International Journal of Fuzzy Logic and Intelligent Systems, vol. 9, no. 4, December 2009 pp. 285−293

Fuzzzy Functions and Fuzzy Partially Ordered Sets

KUL HUR[1], HYO MI JUNG[2] AND Wang Ro LEE[3]

[1,2]Division of Mathematics and Informational Statistics, and Nanoscale Science and Technology Institute, Wonkwang University,Iksan, Chonbuk, Korea 570-749
[3]Faculty of Liberal Education Chonbuk National University Jeonju, Chonbuk, Korea 561-756

Abstract

By using the notion of fuzzy functions introduced by Dib and Youssef, we obtain fuzzy analogues of some results concerning ordinary functions. In particular, we give the definition different from one of invertible fuzzy function introduced by Dib and Youssef. And we show that the two definitions are equivalent. Furthermore, we introduce the concepts of fuzzy increasing functions and fuzzy isomorphisms, and we obtain fuzzy analogues of many results concerning ordinary increasing functions and isomorphisms.

Key words : fuzzy function, fuzzy partially ordered set.

1. Introduction

In the usual set theory, functions are special types of relations and relations are subsets of Cartesian product. Thus the concept of Cartesian product plays an important role in the usual theory of relations and functions. Almost all authors have worked with fuzzy relations without referring to what may be called *fuzzy Cartesian product* (See [1,3,7,8]).

However, in 1991, by using J-fuzzy sets, Dib and Youssef introduced the notion of fuzzy Cartesian product and they defined a fuzzy relation as a subset of the fuzzy Cartesian product (See [2]). This definition is different from all known definitions of fuzzy relations. Also they defined a fuzzy function as a special type of a fuzzy relation. In particular, they defined a fuzzy partially ordered set and investigated some of it′s properties. We can see that this definition generalizes Zadeh′s definition and is different from those in [3]. Moreover, Hur et al. [5], and Lee [6] obtained fuzzy analogues of many results concerning ordinary equivalence relations and functions in the sense of Dib and Youssef.

In section 2, by using the definition of fuzzy functions introduced by Dib and Youssef, we obtain fuzzy analogues of many results concerning ordinary functions. In particular, we give the definition from one of invertible fuzzy function introduced by Dib and Youssef, and consequently prove that the two definitions are equivalent.

In section 3, we investigate fuzzy analogues of some results concerning ordinary partially ordered sets.

In section 4, we introduce the concepts of fuzzy increasing functions and fuzzy isomorphisms. And we study some of their properties.

2. Preliminaries

The totally ordered set $I = [0, 1]$ is a distributive but not complemented lattice under the operations of infimum $\wedge$ and supremum $\vee$. On $J = I \times I$ we define a partial order $\leq$, in terms of the partial order on I, as follows: For every (r_1, r_2), $(s_1, s_2) \in J$,

(i) $(r_1, r_2) \leq (s_1, s_2)$ if and only if $r_1 \leq s_1$, $r_2 \leq s_2$ whenever $s_1 \neq 0$ and $s_2 \neq 0$,

(ii) $(0, 0) = (s_1, s_2)$ whenever $s_1 = 0$ or $s_2 = 0$.

It is clear that J is a distributive but not complemented vector lattice. The operations of infimum and supremum in J are given respectively by: For every (r_1, r_2), $(s_1, s_2) \in J$,

$$(r_1, r_2) \wedge (s_1, s_2) = (r_1 \wedge s_1, r_2 \wedge s_2)$$

Manuscript received Jun. 13, 2009; revised Oct. 20, 2009.

[1] Corresponding Author : Kul Hur, kulhur@wonkwang.ac.kr

2000 Mathematics Subject Classification of AMS: 03E72.
This paper was supported by Wonkwang University in 2009.

and

$(r_1, r_2) \vee (s_1, s_2) \leq (r_1 \vee s_1, r_2 \vee s_2)$,

where the equality holds in the last relation when $r_i \neq 0 \neq s_i$.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping if* $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings, where $f(x) = (f_1(x), f_2(x))$ for each $x \in X$.

Definition 2.1[5]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \eta_A) : X \to J$ is called a *J-fuzzy set* (in short, *fuzzy set*) in X, where $A(x) = (\mu_A(x), \eta_A(x))$ for each $x \in X$, In particular, $\varnothing$ and X denote the *J-fuzzy empty set* and *J-fuzzy whole set* in X defined by $\varnothing(x) = (0, 0)$ and $X(x) = (1, 1)$ for each $x \in X$, respectively.

The notation $\{(x, A(x)) : x \in X\}$ or simply $\{(x, r)\}$, where $r = A(x)$, will be used to denote a fuzzy set in X (see [7]). Similarly, a J-fuzzy set in X, a fuzzy set in $X \times Y$ and a J-fuzzy set in $X \times Y$ will be denoted respectively by $\{(x, (r_1, r_2))\}$, $\{((x, y), r)\}$ and $\{((x, y), (r_1, r_2))\}$. To each fuzzy set $\{(x, r_1)\}$ in X and fuzzy set $\{(y, r_2)\}$ in Y there corresponds a J-fuzzy set $\{((x, y), (r_1, r_2))\}$ in $X \times Y$. Throughout this thesis, the notation $(x, r) \in A$ means that $A(x) = r$, where $A \in I^X$, and X, Y, Z, etc denote ordinary sets.

Definition 2.2[2]. Let X and Y be two ordinary sets. Then the collection of all J-fuzzy sets in $X \times Y$ is called the *fuzzy Cartesian product* of X and Y and is denoted by $X \underline{\times} Y$. Hence $X \underline{\times} Y = J^{X \times Y}$.

The *fuzzy Cartesian product* of a fuzzy set $A = \{(x, r)\}$ in X and a fuzzy set $B = \{(y, s)\}$ in Y is the J-fuzzy set $A \underline{\times} B$ in $X \times Y$ defined by:

$$A \underline{\times} B = \{((x, y), (r, s)) : x \in X, y \in Y\} \equiv \{((x, y), (r, s))\}.$$

It is clear that $A \underline{\times} B \in X \underline{\times} Y$ for each $A \in I^X$ and $B \in I^Y$. The above definitions can be generalized for any finite number of sets. Furthermore, the above definitions can be generalized in an obvious way by replacing the unit interval I by an arbitrary completely distributive lattice.

Definition 2.3[2]. ρ is called a *fuzzy relation from X to Y* if $\rho \subset X \underline{\times} Y$. In particular, ρ is called a *fuzzy relation in X* if $\rho \subset X \underline{\times} X$.

It is clear that $X \underline{\times} Y$ is itself a fuzzy relation from X to Y and any collection of $A \underline{\times} B$, where $A \in I^X$ and $B \in I^Y$, is a fuzzy relation from X to Y.

The fuzzy Cartesian product $X \underline{\times} X$ is called the *universal fuzzy relation in X*. The fuzzy relation $\emptyset \underline{\times} \emptyset = \emptyset$ is called the *empty fuzzy relation*. Between these two extreme cases, lies the *identity fuzzy relation*, denoted by Δ_X, where Δ_X is the fuzzy relation in X whose members are the J-fuzzy sets $\{((x, x), (r, r)) : x \in X$ and $r \in I\}$.

Definition 2.4[2]. Let ρ_1 and ρ_2 be fuzzy relations from X to Y.

(1) We say that ρ_1 *is contained in* ρ_2 if whenever $((x, y), (r_1, r_2)) \in A \in \rho_1$, there exists $B \in \rho_2$ such that $((x, y), (r_1, r_2)) \in B$. In this case, we write $\rho_1 \subset \rho_2$.

(2) We say that ρ_1 and ρ_2 *are equal* if $\rho_1 \subset \rho_2$ and $\rho_2 \subset \rho_1$. In this case, we write $\rho_1 = \rho_2$.

To each J-fuzzy set $C = \{((x, y), (r, s))\}$ in $X \times Y$ we associate a J-fuzzy set C^{-1} in $Y \times X$ defined by $C^{-1} = \{((y, x), (s, r))\}$

Definition 2.5[2]. Let ρ be a fuzzy relation from X to Y. Then the *inverse* of ρ, denoted ρ^{-1}, is the fuzzy relation from Y to X defined by $\rho^{-1} = \{C^{-1} : C \in \rho\}$.

Definition 2.6[2]. Let ρ be a fuzzy relation from X to Y and let σ be a fuzzy relation from Y to Z. Then the *composition* of ρ and σ, denoted $\sigma \circ \rho$, is the fuzzy relation from X to Z whose constituting J-fuzzy sets $C \in X \underline{\times} Z$ are defined as follows:

$((x, z), (r_1, r_3)) \in C$ if and only if there exists $(y, r_2) \in Y \times I$ such that $((x, y), (r_1, r_2)) \in A$ and $((y, z), (r_2, r_3)) \in B$ for some $A \in \rho$ and $B \in \sigma$. Hence $\sigma \circ \rho = \{C \in X \underline{\times} Z : C$ is as defined above$\}$.

It is clear that if ρ is a fuzzy relation in X, then $\Delta_X \circ \rho \subset \rho$ and $\rho \circ \Delta_X \subset \rho$.

Result 2.A[2, Proposition in p.303]. Let $\rho, \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2$ be any fuzzy relations defined on the appropriate sets. Then

(1) $(\rho_1 \circ \rho_2) \circ \rho_3 = \rho_1 \circ (\rho_2 \circ \rho_3)$.

(2) $\rho_1 \subset \rho_2$ and $\sigma_1 \subset \sigma_2 \Rightarrow \rho_1 \circ \sigma_1 \subset \rho_2 \circ \sigma_2$.

(3) $\rho_1 \circ (\rho_2 \cup \rho_3) = (\rho_1 \circ \rho_2) \cup (\rho_1 \circ \rho_3)$.

(4) $\rho_1 \circ (\rho_2 \cap \rho_3) \subset (\rho_1 \circ \rho_2) \cap (\rho_1 \circ \rho_3)$.

(5) $\rho_1 \subset \rho_2 \Rightarrow \rho_1^{-1} \subset \rho_2^{-1}$.

(6) $(\rho^{-1})^{-1} = \rho$ and $(\rho_1 \circ \rho_2)^{-1} = {\rho_2}^{-1} \circ {\rho_1}^{-1}$.

(7) $(\rho_1 \cup \rho_2)^{-1} = {\rho_1}^{-1} \cup {\rho_2}^{-1}$.

(8) $(\rho_1 \cap \rho_2)^{-1} = {\rho_1}^{-1} \cap {\rho_2}^{-1}$.

Definition 2.7[2]. Let ρ be a fuzzy relation in X. Then ρ is said to be:

(1) *reflexive* in X if for each $x \in X$ and $r \in I$, there exists $A \in \rho$ such that $((x, x), (r, r)) \in A$, i.e., $\Delta_X \subset \rho$.

(2) *symmetric* in X if whenever $((x, y), (r, s)) \in A \in \rho$, there exists $B \in \rho$ such that $((y, x), (s, r)) \in B$, i.e., $\rho^{-1} = \rho$.

(3) *transitive* in X if whenever $((x,y),(r,s)) \in A \in \rho$ and $((y,z),(s,t)) \in B \in \rho$, there exists $C \in \rho$ such that $((x,z),(r,t)) \in C$, i.e., $\rho \circ \rho \subset \rho$.
(4) a *fuzzy equivalence relation* in X if it is reflexive, symmetric and transitive.

We will denote the set of all fuzzy equivalence relations in X as $\mathrm{F}Rel_E(X)$. It is clear that $X \underline{\times} X$, $\Delta_X \in \mathrm{F}Rel_E(X)$.

Result 2.B[2, Proposition in p.303]. Let ρ and σ be fuzzy relations on a set X. Then
(1) If ρ is reflexive [resp. symmetric and transitive], then ρ^{-1} is reflexive [resp. symmetric and transitive].
(2) If ρ is reflexive [resp. symmetric and transitive], then $\rho \circ \rho$ is reflexive [resp. symmetric and transitive].
(3) If ρ is reflexive, then $\rho \subset \rho \circ \rho$.
(4) If ρ is symmetric, then $\rho \cup \rho^{-1}$, $\rho \cap \rho^{-1}$ are symmetric and $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho$.
(5) If ρ and σ are reflexive [resp. symmetric and transitive], then $\rho \cap \sigma$ is reflexive [resp. symmetric and transitive].
(6) If ρ and σ are symmetric, then $\rho \cup \sigma$ is symmetric.

3. Fuzzy functions

In the usual set theory, functions are special types of relations. Dib and Youssef[2] defined fuzzy functions analogously as special types of fuzzy relations.

First of all, we list some definitions introduced and some results investigated by Dib and Youssef. Next, we improve another results corresponding to fuzzy functions for ordinary functions.

Definition 3.1[2]. Let X and Y be nonempty sets. Then a fuzzy relation $\mathbb{F}$ from X to Y is called a *fuzzy function from X to Y* if $\mathbb{F} : I^X \to I^Y$ is a function from I^X to I^Y characterized by the ordered pair $(F, \{f_x\}_{x \in X})$, where $F : X \to Y$ is a function from X to Y and $\{f_x\}_{x \in X}$ is a family of functions $f_x : I \to I$ satisfying the conditions:
(α) f_x is nondecreasing on I,
(β) $f_x(0) = 0$ and $f_x(1) = 1$,
such that the image of any fuzzy set A in X under $\mathbb{F}$ is a fuzzy set $\mathbb{F}(A)$ in Y defined as follows: For each $y \in Y$,

$$\mathbb{F}(A) = \begin{cases} \bigvee_{x \in F^{-1}(y)} f_x(A(x)) & \text{if} \quad F^{-1}(y) \neq \emptyset, \\ 0 & \text{if} \quad F^{-1}(y) = \emptyset. \end{cases}$$

In this case, we write $\mathbb{F} = (F, \{f_x\}_{x \in X}) : X \to Y$ or, simply, $\mathbb{F} = (F, f_x) : X \to Y$ to denote a fuzzy function from X to Y and we call the functions f_x, $x \in X$, the *comembership functions associated to* $\mathbb{F}$. If there exists $A \in \mathbb{F}$ such that $(x, F(x), r, f_x(r)) \in A \in \mathbb{F}$ for each $(x,r) \in X \times I$, then we write this in the form $\mathbb{F}(x,r) = (F(x), f_x(r))$.

Two fuzzy functions $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_x) : X \to Y$ are said to be *equal*, denoted by $\mathbb{F} = \mathbb{G}$, if $\mathbb{F}(A) = \mathbb{G}(A)$ for each $A \in I^X$.

Result 3.A[2, Theorem 5]. Let $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_x) : X \to Y$ be fuzzy functions. Then $\mathbb{F} = \mathbb{G}$ if and only if $F = G$ and $f_x = g_x$ for each $x \in X$.

Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. The *inverse image under $\mathbb{F}$ of a fuzzy set B in Y*, denoted by $\mathbb{F}^{-1}(B)$, is a fuzzy set in X defined as follows:

$$\mathbb{F}^{-1}(B) = \bigcup \{C \in I^X \ : \ \mathbb{F}(C) \subset B\}.$$

Result 3.B[2, Proposition in p.311]. Let $\mathbb{F} = (F, f_x) : X \to Y$ be any fuzzy function whose comembership functions f_x are surjective. Then for each fuzzy set B in Y, and each $x \in X$,
$\mathbb{F}^{-1}(B)(x) = \bigvee {f_x}^{-1}[B(F(x))]$, where the supremum is taken over the set of values ${f_x}^{-1}[B(F(x))] \subset I$.

Result 3.C[2, Theorem 6]. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function, let $A, B \in I^X$, $\{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$ and let $C, D \in I^Y$, $\{C_\alpha\}_{\alpha \in \Gamma} \subset I^Y$. Then
(1) $\mathbb{F}(\emptyset) = \emptyset$.
(2) $\mathbb{F}(X) = Y$ if F is surjective.
(3) If $A \subset B$, then $\mathbb{F}(A) \subset \mathbb{F}(B)$.
(4) $\mathbb{F}(A \cup B) = \mathbb{F}(A) \cup \mathbb{F}(B)$.
(5) $\mathbb{F}(A \cap B) \subset \mathbb{F}(A) \cap \mathbb{F}(B)$ (equality holds if F is injective).
(6) $\mathbb{F}^{-1}(Y) = X$.
(7) If $C \subset D$, then $\mathbb{F}^{-1}(C) \subset \mathbb{F}^{-1}(D)$.
(8) $A \subset \mathbb{F}^{-1}(\mathbb{F}(A))$ (equality holds if F is bijective).
If f_x is surjective for each $x \in X$, then
(9) $\mathbb{F}(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} \mathbb{F}(A_\alpha)$.
(10) $\mathbb{F}(\bigcap_{\alpha \in \Gamma} A_\alpha) \subset \bigcap_{\alpha \in \Gamma} \mathbb{F}(A_\alpha)$ (equality holds if F is injective).
(11) $\mathbb{F}^{-1}(\bigcup_{\alpha \in \Gamma} C_\alpha) = \bigcup_{\alpha \in \Gamma} \mathbb{F}^{-1}(C_\alpha)$.
(12) $\mathbb{F}^{-1}(\bigcap_{\alpha \in \Gamma} C_\alpha) = \bigcap_{\alpha \in \Gamma} \mathbb{F}^{-1}(C_\alpha)$.
(13) $\mathbb{F}(\mathbb{F}^{-1}(C)) \subset C$ (equality holds if F is surjective).
If $f_x(1-r) \geq 1 - f_x(r)$ for each $(x,r) \in X \times I$, then
(14) $\mathbb{F}(A^c) \supset (\mathbb{F}(A))^c$ if F is surjective. (Equality holds if F is bijective and $f_x(1-r) = 1 - f_x(r)$.)
If f_x is bijective and $f_x(1-r) = 1 - f_x(r)$ for each $(x,r) \in X \times I$, then

$$\mathbb{F}^{-1}(D^c) = (\mathbb{F}^{-1}(D))^c.$$

The *composition* of two fuzzy functions $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to Z$ is the fuzzy function

$\mathbb{G} \circ \mathbb{F} : X \to Z$ defined by $(\mathbb{G} \circ \mathbb{F})(A) = \mathbb{G}(\mathbb{F}(A))$ for each $A \in I^X$.

Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. Then $\mathbb{F}$ is said to be *injective* or *one-to-one* if for any fuzzy sets A_1 and A_2 in X, $\mathbb{F}(A_1) = \mathbb{F}(A_2)$ implies $A_1 = A_2$. *Surjective* and *bijective* fuzzy functions can be defined similarly in an obvious manner. It is clear that the *fuzzy identity function* $\mathbf{id}_X = (id_X, id_I) : X \to X$ is bijective, where $id_X : X \to X$ and $id_I : I \to I$ are identity functions, respectively.

A fuzzy function $\mathbb{F} = (F, f_x) : X \to Y$ is said to be *invertible* if there exists a fuzzy function $\mathbb{G} = (G, g_y) : Y \to X$ such that $\mathbb{G} \circ \mathbb{F} = \mathbf{id}_X$ and $\mathbb{F} \circ \mathbb{G} = \mathbf{id}_Y$. In this case, the fuzzy function $\mathbb{G}$ is called the *inverse* of $\mathbb{F}$ and is denoted by $\mathbb{F}^{-1}$.

Result 3.D[2, Theorem 7]. Let $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to Z$ be fuzzy functions. Let g_y be surjective for each $y \in Y$. Then

(1) The composition $\mathbb{G} \circ \mathbb{F} : X \to Z$ of $\mathbb{F}$ and $\mathbb{G}$ is given by

$$\mathbb{G} \circ \mathbb{F} = (G \circ F,\ g_{F(x)} \circ f_x).$$

(2) $\mathbb{F} = (F, f_x)$ is surjective [resp. injective] if and only if F and f_x, $x \in X$, are surjective [resp. injective].

(3) $\mathbb{F} = (F, f_x)$ is invertible if and only if F and f_x are invertible. The inverse $\mathbb{F}^{-1}$ of $\mathbb{F}$ is given by $\mathbb{F}^{-1} = (F^{-1}, {f_x}^{-1})$.

Now we will obtain some another properties of fuzzy functions.

The following is the immediate result of Result 3.A.

Proposition 3.2. If $\mathbb{F} = (F, f_x) : X \to Y$ is a fuzzy function, then $\mathbf{id}_Y \circ \mathbb{F} = \mathbb{F} = \mathbb{F} \circ \mathbf{id}_X$.

Proposition 3.3. Let $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to Z$ be fuzzy functions.

(1) If $\mathbb{F}$ and $\mathbb{G}$ are injective and g_y is surjective for each $y \in Y$, then $\mathbb{G} \circ \mathbb{F}$ is injective.

(2) If $\mathbb{F}$ and $\mathbb{G}$ are surjective, then $\mathbb{G} \circ \mathbb{F}$ is surjective.

(3) If $\mathbb{F}$ and $\mathbb{G}$ are bijective, then $\mathbb{G} \circ \mathbb{F}$ is bijective.

Proof. (1) Since g_y is surjective for each $y \in Y$, by Result 3.D(1), $\mathbb{G} \circ \mathbb{F} = (G \circ F,\ g_{F(x)} \circ f_x)$.
Since $\mathbb{F}$ and $\mathbb{G}$ are injective, by Result 2.D(2), F and f_x, $x \in X$, are injective, and G and $g_{F(x)}$, $x \in X$, are injective. Thus $G \circ F : X \to Z$ and $g_{F(x)} \circ f_x$ are injective for each $x \in X$. So, by Result 3.D(2), $\mathbb{G} \circ \mathbb{F} = (G \circ F, g_{F(x)} \circ f_x)$ is injective.

(2) Since $\mathbb{G}$ is surjective, by Result 3.D(2), G and g_y, $y \in Y$, are surjective. Then $\mathbb{G} \circ \mathbb{F} = (G \circ F, g_{F(x)} \circ f_x)$. Since $\mathbb{F}$ is surjective, by Result 3.D(2), F and f_x, $x \in X$, are surjective. Moreover, $g_{F(x)}$ is surjective for each $x \in X$. Thus $G \circ F$ and $g_{F(x)} \circ f_x$, $x \in X$, are surjective. So, by Result 3.D(2), $\mathbb{G} \circ \mathbb{F}$ is surjective.

(3) It is clear that $\mathbb{G} \circ \mathbb{F}$ is bijective from (1) and (2). $\square$

The following is the immediate result of Result 2.D.

Proposition 3.4. Let $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to Z$ be fuzzy functions.

(1) If $\mathbb{G} \circ F$ is injective and g_y is surjective for each $y \in Y$, then $\mathbb{F}$ is injective.

(2) If $\mathbb{G} \circ F$ is surjective and g_y is surjective for each $y \in Y$, then $\mathbb{G}$ is surjective.

(3) If $\mathbb{G} \circ F$ is bijective and g_y is surjective for each $y \in Y$, then $\mathbb{F}$ is injective and $\mathbb{G}$ is surjective.

Now we give the definition different from one of invertible fuzzy function introduced by Dib and Yoyssef[2].

Definition 3.5. A fuzzy function $\mathbb{F} = (F, f_x) : X \to Y$ is said to be *invertible* if $\mathbb{F}^{-1} = (F^{-1}, {f_x}^{-1}) : Y \to X$ is a fuzzy function.

Let $\mathbb{F} = (F, f_x) : X \to Y$ be an invertible fuzzy function. Then $(x, y, r, s) \in A \in \mathbb{F}$ if and only if $(y, x, s, r) \in A^{-1} \in \mathbb{F}^{-1}$. Thus $\mathbb{F}(x, r) = (y, s)$ if and only if $\mathbb{F}^{-1}(y, s) = (x, r)$.

The following is the immediate result of Definition 3.5.

Proposition 3.6. Let $\mathbb{F} = (F, f_x) : X \to Y$ be an invertible fuzzy function. Then for each $(x, r) \in X \times I, (y, s) = \mathbb{F}(x, r)$ if and only $(x, r) = \mathbb{F}^{-1}(y, s)$, equivalently, $y = F(x)$ and $s = f_x(r)$ if and only if $x = F^{-1}(y)$ and $r = f_x^{-1}(s)$.

The next two lemmas give a necessary and sufficient condition for a fuzzy function to be invertible.

Lemma 3.7. If $\mathbb{F} = (F, f_x) : X \to Y$ is a bijective fuzzy function, then $\mathbb{F}^{-1} = (F^{-1}, {f_x}^{-1}) : Y \to X$ is a bijective fuzzy function.

Proof. Suppose $\mathbb{F} = (F, f_x)$ is bijective. Then, by Result 3.D(2), F and f_x, $x \in X$, are bijective. Thus $F^{-1} : Y \to X$ and ${f_x}^{-1} : I \to I$, $x \in X$, are bijective. Hence, by Result 3.D(2), $\mathbb{F}^{-1} = (F^{-1}, {f_x}^{-1}) : Y \to X$ is bijective. $\square$

Lemma 3.8. If $\mathbb{F} = (F, f_x) : X \to Y$ is invertible, then $\mathbb{F}$ is bijective.

Proof. Suppose $\mathbb{F} = (F, f_x) : X \to Y$ is invertible. Then $\mathbb{F}^{-1} = (F^{-1}, {f_x}^{-1}) : Y \to X$ is a fuzzy function. Thus, by Definition 2.1, $F^{-1} : Y \to X$ and $f_x^{-1} : I \to I$ are functions. So $F : X \to Y$ and $f_x : I \to I$ are invertible. Hence F and f_x are bijective. Therefore, by Result 3.D(2), $\mathbb{F}$ is bijective. □

Lemmas 3.7 and 3.8 may be summarized as follows.

Theorem 3.9. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. Then $\mathbb{F}$ is invertible if and only if it is bijective ; furthermore, if $\mathbb{F} : X \to Y$ is invertible, then $\mathbb{F}^{-1} = (F^{-1}, f_x^{-1}) : Y \to X$ is bijective.

The next two lemmas give another useful characterization of invertible fuzzy functions. Moreover, we can see that this characterization is the definition introduced by Dib and Youssef.

Lemma 3.10. Let $\mathbb{F} = (F, f_x) : X \to Y$ be an invertible fuzzy function. Then $\mathbb{F}^{-1} \circ \mathbb{F} = \mathbf{id}_X$ and $\mathbb{F} \circ \mathbb{F}^{-1} = \mathbf{id}_Y$.

Proof. Suppose $\mathbb{F} = (F, f_x) : X \to Y$ is invertible. Then, by Theorem 3.9, $\mathbb{F}$ and $\mathbb{F}^{-1}$ are bijective. Thus by Result 3.D(1),
$\mathbb{F}^{-1} \circ \mathbb{F} = (F^{-1} \circ F, f_x^{-1} \circ f_x)$ and $\mathbb{F} \circ \mathbb{F}^{-1} = (F \circ F^{-1}, f_y \circ f_y^{-1})$.
Moreover, $F^{-1} \circ F = id_X, f_x^{-1} \circ f_x = id_I$ and $F \circ F^{-1} = id_Y, f_y \circ f_y^{-1} = id_I$. Hence, by Result 3.A, $\mathbb{F}^{-1} \circ \mathbb{F} = \mathbf{id}_X$ and $\mathbb{F} \circ \mathbb{F}^{-1} = \mathbf{id}_Y$. □

The following is the immediate result of proposition 3.4 and Result 3.D.

Lemma 3.11. Let $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to X$ be fuzzy functions. If $\mathbb{G} \circ F = \mathbf{id}_X$ and $\mathbb{F} \circ G = \mathbf{id}_Y$, then $\mathbb{F}$ is bijective (hence invertible), and $\mathbb{G} = \mathbb{F}^{-1}$.

Lemmas 3.10 and 3.11 may be summarized as follows. Thus we can see that this result is the definition of invertible fuzzy functions introduced by Dib and Youssef.

Theorem 3.12. Let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function. Then $\mathbb{F}$ is invertible if and only if there exists a fuzzy function $\mathbb{G} = (G, g_y) : Y \to X$ such that $\mathbb{G} \circ F = \mathbf{id}_X$ and $\mathbb{F} \circ G = \mathbf{id}_Y$.

4. Fuzzy partial orders

Definition 4.1[2]. A fuzzy relation ρ in X is said to be *antisymmetric* if $((x, y), (r, t)) \in A \in \rho$ implies that there is no $B \in \rho$ such that $((y, x), (t, r)) \in B$ or, equivalently, $((x, y), (r, t)) \in A \in \rho$ and $((y, x), (t, r)) \in B \in \rho$ imply that $x = y$ and $r = t$.

The following is the immediate result of Definition 4.1.

Proposition 4.2. Let ρ be a fuzzy relation in X. If ρ is antisymmetric, then ρ^{-1} is antisymmetric.

Definition 4.3[2]. A fuzzy relation ρ in X is called a *fuzzy partial order* or, simply, a *fuzzy order* if it is reflexive, antisymmetric and transitive.

A nonempty set X in which a fuzzy partial order is defined is called a *fuzzy partially ordered set* or, simply, a *fuzzy ordered set.*

Let X be a fuzzy partially ordered set with the fuzzy order ρ. Then we will write $(x, r) \lesssim_\rho (y, t)$, simply, $(x, r) \lesssim (y, t)$ to denote the fact that there exist $A \in \rho$ such that $((x, y), (r, t)) \in A$ for any $x, y \in X$ and $r, t \in I$. We further agree that $(y, t) \gtrsim (x, r)$ has the same meaning as $(x, r) \lesssim (y, t)$ and that $(x, r) \not\lesssim (y, t)$ means that there is no $A \in \rho$ such that $((x, y), (r, t)) \in A$.

If $(x, r), (y, t) \in X \times I$ and $(x, r) \lesssim (y, t)$, then we say that "(x, r) is fuzzy less than or fuzzy equal to (y, t)." We agree that $(x, r) < (y, t)$ is an abbreviation for "$(x, r) \lesssim (y, t)$ and $(x, r) \neq (y, t)$." If $(x, r) < (y, t)$, then we say that "(x, r) is strictly fuzzy less than (y, t)."

Example 4.3. Let $X = \{a, b, c, d, e\}$ be a set and let $\rho = \triangle_X \cup \{A_1, A_2, A_3\}$ be the fuzzy relation in defined X as follows:

A_1	a	b	c	d	e
a	(r_0, r_0)	(r_1, r_1)	(r_1, r_1)	(r_1, r_1)	(r_1, r_1)
b	(r_0, r_1)	(r_0, r_0)	(r_0, r_0)	(r_1, r_0)	(r_1, r_0)
c	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_1, r_1)	(r_1, r_1)
d	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_1, r_1)
e	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0),

A_2	a	b	c	d	e
a	(r_0, r_0)	(r_1, r_1)	(r_1, r_1)	(r_1, r_1)	(r_1, r_1)
b	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_0, r_0)
c	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_1, r_1)	(r_1, r_1)
d	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_1, r_1)
e	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0)	(r_0, r_0),

A_3	a	b	c	d	e
a	(r_0, r_1)	(r_1, r_1)	(r_1, r_1)	(r_1, r_1)	(r_1, r_1)
b	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_1, r_1)	(r_1, r_1)
c	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)
d	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)
e	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_0, r_1)	(r_0, r_1),

where $r_0 \neq r_1 \in I$. Then we can easily see that ρ is a fuzzy partial order in X. □

The following is the immediate result of Definition 4.3 and Proposition 4.2.

Proposition 4.4. Let ρ be a fuzzy partial order in X. Then ρ^{-1} is a fuzzy partial order in X.

Definition 4.5[2]. A fuzzy relation ρ in X is called a *fuzzy total* (or *linear*) *order* if for any $x, y \in X$ and $r, t \in I$ there exists $A \in \rho$ such that either $((x, y), (r, t)) \in A$ or $((y, x), (t, r)) \in A$.

A nonempty set X in which a fuzzy total order is defined is called a *fuzzy totally* (or *linearly*) *ordered set.*

Result 4.A[2, Theorem 4]. Let ρ be a fuzzy order in X. Then

(1) For each $x_0 \in X$, ρ induces an (ordinary) order $\rho_I(x_0)$ in I defined by

$$\rho_I(x_0) = \{(r, s) \in J : \exists A \in \rho \text{ s.t. } ((x_0, x_0), (r, s)) \in A\}.$$

(2) For each $r_0 \in I$, ρ induces an (ordinary) order $\rho_X(r_0)$ in X defined by

$$\rho_X(r_0) = \{(x, y) \in X \times X : \exists A \in \rho \text{ s.t. } ((x, y), (r_0, r_0)) \in A\}.$$

Example 4.A. Let ρ be the fuzzy partial order in X in Example 4.3. Then

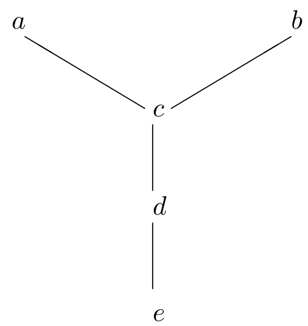
$$\rho_X(r_0) = \{(a,a),(b,b),(c,c),(d,d),(e,e),(c,a),(c,b),(d,a),(d,b),(d,c),(e,a),(e,b),(e,c),(e,d)\}$$

and

$$\rho_X(r_1) = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(a,e),(b,d),(b,e),(c,d),(c,e),(d,e)\}$$

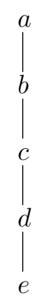
are ordinary partial orders in X with following diagrams:

$\rho_X(r_0)$:



,

$\rho_X(r_1)$:



. □

Furthermore,

$\rho_I(a) = \rho_I(b) = \rho_I(c) = \rho_I(d) = \rho_I(e) = \{(r, r) : r \in I\} \cup \{(r_0, r_1)\}$

are ordinary partial orders in I. □

Result 4.B[2, Corollary in p.308]. (1) To each fuzzy partial order ρ in X there are associated an ordinary partial order ρ_I in I and an ordinary partial order ρ_X in X. In fact, $\rho_I = \bigcap_{x \in X} \rho_I(x)$ and $\rho_X = \bigcap_{r \in I} \rho_X(r)$. In this case, ρ_I [resp. ρ_X] is called the *partial order in* I [resp. X] *associated to the fuzzy partial order* ρ.

(2) If ρ is a fuzzy total order in X, then $\rho_I(x)$ [resp. $\rho_X(r)$] is a total order in I [resp. X] for each $x \in X$ [resp. $r \in I$]. In this case, ρ_I and ρ_X are evidently total order relations.

Proposition 4.6. Let $\{\rho_\alpha\}_{\alpha \in \Gamma}$ be an indexed family of fuzzy partial orders in X. Then $\bigcap_{\alpha \in \Gamma} \rho_\alpha$ is a fuzzy partial order in X.

Proof. Let $\rho = \bigcap_{\alpha \in \Gamma} \rho_\alpha$ and let $(x, r) \in X \times I$. Since $\Delta_X \subset \rho_\alpha$ for each $\alpha \in \Gamma$, there exists $A_\alpha \in \rho_\alpha$ such that $((x, x), (r, r)) \in A_\alpha$ for each $\alpha \in \Gamma$. Then $((x, x), (r, r)) \in \bigcap_{\alpha \in \Gamma} A_\alpha \in \rho$. Thus $\Delta_X \subset \rho$. So ρ is reflexive. Suppose $((x, y), (r, t)) \in A \in \rho$ and $((y, x), (t, r)) \in A \in \rho$ for some $A \in \rho$. Since $\rho = \bigcap_{\alpha \in \Gamma} \rho_\alpha$, there exists $A_\alpha \in \rho_\alpha$ such that $A = \bigcap_{\alpha \in \Gamma} A_\alpha$. Then $((x, y), (r, t)) \in A_\alpha$ or $((y, x), (t, r)) \in A_\alpha$. Since ρ_α is antisymmetric, $x = y$ and $r = t$. Thus ρ is antisymmetric. Suppose $((x, y), (r, s)) \in A \in \rho$ and $((y, z), (s, t)) \in B \in \rho$. Then there exist $A_\alpha \in \rho_\alpha$ and $B_\alpha \in \rho_\alpha$ such that $((x, y), (r, s)) \in A_\alpha$ and $((y, z), (s, t)) \in B_\alpha$ for each $\alpha \in \Gamma$. Since ρ_α is transitive for each $\alpha \in \Gamma$, there exists $C_\alpha \in \rho_\alpha$ such that $((x, z), (r, t)) \in C_\alpha$ for each $\alpha \in \Gamma$. Let $C = \bigcap_{\alpha \in \Gamma} C_\alpha$. Then clearly $((x, z), (r, t)) \in C \in \rho$. Thus ρ is transitive. Hence ρ is a fuzzy partial order in X. □

Theorem 4.7. Let ρ be a fuzzy relation in X. Then ρ is a fuzzy partial order in X if and only if $\rho \cap \rho^{-1} = \Delta_X$ and $\rho \circ \rho = \rho$.

Proof. ($\Rightarrow$): Suppose ρ is a fuzzy partial order. Since ρ is reflexive, by Result 1.A(1), ρ^{-1} is reflexive. Then $\Delta_X \subset \rho$ and $\Delta_X \subset \rho^{-1}$. Thus $\Delta_X \subset \rho \cap \rho^{-1}$. Let $((x, y), (r, t)) \in A \in \rho \cap \rho^{-1}$. Then there exist $B, C \in \rho$ such that $A = B \cap C^{-1}$. Thus $((x, y), (r, t)) \in B$ and $((x, y), (r, t)) \in C^{-1}$, i.e., $((y, x), (t, r)) \in C$. Since ρ is antisymmetric, $x = y$ and $r = t$. Thus $\rho \cap \rho^{-1} \subset \Delta_X$. So $\rho \cap \rho^{-1} = \Delta_X$. Since ρ is transitive, $\rho \circ \rho \subset \rho$. Since ρ is reflexive, by Result 2.B(3), $\rho \subset \rho \circ \rho$. Thus $\rho \circ \rho = \rho$.

($\Leftarrow$): Suppose $\rho \cap \rho^{-1} = \Delta_X$ and $\rho \circ \rho = \rho$. Then clearly $\Delta_X \subset \rho$ and $\rho \circ \rho \subset \rho$. Thus ρ is reflexive and transitive. Now suppose $((x, y), (r, t)) \in A \in \rho$ and $((y, x), (t, r)) \in B \in \rho$. Then $((x, y), (r, t)) \in A \cap B^{-1} \in \rho \cap \rho^{-1} = \Delta_X$. Thus $x = y$ and $r = t$. So ρ is antisymmetric. This completes the proof. □

5. Fuzzy order preserving fuzzy functions

Definition 5.1. Let X and Y be fuzzy partially ordered sets. Then a fuzzy function $\mathbb{F} = (F, f_x) : X \to Y$ is said to be *fuzzy increasing* or *fuzzy order preserving* if it satisfies the following condition: For any $(x, r), (y, t) \in X \times I$,

$$(x, r) \lesssim (y, t) \;\Rightarrow\; \mathbb{F}(x, r) \lesssim \mathbb{F}(y, t).$$

Theorem 5.2. Let (X, ρ) and (Y, σ) be fuzzy partially ordered sets, let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy function and let f_x be identical for all $x \in X$. Then $\mathbb{F}$ is fuzzy increasing if and only if $F : (X, \rho_X) \to (Y, \sigma_Y)$ and $f_x : (I, \rho_I) \to (I, \sigma_I)$ are increasing.

Proof. ($\Rightarrow$): Suppose $\mathbb{F}$ is fuzzy increasing. For any $x, y \in X$. let $(x, y) \in \rho_X$ and let $r_0 \in I$. Then, by Results 4.A(2) and 4.B(1), there exists $A \in \rho$ such that $((x, y), (r_0, r_0)) \in A$. Thus $(x, r_0) \lesssim (y, r_0)$. Since $\mathbb{F}$ is fuzzy increasing, $\mathbb{F}(x, r_0) \lesssim \mathbb{F}(y, r_0)$, i,e., $(F(x), f_x(r_0)) \lesssim (F(y), f_y(r_0))$. Since f_x are identical for all $x \in X$, $f_x(r_0) = f_y(r_0)$. Thus there exists $B \in \sigma$ such that $((F(x), F(y)), (f_x(r_0), f_y(r_0)) \in B$. So $(F(x), F(y)) \in \sigma_Y$. Hence $F : (X, \rho_X) \to (Y, \sigma_Y)$ is increasing.

Now for any $r, t \in I$, let $(r, t) \in \rho_I$ and let $x_0 \in X$. Then, by Results 4.A(1) and 4.B(1), there exists $C \in \rho$ such that $((x_0, x_0), (r, t)) \in C$. Thus $(x_0, r) \lesssim (x_0, t)$. Since $\mathbb{F}$ is fuzzy increasing, $\mathbb{F}(x_0, r) \lesssim \mathbb{F}(x_0, t)$, i,e., $(F(x_0), f_{x_0}(r)) \lesssim (F(x_0), f_{x_0}(t))$. Thus there exists $D \in \sigma$ such that $(F(x_0), F(x_0)), (f_{x_0}(r), f_{x_0}(t)) \in D$. So $(f_{x_0}(r), f_{x_0}(t)) \in \sigma_I$. Hence $f_x : (I, \rho_I) \to (I, \sigma_I)$ is increasing.

($\Leftarrow$): Obvious. □

Proposition 5.3. Let $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to Z$ be fuzzy increasing. If $g_y : I \to I$ is surjective for each $y \in Y$, then $\mathbb{G} \circ \mathbb{F} : X \to Z$ is fuzzy increasing.

Proof. Suppose $(x, r) \lesssim (y, t)$ for any $(x, r), (y, t) \in X \times I$. Since $\mathbb{F}$ is fuzzy increasing, $\mathbb{F}(x, r) \lesssim \mathbb{F}(y, t)$, i.e., $(F(x), f_x(r)) \lesssim (F(y), f_y(t))$ in Y. Since $\mathbb{G}$ is fuzzy increasing and $g_y : I \to I$ is surjective for each $y \in Y$, $\mathbb{G}(F(x), f_x(r)) \lesssim \mathbb{G}(F(y), f_y(t))$, i,e., $(G(F(x)),\ g_{F(x)}(f_x(r))) \lesssim (G(F(y)), g_{F(y)}(f_y(t)))$ in Z. Thus $(\mathbb{G} \circ \mathbb{F})(x, r) \lesssim (\mathbb{G} \circ \mathbb{F})(y, t)$. Hence $\mathbb{G} \circ \mathbb{F}$ is fuzzy increasing. □

Definition 5.4. Let (X, ρ) be a fuzzy partially ordered set and let $Y \in P(X)$, where $P(X)$ denote the set of all subsets of X. Then σ is called a *fuzzy partial suborder* of ρ if $\sigma = \rho|_{Y \times Y}$ is a fuzzy partial order in Y, where $\rho|_{Y \times Y} = \{A|_{Y \times Y} : A \in \rho\}$. In this case, (Y, σ) is called a *fuzzy partially ordered subset* of X.

Let Y be a fuzzy partially ordered subset of a fuzzy partially ordered set X. If $(x, r) \in Y \times I$ and $(y, t) \in Y \times I$, then we let $(x, r) \lesssim (y, t)$ in Y if and only if $(x, r) \lesssim (y, t)$ in X.

Let X be a fuzzy partially ordered set. Two elements (x, r) and (y, t) in $X \times I$ are said to be *fuzzy comparable* if either $(x, r) \lesssim (y, t)$ or $(y, t) \lesssim (x, r)$; otherwise, they are said to be *fuzzy incomparable.*

Definition 5.5. Let X be a fuzzy partially ordered set and let Y be a fuzzy partially ordered subset of X. Then Y is called a *fuzzy fully ordered subset* of X, or a *fuzzy linearly ordered subset* of X or a *fuzzy chain* of X if any two elements of $Y \times I$ are fuzzy comparable.

Proposition 5.6. Let $\mathbb{F} = (F, f_x) : X \to Y$ be an increasing fuzzy function and let $f_x : I \to I$ be surjective for each $x \in X$. If C is a fuzzy chain of X, then $F(C)$ is a fuzzy chain of Y.

Proof. Let $(y, u), (y', v) \in F(C) \times I$. Then there exist $x, x' \in C$ such that $y = F(x)$ and $y' = F(x')$. Since $f_x : I \to I$ is surjective for each $x \in X$, there exist $r, t \in I$ such that $u = f_x(r)$ and $v = f_{x'}(t)$. Since $\mathbb{F}$ is a fuzzy function, $\mathbb{F}(x, r) = (y, u)$ and $\mathbb{F}(x', t) = (y', v)$. Since C is a fuzzy chain of X, either $(x, r) \lesssim (x', t)$ or $(x', t) \lesssim (x, r)$. Since $\mathbb{F}$ is fuzzy increasing, either $\mathbb{F}(x, r) \lesssim \mathbb{F}(x', t)$ or $\mathbb{F}(x', t) \lesssim \mathbb{F}(x, r)$. Thus either $(y, u) \lesssim (y', v)$ or $(y', v) \lesssim (y, u)$. Hence $F(C)$ is a fuzzy chain of Y. □

Definition 5.7. Let C be a fuzzy partially ordered subset of a fuzzy partially ordered set X. Then C is said to be *fuzzy convex* if it satisfies the following condition: $(a, r), (b, t) \in C \times I$ and $(a, r) \lesssim (x, s) \lesssim (b, t)$ imply $(x, s) \in C \times I$.

Proposition 5.8. Let X and Y be fuzzy ordered sets, let $\mathbb{F} = (F, f_x) : X \to Y$ be an increasing fuzzy function and let C be a fuzzy convex ordered subset of Y. Then $F^{-1}(C)$ is a fuzzy convex ordered subset of X.

Proof. Clearly $F^{-1}(C)$ is a fuzzy ordered subset of X. Suppose $(a,r),(b,t) \in F^{-1}(C) \times I$ and $(a,r) \lesssim (x,s) \lesssim (b,t)$. Since $\mathbb{F} = (F, f_x) : X \to Y$ is an increasing fuzzy function, $\mathbb{F}(a,r) \lesssim \mathbb{F}(x,s) \lesssim \mathbb{F}(b,t)$, i.e., $(F(a), f_a(r)) \lesssim (F(x), f_x(s)) \lesssim (F(b), f_b(t))$. Since $(a,r),(b,t) \in F^{-1}(C) \times I$, $(F(a), f_a(r)) \in C \times I$ and $(F(b), f_b(t)) \in C \times I$. Since C is a fuzzy convex ordered subset of Y, $(F(x), f_x(s)) \in C \times I$. Thus $(x,s) \in F^{-1}(C) \times I$. Hence $F^{-1}(C)$ is a fuzzy convex ordered subset of X. □

Definition 5.9. Let X and Y be fuzzy partially ordered sets. Then a fuzzy function $\mathbb{F} = (F, f_x) : X \to Y$ is called a *fuzzy isomorphism* if it is fuzzy bijective and satisfies the following condition: For any two elements $(x,r) \in X \times I$ and $(y,t) \in X \times I$,

$$(x,r) \lesssim (y,t) \;\Leftrightarrow\; \mathbb{F}(x,r) \lesssim \mathbb{F}(y,t).$$

We say that X is *fuzzy isomorphic with* Y, denoted by $X \simeq Y$, if there exists a fuzzy isomorphism from X to Y.

Theorem 5.10. Let X and Y be fuzzy partially ordered sets and let $\mathbb{F} = (F, f_x) : X \to Y$ be a fuzzy bijective function. Then $\mathbb{F} : X \to Y$ is a fuzzy isomorphism if and only if $\mathbb{F} : X \to Y$ and $\mathbb{F}^{-1} : Y \to X$ are increasing fuzzy functions.

Proof. Since $\mathbb{F}$ is fuzzy bijective, by Theorem 3.9, $\mathbb{F}$ is invertible. Then $\mathbb{F}^{-1}(\mathbb{F}(x,r)) = (x,r)$ for each $(x,r) \in X \times I$.

($\Leftarrow$): Suppose $\mathbb{F}$ and $\mathbb{F}^{-1}$ are fuzzy increasing. If $(x,r) \lesssim (y,t)$ in X, then clearly $\mathbb{F}(x,r) \lesssim \mathbb{F}(y,t)$. Now if $\mathbb{F}(x,r) \lesssim \mathbb{F}(y,t)$ in Y, then $\mathbb{F}^{-1}(\mathbb{F}(x,r)) \lesssim \mathbb{F}^{-1}(\mathbb{F}(y,t))$ in X since $\mathbb{F}^{-1}$ is fuzzy increasing. Thus $(x,r) \lesssim (y,t)$ in X. So $\mathbb{F}$ is a fuzzy isomorphism.

($\Rightarrow$): Suppose $\mathbb{F}$ is a fuzzy isomorphism. Then clearly $\mathbb{F}$ is fuzzy increasing. Let $\mathbb{F}(x,r)$ and $\mathbb{F}(y,t)$ be any elements of $Y \times I$. Then

$$\mathbb{F}(x,r) \lesssim \mathbb{F}(y,t) \;\Rightarrow\; (x,r) \lesssim (y,t) \;\Rightarrow\; \mathbb{F}^{-1}(\mathbb{F}(x,r)) \lesssim \mathbb{F}^{-1}(\mathbb{F}(y,t)).$$

So $\mathbb{F}^{-1}$ is fuzzy increasing. This completes the proof. □

Theorem 5.11. Let X, Y, Z be fuzzy partially ordered sets.

(1) The fuzzy identity function $\mathbf{id}_X = (id_X, id_I) : X \to X$ is a fuzzy isomorphism.

(2) If $\mathbb{F} = (F, f_x) : X \to Y$ is a fuzzy isomorphism, then $\mathbb{F}^{-1} = (F^{-1}, {f_x}^{-1}) : Y \to X$ is a fuzzy isomorphism.

(3) If $\mathbb{F} = (F, f_x) : X \to Y$ and $\mathbb{G} = (G, g_y) : Y \to Z$ are fuzzy isomorphisms, then $\mathbb{G} \circ \mathbb{F} : X \to Z$ is a fuzzy isomorphism.

Proof. (1) It is clear that, $\mathbf{id}_X : X \to X$ is fuzzy bijective. Moreover, $\mathbf{id}_X$ and $\mathbf{id}_X^{-1}$ are fuzzy increasing. So, by Theorem 5.10, $\mathbf{id}_X$ is a fuzzy isomorphism.

(2) From Theorem 5.10 and Theorem 3.9, we can easily see that $\mathbb{F}^{-1}$ is a fuzzy isomorphism.

(3) By Proposition 5.2, $\mathbb{G} \circ \mathbb{F} = (G \circ F, g_{F(x)} \circ f_x) : X \to Z$ is fuzzy increasing. By Proposition 3.3(3), $(\mathbb{G} \circ \mathbb{F})$ is fuzzy bijective. Moreover, we can easily see that $\mathbb{G} \circ F^{-1}$ is fuzzy increasing. So, by Theorem 5.10, $\mathbb{G} \circ F$ is a fuzzy isomorphism. This completes the proof. □

References

1. J. C. Bezdek and J. D. Harris, "Fuzzy partitions and relations; an axiomatic basis for clustering", *Fuzzy Sets and Systems* **1**, pp. 111-127, 1978.

2. K. A. Dib and Nabil L. Youssef, "Fuzzy cartesian product, fuzzy relations and fuzzy functions", *Fuzzy Sets and Systems* **41**, pp. 299-315, 1991.

3. M. A. Erceg, "Functions, equivalence relations, quotient spaces and subsets in fuzzy set theory", *Fuzzy Sets and Systems* **3**, pp. 75-92, 1980.

4. J. A. Goguen, L-fuzzy sets, *J. Math. Anal. Appl.* **8**, pp. 145-174, 1967.

5. K.Hur, H.W.Kang and K.C.Lee, "Fuzzy equivalence relations and fuzzy partitions", *Honam Math. J.* **28(3)** pp. 291-315, 2006.

6. K.C.Lee, "Fuzzy equivalence relations and fuzzy functions", *International J, of Fuzzy Logic and Intelligent Systems* **9(1)**, pp. 20-29, 2009.

7. V. Murali, "Fuzzy equivalence relations", *Fuzzy Sets and Systems*, **30**, pp. 155-163, 1989.

8. W. C. Nemitz, "Fuzzy relations and fuzzy functions", *Fuzzy Sets and Systems* **19**, pp. 177-191, 1986.

9. L. A. Zadeh, "Fuzzy sets", *Inform. and Control* **8**, pp. 338-353, 1965.

[1] Kul Hur
Professor of Wonkwang University
E-mail : kulhur@wonkwang.ac.kr

[2] Hyo Mi Jung
E-mail : hope1082@nate.com

[3] Wang Ro Lee
Professor of Chonbuk University
E-mail : wrlee@jbnu.ac.kr