Fuzzy r-Compactness on Fuzzy r-Minimal Spaces

Jung Il Kim[1] , Won Keun Min[2] and Young Ho Yoo[3]

Department of Statics, Kangwon National University, Chuncheon, 200-701, Korea
Department of Mathematics, Kangwon National University, Chuncheon, 200-701, Korea
Department of Statics, Kangwon National University, Chuncheon, 200-701, Korea

Abstract

In [8], we introduced the concept of fuzzy r-minimal structure which is an extension of smooth fuzzy topological spaces and fuzzy topological spaces in Chang's sense. And we also introduced and studied the fuzzy r-M continuity. In this paper, we introduce the concepts of fuzzy r-minimal compactness, almost fuzzy r-minimal compactness and nearly fuzzy r-minimal compactness on fuzzy r-minimal spaces and investigate the relationships between fuzzy r-M continuous mappings and such types of fuzzy r-minimal compactness.

Key words : fuzzy r-minimal spaces, fuzzy r-M open mapping, fuzzy r-M continuous, fuzzy r-minimal compact, almost fuzzy r-minimal compact and nearly fuzzy r-minimal compact

1. Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Chang [2] defined fuzzy topological spaces using fuzzy sets. In [3, 7], Chattopadhyay, Hazra and Samanta introduced smooth fuzzy topological spaces which are a generalization of fuzzy topological spaces.

In [8], we introduced the concept of fuzzy r-minimal space which is an extension of the smooth fuzzy topological space. The concepts of fuzzy r-open sets, fuzzy r-semiopen sets, fuzzy r-preopen sets, r-fuzzy β-open sets and fuzzy r-regular open sets are introduced in [1, 4, 5, 6], which are a kind of fuzzy r-minimal structures. We also introduced and studied the concepts of fuzzy r-M continuity, fuzzy r-M open maps and fuzzy r-M closed maps. In this paper, we introduce the concepts of fuzzy r-minimal compactness, almost fuzzy r-minimal compactness and nearly fuzzy r-minimal compactness on fuzzy r-minimal spaces and investigate the relationships between fuzzy r-M continuous mappings and such types of fuzzy r-minimal compactness.

2. Preliminaries

Let I be the unit interval $[0, 1]$ of the real line. A member μ of I^X is called a fuzzy set of X. By $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$ we denote constant maps on X with value 0 and 1, respectively. For any $\mu \in I^X$, μ^c denotes the complement $\tilde{\mathbf{1}} - \mu$. All other notations are standard notations of fuzzy set theory.

Manuscript received Feb. 9, 2009; revised Aug. 20, 2009.

Corresponding Author : wkmin@kangwon.ac.kr (Won Keun Min)

A *smooth fuzzy topology* [7] on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{\mathbf{0}}) = \mathcal{T}(\tilde{\mathbf{1}}) = 1$.
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.
(3) $\mathcal{T}(\vee \mu_i) \geq \wedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *smooth fuzzy topological space*.

Let A be a fuzzy set in a smooth fuzzy topological spaces $(X, \mathcal{T})$ and $r \in I$. Then A is said to be *fuzzy r-semiopen* [5] (resp., *fuzzy r-preopen* [4], *r-fuzzy β-open* [1]) if $A \subseteq cl(int(A, r), r)$ (resp., $A \subseteq int(cl(A, r), r)$, $A \subseteq cl(int(cl(A, r), r), r)$).

Definition 2.1. ([8]) Let X be a nonempty set and $r \in (0, 1] = I_0$. A fuzzy family $\mathcal{M} : I^X \to I$ on X is said to have a *fuzzy r-minimal structure* if the family

$$\mathcal{M}_r = \{A \in I^X \mid \mathcal{M}(A) \geq r\}$$

contains $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$.

Then the $(X, \mathcal{M})$ is called a *fuzzy r-minimal space* (simply r-FMS). Every member of $\mathcal{M}_r$ is called a *fuzzy r-minimal open* set. A fuzzy set A is called a *fuzzy r-minimal closed* set if the complement of A (simply, A^c) is a fuzzy r-minimal open set.

Let $(X, \mathcal{M})$ be an r-FMS and $r \in I_0$. The fuzzy r-minimal closure and the fuzzy r-minimal interior of A [8], denoted by $mC(A,r)$ and $mI(A,r)$, respectively, are defined as

$$mC(A,r) = \cap\{B \in I^X : B^c \in \mathcal{M}_r \text{ and } A \subseteq B\},$$

$$mI(A,r) = \cup\{B \in I^X : B \in \mathcal{M}_r \text{ and } B \subseteq A\}.$$

Theorem 2.2. ([8]) Let $(X,\mathcal{M})$ be an r-FMS and A, B in I^X.

(1) $mI(A,r) \subseteq A$ and if A is a fuzzy r-minimal open set, then $mI(A,r) = A$.

(2) $A \subseteq mC(A,r)$ and if A is a fuzzy r-minimal closed set, then $mC(A,r) = A$.

(3) If $A \subseteq B$, then $mI(A,r) \subseteq mI(B,r)$ and $mC(A,r) \subseteq mC(B,r)$.

(4) $mI(A,r) \cap mI(B,r) \supseteq mI(A \cap B,r)$ and $mC(A,r) \cup mC(B,r) \subseteq mC(A \cup B,r)$.

(5) $mI(mI(A,r),r) = mI(A,r)$ and $mC(mC(A,r),r) = mC(A,r)$.

(6) $\tilde{\mathbf{1}} - mC(A,r) = mI(\tilde{\mathbf{1}} - A,r)$ and $\tilde{\mathbf{1}} - mI(A,r) = mC(\tilde{\mathbf{1}} - A,r)$.

Definition 2.3. ([8]) Let $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ be two r-FMS's. Then $f : X \to Y$ is said to be

(1) *fuzzy r-M continuous* mapping if for every $A \in \mathcal{N}_r$, $f^{-1}(A)$ is in $\mathcal{M}_r$,

(2) *fuzzy r-M open* if for every $A \in \mathcal{M}_r$, $f(A)$ is in $\mathcal{N}_r$.

3. Fuzzy r-Minimal Compactness

Definition 3.1. Let $(X,\mathcal{M})$ be an r-FMS and $\mathcal{A} = \{A_i \in I^X : i \in J\}$. $\mathcal{A}$ is called a *fuzzy r-minimal cover* if $\cup\{A_i : i \in J\} = \tilde{\mathbf{1}}$. It is a *fuzzy r-minimal open cover* if each A_i is a fuzzy r-minimal open set. A subcover of a fuzzy r-minimal cover $\mathcal{A}$ is a subfamily of it which also is a fuzzy r-minimal cover.

Definition 3.2. Let $(X,\mathcal{M})$ be an r-FMS. A fuzzy set A in X is said to be *fuzzy r-minimal compact* if every fuzzy r-minimal open cover $\mathcal{A} = \{A_i \in \mathcal{M}_r : i \in J\}$ of A has a finite subcover.

Theorem 3.3. Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a fuzzy r-M continuous mapping on two r-FMS's. If A is a fuzzy r-minimal compact set, then $f(A)$ is also a fuzzy r-minimal compact set.

Proof. Let $\{B_i \in I^Y : i \in J\}$ be a fuzzy r-minimal open cover of $f(A)$ in Y. Then since f is a fuzzy r-M continuous mapping, $\{f^{-1}(B_i) : i \in J\}$ is a fuzzy r-minimal open cover of A in X. By fuzzy r-minimal compactness, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} f^{-1}(B_i)$. Hence $f(A) \subseteq \cup_{i \in J_0} B_i$. □

Definition 3.4. Let $(X,\mathcal{M})$ be an r-FMS. A fuzzy set A in X is said to be *almost fuzzy r-minimal compact* if for every fuzzy r-minimal open cover $\mathcal{A} = \{A_i \in I^X : i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} mC(A_i, r)$.

Theorem 3.5. Let $(X,\mathcal{M})$ be an r-FMS. If a fuzzy set A in X is fuzzy r-minimal compact, then it is also almost fuzzy r-minimal compact.

Proof. Obvious. □

In Theorem 3.5, the converse is not always true as shown the next example.

Example 3.6. Let $X = I$ and $n \in N - \{1\}$. Let A_1 and A_n be fuzzy sets defined as follows

$$A_n(x) = \begin{cases} 0.8, & \text{if } x = 0, \\ nx, & \text{if } 0 < x \leq \frac{1}{n}, \\ 1, & \text{if } \frac{1}{n} < x \leq 1; \end{cases}$$

$$A_1(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Consider a fuzzy r-minimal structure $\mathcal{M} : I^X \to I$ on X as follows

$$\mathcal{M}(A) = \begin{cases} \frac{4}{5}, & \text{if } A = \tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \\ \frac{n}{n+1}, & \text{if } A = A_n, \\ \frac{2}{3}, & \text{if } A = A_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = \{A_n : n \in N\}$ be a fuzzy $\frac{1}{2}$-minimal open cover of X. Then there does not exist a finite subcover of $\mathcal{A}$. Thus X is not fuzzy $\frac{1}{2}$-minimal compact. But X is almost fuzzy $\frac{1}{2}$-minimal compact.

Theorem 3.7. ([8]) Let $f : X \to Y$ be a mapping on two r-FMS's $(X,\mathcal{M})$ and $(Y,\mathcal{N})$.

(1) f is fuzzy r-M continuous.

(2) $f^{-1}(B)$ is a fuzzy r-minimal closed set, for each fuzzy r-minimal closed set B in Y.

(3) $f(mC(A,r)) \subseteq mC(f(A),r)$ for $A \in I^X$.

(4) $mC(f^{-1}(B),r) \subseteq f^{-1}(mC(B,r))$ for $B \in I^Y$.

(5) $f^{-1}(mI(B,r)) \subseteq mI(f^{-1}(B),r)$ for $B \in I^Y$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

Theorem 3.8. Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a fuzzy r-M continuous mapping on two r-FMS's. If A is an almost fuzzy r-minimal compact set, then $f(A)$ is also an almost fuzzy r-minimal compact set.

Proof. Let $\{B_i \in I^Y : i \in J\}$ be a fuzzy r-minimal open cover of $f(A)$ in Y. Then $\{f^{-1}(B_i) : i \in J\}$ is a fuzzy r-minimal open cover of A in X. By almost fuzzy r-minimal compact, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that

$A \subseteq \cup_{i\in J_0} mC(f^{-1}(B_i), r)$. From Theorem 3.7, it follows

$$\begin{aligned} \cup_{i\in J_0} mC(f^{-1}(B_i, r)) &\subseteq \cup_{i\in J_0} f^{-1}(mC(B_i, r)) \\ &= f^{-1}(\cup_{i\in J_0} mC(B_i, r)). \end{aligned}$$

Hence $f(A) \subseteq \cup_{i\in J_0} mC(B_i, r)$. □

Definition 3.9. Let $(X, \mathcal{M})$ be an r-FMS. A fuzzy set A in X is said to be *nearly fuzzy r-minimal compact* if for every fuzzy r-minimal open cover $\mathcal{A} = \{A_i : i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{i\in J_0} mI(mC(A_i, r), r)$.

Example 3.10. (1) Let $X = I$. Consider the fuzzy minimal structure $\mathcal{M}$ defined in Example 3.6. The fuzzy set $\tilde{\mathbf{1}}$ is an almost fuzzy $\frac{1}{2}$-minimal compact set but it is not nearly fuzzy $\frac{1}{2}$-minimal compact in $(X, \mathcal{M})$.

(2) Let $X = I$. Consider fuzzy sets for $0 < n < 1$,

$$\sigma_n(x) = \begin{cases} \frac{1}{n}x, & \text{if } 0 \le x \le n, \\ -\frac{x-1}{1-n}, & \text{if } n < x \le 1; \end{cases}$$

$$\alpha(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } 0 < x \le 1; \end{cases}$$

$$\beta(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

And consider a fuzzy minimal structure

$$\mathcal{N}(\mu) = \begin{cases} \max(\{1-n, n\}), & \text{if } \mu = \alpha_n, \\ 1, & \text{if } \mu = \alpha, \beta, \tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then X is nearly fuzzy $\frac{1}{2}$-minimal compact but not fuzzy $\frac{1}{2}$-minimal compact.

Theorem 3.11. Let $(X, \mathcal{M})$ be an r-FMS. If a fuzzy set A in X is fuzzy r-minimal compact, then it is nearly fuzzy r-minimal compact.

Proof. For any a fuzzy r-minimal open set U in X, from Theorem 2.2, it follows $U = mI(U, r) \subseteq mI(mC(U, r), r)$. Thus we get the result. □

In Theorem 3.11, the converse implication is not true always true as shown in the Example 3.10. Hence the following implications are obtained:

fuzzy r-minimal compact $\Rightarrow$ nearly fuzzy r-minimal compact $\Rightarrow$ almost fuzzy r-minimal compact

Theorem 3.12. ([8]) Let $f : X \to Y$ be a mapping on two r-FMS's $(X, \mathcal{M})$ and $(Y, \mathcal{N})$. Then
(1) f is fuzzy r-M open.
(2) $f(mI(A), r) \subseteq mI(f(A), r)$ for $A \in I^X$.
(3) $mI(f^{-1}(B), r) \subseteq f^{-1}(mI(B), r)$ for $B \in I^Y$.
Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Theorem 3.13. Let a mapping $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be fuzzy r-M continuous and fuzzy r-M open on two r-FMS's. If A is a nearly fuzzy r-minimal compact set, then $f(A)$ is a nearly fuzzy r-minimal compact set.

Proof. Let $\{B_i \in I^Y : i \in J\}$ be a fuzzy r-minimal open cover of $f(A)$ in Y. Then $\{f^{-1}(B_i) : i \in J\}$ is a fuzzy r-minimal open cover of A in X. By nearly fuzzy r-minimal compactness, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{i\in J_0} mI(mC(f^{-1}(B_i), r), r)$. From Theorem 3.7 and Theorem 3.12, it follows

$$\begin{aligned} f(A) &\subseteq \cup_{i\in J_0} f(mI(mC(f^{-1}(B_i), r), r)) \\ &\subseteq \cup_{i\in J_0} mI(f(mC(f^{-1}(B_i), r)), r) \\ &\subseteq \cup_{i\in J_0} mI(f(f^{-1}(mC(B_i, r))), r) \\ &\subseteq \cup_{i\in J_0} mI(mC(B_i, r), r). \end{aligned}$$

Hence $f(A)$ is a nearly fuzzy r-minimal compact set. □

Remark 3.14. In Theorem 3.13, the fuzzy r-M continuity and fuzzy r-M openness of the mapping f are necessary conditions as shown in the next example.

Example 3.15. Let $X = I$. Consider fuzzy sets for $0 < n < 1$,

$$\sigma_n(x) = \begin{cases} \frac{1}{n}x, & \text{if } 0 \le x \le n, \\ -\frac{x-1}{1-n}, & \text{if } n < x \le 1; \end{cases}$$

$$\alpha(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } 0 < x \le 1; \end{cases}$$

$$\beta(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1; \end{cases}$$

$$\gamma(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } 0 < x \le 1; \end{cases}$$

$$\eta(x) = \begin{cases} 1, & \text{if } 0 \le x < 1, \\ 0, & \text{if } x = 1. \end{cases}$$

And consider fuzzy minimal structures

$$\mathcal{L}(\mu) = \begin{cases} \max(\{1-n, n\}), & \text{if } \mu = \alpha_n, \\ 1, & \text{if } \mu = \alpha, \beta, \gamma, \eta, \tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \\ 0, & \text{otherwise;} \end{cases}$$

$$\mathcal{M}(\mu) = \begin{cases} \max(\{1-n, n\}), & \text{if } \mu = \alpha_n, \\ 1, & \text{if } \mu = \alpha, \beta, \tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \\ 0, & \text{otherwise;} \end{cases}$$

$$\mathcal{N}(\mu) = \begin{cases} \max(\{1-n, n\}), & \text{if } \mu = \alpha_n, \\ 1, & \text{if } \mu = \tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $f : (X, \mathcal{L}) \to (X, \mathcal{M})$ be the identity mapping. It is obvious that f is fuzzy $\frac{1}{2}$-M continuous. X is nearly fuzzy $\frac{1}{2}$-minimal compact on $(X, \mathcal{L})$ but $f(X)$ is not nearly fuzzy $\frac{1}{2}$-minimal compact on $(X, \mathcal{M})$.

Now let $f : (X, \mathcal{N}) \to (X, \mathcal{M})$ be the identity mapping. Then f is fuzzy $\frac{1}{2}$-M open. Consider a fuzzy set A defined as follows

$$A(x) = \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{if } x = 0, 1. \end{cases}$$

Then A is nearly fuzzy $\frac{1}{2}$-minimal compact on $(X, \mathcal{N})$ but $f(A)$ is not nearly fuzzy $\frac{1}{2}$-minimal compact $(X, \mathcal{M})$.

References

[1] S. E. Abbas, ”Fuzzy β-irresolute functions”, *Applied Mathematics and Computation*, vol. 157, pp. 369-380, 2004.

[2] C. L. Chang, ”Fuzzy topological spaces”, *J. Math. Anal. Appl.*, vol. 24, pp. 182-190, 1968.

[3] K. C. Chattopadhyay, R. N. Hazra, and S. K. Samanta, ”Gradation of openness : Fuzzy topology”, *Fuzzy Sets and Systems*, vol. 49, pp. 237-242, 1992.

[4] S. J. Lee and E. P. Lee, ”Fuzzy r-preopen and fuzzy r-precontinuous maps”, *Bull. Korean Math. Soc.*, vol. 36, pp. 91-108, 1999.

[5] ———-, ”Fuzzy r-continuous and fuzzy r-semicontinuous maps” , *Int. J. Math. Math. Sci.*, vol. 27, pp. 53-63, 2001.

[6] ———, ”Fuzzy r-regular open sets and fuzzy almost r-continuous maps” , *Bull. Korean Math. Soc.*, vol. 39, pp. 91-108, 2002.

[7] A. A. Ramadan, ”Smooth topological spaces”, *Fuzzy Sets and Systems*, vol. 48, pp. 371-375, 1992.

[8] Young Ho Yoo, Won Keun Min and Jung IL Kim. ”Fuzzy r-Minimal Structures and Fuzzy r-Minimal Spaces”, *Far East Journal of MAthematical Sciences*, vol. 33, no. 2, pp. 193-205, 2009.

[9] L. A. Zadeh, ”Fuzzy sets”, *Information and Control*, vol. 8, pp. 338-353, 1965.

Jung Il Kim[1]
Professor of Kangwon National University
Research Area: Fuzzy logic, Statics
E-mail : jikim@kangwon.ac.kr

Won Keun Min[2]
Professor of Kangwon National University
Research Area: Fuzzy topology, General topology
E-mail : wkmin@kangwon.ac.kr

Young Ho Yoo[3]
Professor of Kangwon National University
Research Area: Fuzzy logic, Statics
E-mail : yhyoo@kangwon.ac.kr