1}(int(F, r)), r) \subseteq f^{-1}(F)$ for each fuzzy r-closed set F in Y.

(3) $scl(f^{-1}(int(cl(B, r), r)), r) \subseteq f^{-1}(cl(B, r))$ for each fuzzy set B in Y.

(4) $f^{-1}(int(B, r)) \subseteq sint(f^{-1}(cl(int(B, r), r)), r)$ for each fuzzy set B in Y.

(5) $scl(f^{-1}(V), r) \subseteq f^{-1}(cl(V, r))$ for a fuzzy r-open set V in Y.

Proof. (1) $\Rightarrow$ (2) Let F be any fuzzy r-closed set of Y. Then since $\tilde{1} - F$ is a fuzzy r-open set in Y,

$$\begin{aligned} f^{-1}(\tilde{1} - F) &\subseteq sint(f^{-1}(cl(\tilde{1} - F, r)), r) \\ &= sint(f^{-1}(\tilde{1} - int(F, r)), r) \\ &= sint(\tilde{1} - f^{-1}(int(F, r)), r) \\ &= \tilde{1} - scl(f^{-1}(int(F, r)), r). \end{aligned}$$

Hence we have $scl(f^{-1}(int(F, r)), r) \subseteq f^{-1}(F)$.

(2) $\Rightarrow$ (3) Let $B \in I^Y$. Since $cl(B, r)$ is a fuzzy r-closed set in Y, by (2),

$$scl(f^{-1}(int(cl(B, r), r)) \subseteq f^{-1}(cl(B, r)).$$

(3) $\Rightarrow$ (4) For $B \in I^Y$,

$$\begin{aligned} & f^{-1}(int(B, r)) = \tilde{1} - (f^{-1}(cl(\tilde{1} - B, r))) \\ & \subseteq \tilde{1} - scl(f^{-1}(int(cl(\tilde{1} - B, r), r)), r) \\ & = sint(f^{-1}(cl(int(B, r), r)), r). \end{aligned}$$

Hence,

$$f^{-1}(int(B, r)) \subseteq sint(f^{-1}(cl(int(B, r), r)), r).$$

(4) $\Rightarrow$ (5) Let V be any fuzzy r-open set of Y. Then from $(V, r) \subseteq int(cl(V, r), r)$, it follows

$$\begin{aligned} & \tilde{1} - f^{-1}(cl(V, r)) = f^{-1}(int(\tilde{1} - V, r)) \\ & \subseteq sint(f^{-1}(cl(int(\tilde{1} - V, r), r)), r) \\ & = sint(\tilde{1} - (f^{-1}(int(cl(V, r), r))), r) \\ & = \tilde{1} - scl(f^{-1}(int(cl(V, r), r)), r) \\ & \subseteq \tilde{1} - scl(f^{-1}(V), r). \end{aligned}$$

Hence we have $scl(f^{-1}(V), r) \subseteq f^{-1}(cl(V, r))$.

(5)$\Rightarrow$ (1) Let V be a fuzzy r-open set in Y. From $(V, r) \subseteq int(cl(V, r), r)$, we have

$$\begin{aligned} f^{-1}(V) &\subseteq f^{-1}(int(cl(V, r), r)) \\ &= \tilde{1} - f^{-1}(cl(\tilde{1} - cl(V, r), r)) \\ &\subseteq \tilde{1} - scl(f^{-1}(\tilde{1} - cl(V, r)), r) \\ &= sint(f^{-1}(cl(V, r)), r). \end{aligned}$$

Hence f is a fuzzy S-weakly r-continuous mapping. □

Theorem 3.8. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a mapping on FTS's $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ ($r \in I_0$). Then the following statements are equivalent:

(1) f is fuzzy S-weakly r-continuous.

(2) $scl(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r))$ for each fuzzy r-open set G in Y.

(3) $scl(f^{-1}(int(cl(V, r), r)), r) \subseteq f^{-1}(cl(V, r))$ for each fuzzy r-preopen set V in Y.

(4) $scl(f^{-1}(int(K, r)), r) \subseteq f^{-1}(K)$ for each fuzzy r-regular closed set K in Y.

(5) $scl(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r))$ for each fuzzy r-semiopen set G in Y.

Proof. (1) $\Rightarrow$ (2) Let G be a fuzzy r-open set of Y; then by Theorem 3.7 (3), we have $scl(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r))$.

(2) $\Rightarrow$ (3) Let V be a fuzzy r-preopen of Y. Then $V \subseteq int(cl(V, r), r)$. Set $A = int(cl(V, r), r)$. Since A is a fuzzy r-open set, it follows

$$scl(f^{-1}(int(cl(A, r), r)), r, s) \subseteq f^{-1}(cl(A, r)).$$

Since $cl(A, r) = cl(V, r)$, $scl(f^{-1}(int(cl(V, r), r)), r) \subseteq f^{-1}(cl(V, r))$.

(3)$\Rightarrow$ (4) Let K be a fuzzy r-regular closed set of Y. Since $int(K, r)$ is a fuzzy r-preopen set,

$$scl(f^{-1}(int(cl(int(K, r), r), r)), r) \subseteq f^{-1}(cl(int(K, r), r)).$$

From $int(K, r) = int(cl(int(K, r), r), r)$ and $K = cl(int(K, r), r)$, it follows $scl(f^{-1}(int(K, r)), r) \subseteq f^{-1}(K)$.

(4) $\Rightarrow$ (5) Let G be a fuzzy r-semiopen set. Then $G \subseteq cl(int(G, r), r) \subseteq cl(int(cl(G, r), r), r) \subseteq cl(G, r)$, and so $cl(G, r)$ is a fuzzy r-regular closed set. Hence we have $scl(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r))$.

(5) $\Rightarrow$ (1) Let V be a fuzzy r-open set; then since V is a fuzzy r-semiopen set, from $V \subseteq int(cl(V, r), r)$, we have

$$\begin{aligned} scl(f^{-1}(V), r) &\subseteq scl(f^{-1}(int(cl(V, r), r)), r) \\ &\subseteq f^{-1}(cl(V, r)). \end{aligned}$$

Hence, by Theorem 3.7 (5), f is a fuzzy S-weakly r-continuous mapping. □

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