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The existence and uniqueness of fuzzy solutions for semilinear fuzzy integrodifferential equations using integral contractor

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Abstract

In this paper, we investigate the existence and uniqueness of fuzzy solutions for semilinear fuzzy integrodifferential equations using integral contractor. The notion of 'bounded integral contractor', introduced by Altman [1], is weaker than Lipschitz condition.

Key words : fuzzy solution, semilinear fuzzy integrodifferential equation, integral contractor, fuzzy number

1. Introduction

Many authors have studied several concepts of fuzzy systems using Lipschitz condition. Kaleva [4] studied the existence and uniqueness of solution for the fuzzy differential equation on E^n where E^n is normal, convex, upper semicontinuous and compactly supported fuzzy sets in R^n. Seikkala [8] proved the existence and uniqueness of fuzzy solution for the following equation:

$$\dot{x}(t) = f(t, x(t)) \ , \ \ x(0) = x_0,$$

where f is a continuous mapping from $R^+ \times R$ into R and x_0 is a fuzzy number in E^1. Diamond and Kloeden [3] proved the fuzzy optimal control for the following system:

$$\dot{x}(t) = a(t)x(t) + u(t), \ \ x(0) = x_0,$$

where $x(\cdot), u(\cdot)$ are nonempty compact interval-valued functions on E^1. Kwun and Park [5] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in E^1_N using by Kuhn-Tucker theorems. Balasubramaniam and Muralisankar [2] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Recently, Park, Park and Kwun [7] find the sufficient condition of nonlocal controllability for the semilinear fuzzy integrodifferential equation with nonlocal initial condition.

In this paper, we study the existence and uniqueness of solutions for the semilinear fuzzy integrodifferential equations by replacing Lipschitz condition with integral contractor condition

$$\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] \qquad (1)$$
$$+f(t, x(t)), \ t \in I = [0, T],$$
$$x(0) = x_0 \in E_N, \qquad (2)$$

where $A : I \to E_N$ is a fuzzy coefficient, E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α-level intervals, $f : I \times E_N \to E_N$ is nonlinear continuous functions, $G(t)$ is $n \times n$ continuous matrix such that $\frac{dG(t)x}{dt}$ is continuous for $x \in E_N$ and $t \in I$ with $\|G(t)\| \leq k$, $k > 0$.

2. Preliminaries

A fuzzy subset of R^n is defined in terms of membership function which assigns to each point $x \in R^n$ a grade of membership in the fuzzy set. Such a membership function $m : R^n \to [0, 1]$ is used synonymously to denote the corresponding fuzzy set. We shall restrict attention here to the normal fuzzy sets which satisfy

Assumption 1. m maps R^n onto $[0, 1]$.
Assumption 2. $[m]^0$ is a bounded subset of R^n.
Assumption 3. m is upper semicontinuous.
Assumption 4. m is fuzzy convex.

We denote by E^n the space of all fuzzy subsets m of R^n which satisfy assumptions 1-4; that is, normal, fuzzy convex and upper semicontinuous fuzzy sets with bounded

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supports. In particular, we denoted by E^1 the space of all fuzzy subsets m of R which satisfy assumptions 1-4 [3].

A fuzzy number a in real line R is a fuzzy set characterized by a membership function m_a as $m_a : R \to [0,1]$. A fuzzy number a is expressed as $a = \int_{x\in R} m_a(x)/x$, with the understanding that $m_a(x) \in [0,1]$ represent the grade of membership of x in a and $\int$ denotes the union of $m_a(x)/x$'s [6].

Let E_N be the set of all upper semicontinuous convex normal fuzzy number with bounded α-level intervals. This means that if $a \in E_N$ then the α-level set

$$[a]^\alpha = \{x \in R : m_a(x) \geq \alpha,\ 0 < \alpha \leq 1\}$$

is a closed bounded interval which we denote by

$$[a]^\alpha = [a_l^\alpha, a_r^\alpha]$$

and there exists a $t_0 \in R$ such that $a(t_0) = 1$ [5].

The support Γ_a of a fuzzy number a is defined, as a special case of level set, by the following

$$\Gamma_a = \{x \in R : m_a(x) > 0\}.$$

Two fuzzy numbers a and b are called equal $a = b$, if $m_a(x) = m_b(x)$ for all $x \in R$. It follows that

$$a = b \Leftrightarrow [a]^\alpha = [b]^\alpha \text{ for all } \alpha \in (0,1].$$

A fuzzy number a may be decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha[a]^\alpha,$$

where $\alpha[a]^\alpha$ is the product of a scalar α with the set $[a]^\alpha$ and $\int$ is the union of $[a]^\alpha$'s with α ranging from 0 to 1.

We denote the suprimum metric d_∞ on E^n and the suprimum metric H_1 on $C(I : E^n)$.

Definition 1. Let $a, b \in E^n$.

$$d_\infty(a,b) = \sup\{d_H([a]^\alpha, [b]^\alpha) : \alpha \in (0,1]\},$$

where d_H is the Hausdorff distance.

Definition 2. Let $x, y \in C(I : E^n)$

$$H_1(x,y) = \sup\{d_\infty(x(t), y(t)) : t \in I\}.$$

Let I be a real interval. A mapping $x : I \to E_N$ is called a fuzzy process. We denote

$$[x(t)]^\alpha = [x_l^\alpha(t), x_r^\alpha(t)],\ t \in I,\ 0 < \alpha \leq 1.$$

The derivative $x'(t)$ of a fuzzy process x is defined by

$$[x'(t)]^\alpha = [(x_l^\alpha)'(t), (x_r^\alpha)'(t)],\ \ 0 < \alpha \leq 1$$

provided that is equation defines a fuzzy $x'(t) \in E_N$.

The fuzzy integral

$$\int_a^b x(t)dt, \quad a, b \in I$$

is defined by

$$\left[\int_a^b x(t)dt\right]^\alpha = \left[\int_a^b x_l^\alpha(t)dt, \int_a^b x_r^\alpha(t)dt\right]$$

provided that the Lebesgue integrals on the right exist.

Definition 3. [2] The fuzzy process $x : I \to E_N$ is a solution of equations (1)-(2) without the inhomogeneous term if and only if

$$\begin{aligned}(\dot{x}_l^\alpha)(t) = \min\Big\{&A_i^\alpha(t)\big[x_j^\alpha(t) \\ &+ \int_0^t G(t-s)x_j^\alpha(s)ds\big], i,j = l,r\Big\},\\ (\dot{x}_r^\alpha)(t) = \max\Big\{&A_i^\alpha(t)\big[x_j^\alpha(t) \\ &+ \int_0^t G(t-s)x_j^\alpha(s)ds\big], i,j = l,r\Big\},\end{aligned}$$

and

$$(x_l^\alpha)(0) = x_{0l}^\alpha, \quad (x_r^\alpha)(0) = x_{0r}^\alpha.$$

Now we assume the following:

(H1) $S(t)$ is a fuzzy number satisfying, for $y \in E_N$ and $S'(t)y \in C^1(I : E_N) \cap C(I : E_N)$, the equation

$$\begin{aligned}\frac{d}{dt}S(t)y &= A\left[S(t)y + \int_0^t G(t-s)S(s)yds\right]\\ &= S(t)Ay + \int_0^t S(t-s)AG(s)yds,\ \ t \in I,\end{aligned}$$

such that

$$[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)],$$

and $S_i^\alpha(t)$ $(i = l, r)$ is continuous. That is, there exists a constant $c > 0$ such that $|S_i^\alpha(t)| \leq c$ for all $t \in I$.

3. Existence and Uniqueness

In this section, we consider the existence and uniqueness of fuzzy solution for the equations (1)-(2).

The equations (1)-(2) is related to the following fuzzy integral equation:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s))ds, \qquad (3)$$

where $S(t)$ is satisfy (H1).

Definition 4. Suppose $\Gamma : [0,T] \times E_N \to E_N$ is a bounded continuous function and there exists a positive number k such that for any x and y in $C([0,T] : E_N)$ and $t \in [0,T]$

$$\begin{aligned} & d_H\Big(\big[f(t,x(t)+y(t)+\int_0^t S(t-s)\Gamma(s,x(s)) \\ & \times y(s)ds) - f(t,x(t)) - \Gamma(t,x(t))y(t)\big]^\alpha, \{0\}\Big) \\ \leq \ & kd_H([y(t)]^\alpha, \{0\}). \end{aligned} \qquad (4)$$

Then we say that $f(t,x(t))$ has a bounded integral contractor $\{I + \int S\Gamma\}$ with respect to $S(t-s)$.

Remark If $\Gamma \equiv 0$, the condition (4) reduces to the Lipschitz condition. It should be remarked here that the Lipschitz condition gives rise to a unique solution, where as the condition (4) does not yield, in general, to a unique solution. Therefore, we define the regularity of integral contractors which ensures the uniqueness of the solution.

Definition 5. A bounded integral contractor Γ is said to be regular if the integral equation

$$y(t) + \int_0^t S(t-s)\Gamma(s,x(s))y(s)ds = z(t) \qquad (5)$$

has a solution in $C([0,T] : E_N)$ for any $x, z \in C([0,T] : E_N)$.

Now we prove the existence and uniqueness of solution of (3) using the idea of integral contractors.

Theorem 1. Let $T > 0$ and hypothesis (H) hold and the nonlinear function $f(t,x(t))$ has a regular integral contractor. Then for every $x_0 \in E_N$, the equation (3) has a unique fuzzy solution $x \in C([0,T] : E_N)$.

Proof. The main idea is to use the following iteration procedure to define

$$x_0(t) = S(t)x_0, \qquad (6)$$

$$\begin{aligned} x_{n+1}(t) = x_n(t) - \big[y_n(t) \qquad (7) \\ + \int_0^t S(t-s)\Gamma(s,x_n(s))y_n(s)ds\big], \quad n \geq 0 \end{aligned}$$

$$y_n(t) = x_n(t) - \int_0^t S(t-s)f(s,x_n(s))ds - x_0(t). \quad (8)$$

Hence, it follows from (7) and (8) that

$$\begin{aligned} y_{n+1}(t) &= x_n(t) - y_n(t) \\ &\quad - \int_0^t S(t-s)\Gamma(s,x_n(s))ds \\ &\quad - \int_0^t S(t-s)f(s,x_{n+1}(s))ds - x_0(t) \\ &= \int_0^t S(t-s)[f(s,x_n(s)) - f(s,x_{n+1}(s)) \\ &\quad -\Gamma(s,x_n(s))y_n(s)]ds \\ &= -\int_0^t S(t-s)\Big[f\Big(s,x_n(s) - y_n(s) \\ &\quad - \int_0^s S(s-\tau)\Gamma(\tau,x_n(\tau))y_n(\tau)d\tau\Big) \\ &\quad -f(s,x_n(s)) + \Gamma(s,x_n(s))y_n(s)\Big]ds \end{aligned}$$

Now apply to Definition 4 with $x_n = x$ and $y_n = -y$, we have that

$$d_H([y_{n+1}(t)]^\alpha, \{0\}) \leq ck \int_0^t d_H([y_n(s)]^\alpha, \{0\})ds.$$

We can derive inductively the following inequality

$$\begin{aligned} & d_H([y_{n+1}(t)]^\alpha, \{0\}) \\ \leq \ & \frac{(ck)^n}{n!}\int_0^t s^n d_H([S(t-s)f(s,x_0(s))]^\alpha, \{0\})ds. \end{aligned}$$

Therefore, we get

$$d_H([y_{n+1}(t)]^\alpha, \{0\}) \leq \frac{\beta(ckT)^n cT}{(n+1)!} \qquad (9)$$

where $\beta = d_H([f(t,x_0(t))]^\alpha, \{0\})$, $t \in [0,T]$. Hence $y_n(\cdot)$ converges to 0 in $C([0,T] : E_N)$, as $n \to \infty$.

From (7) we have

$$\begin{aligned} & x_{n+1}(t) - x_n(t) \\ = \ & -y_n(t) - \int_0^t S(t-s)\Gamma(s,x_n(s))y_n(s)ds. \end{aligned}$$

Using (9) we have the estimate

$$d_H([x_{n+1}(t) - x_n(t)]^\alpha, \{0\}) \leq \frac{\beta(ckT)^n cT}{(n+1)!}(1 + \gamma cT)$$

where $\gamma = d_H([\Gamma(t,x(t))]^\alpha, \{0\})$, $t \in [0,T]$. Therefore x_n converges to x^* in $C[0,T] : E_N)$ and by (8) we have that

$$x^*(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x^*(s))ds,$$

That is, x^* is a solution of (3).

We now show the uniqueness of solutions by the regularity of integral contractor. Let x_1 and x_2 be two solutions of (3) with a given x_0. By the regularity condition (5) with $x = x_1$ and $z = x_2 - x_1$, there exists $y \in C[0,T] : E_N)$ such that

$$\begin{aligned} & y(t) + \int_0^t S(t-s)\Gamma(s, x_1(s))y(s)ds \quad (10) \\ = \;& x_2(t) - x_1(t). \end{aligned}$$

By (4) we have

$$\begin{aligned} & d_H\Big(\big[f(t, x_1(t) + y(t) + \int_0^t S(t-s)\Gamma(s, x_1(s)) \\ & \times y(s)ds) - f(t, x_1(t)) - \Gamma(t, x_1(t))y(t)\big]^\alpha, \{0\}\Big) \\ \leq \;& k d_H([y(t)]^\alpha, \{0\}). \end{aligned}$$

By (10) we get

$$\begin{aligned} & d_H([f(t, x_2(t)) - f(t, x_1(t)) \\ & \qquad -\Gamma(t, x_1(t))y(t)]^\alpha, \{0\}) \\ \leq \;& k d_H([y(t)]^\alpha, \{0\}). \qquad (11) \end{aligned}$$

Again from (10) and (3) we have

$$\begin{aligned} y(t) &= x_2(t) - x_1(t) - \int_0^t S(t-s)\Gamma(s, x_1(s))y(s)ds \\ &= \int_0^t S(t-s)[f(s, x_2(s)) - f(s, x_1(s)) \\ & \qquad -\Gamma(s, x_1(s))y(s)]ds \end{aligned}$$

Now (11) implies

$$d_H([y(t)]^\alpha, \{0\}) \leq ck \int_0^t d_H([y(s)]^\alpha, \{0\})ds.$$

Then by using Grownwall's inequality we get $d_H([y(t)]^\alpha, \{0\}) = 0$. Therefore, by (10) we have that $x_1 = x_2$.

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