Calculation of Data Reliability with Entropy for Fuzzy Sets

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Abstract

Measuring uncertainty for fuzzy sets has been carried out by calculating fuzzy entropy. Fuzzy entropy of fuzzy set is derived with the help of distance measure. The distance proportional value between the fuzzy set and the corresponding crisp set is designed as the fuzzy entropy. The usefulness is verified by proving the proposed entropy. Generally, fuzzy entropy contains the complementary characteristics that the fuzzy entropies of fuzzy set and complementary fuzzy set have the same entropies. Discrepancy that low fuzzy entropy did not guarantee the data certainty was overcome by modifying fuzzy entropy formulation. Obtained fuzzy entropy is analyzed and discussed through simple example.

Key Words : Uncertainty, Fuzzy entropy, distance measure

1. Introduction

The characterization and quantification of data fuzziness are important and interesting in the modeling and system designs, and they affect the management of uncertainty to the related fields. The fact that entropy of a fuzzy set means a measure of its fuzziness has been established by previous researchers [1–6]. Zadeh first proposed fuzzy entropy as a measure of fuzziness; Pal and Pal analyzed classical Shannon information entropy; Kosko considered the relationship between distance measure and fuzzy entropy; Liu proposed axiomatic definitions of entropy, distance measures, and similarity measures and discussed the relationships among these three concepts. Bhandari and Pal presented a measure of fuzzy information for distinguishing between fuzzy sets. Further, Ghosh used fuzzy entropy in neural networks.

Two measures, entropy and similarity, represent the uncertainty and similarity with respect to the corresponding crisp set, respectively. Considering fuzzy entropy represents the degree of fuzziness of data, low fuzzy entropy value of data is close to the deterministic value. Fuzzy entropy is applicable to data selection problem, calculation of degree of fuzziness, or other uncertainty calculation. However, owing to the complementary characteristic of entropy definition, confusion is sometimes shown. In data selection problem, low fuzzy entropy value does not guarantee the certainty of data. In this paper, modified fuzzy entropy is proposed to overcome the drawback of complementary characteristic. Obtained fuzzy entropy is considered by adjusting corresponding crisp set. By the analysis of corresponding crisp set, data quantifying process is discussed clearly. Reliable data selection with fuzzy entropy is also discussed through analyzing fuzzy membership function and comparing the computation results.

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In the following chapter, fuzzy set entropy is explained through fuzzy membership function and corresponding crisp set. Normal fuzzy entropy is designed explicitly, which is based on the distance measure definition. The corresponding crisp set for fuzzy set satisfying minimum fuzzy entropy is derived through entropy formulation. Proposed fuzzy entropy is verified by proving definition. Furthermore, reliable data selection problem is carried out. In Chapter 3, the modified fuzzy entropy to overcome the drawback of conventional fuzzy entropy is considered. Modified fuzzy entropy is designed through changing corresponding crisp set. To obtain the modified fuzzy entropy, assumptions are needed. In assumptions, membership function symmetricity is required. Discussions are also followed. The conclusions are stated in Chapter 4.

2. Fuzzy Entropy for Fuzzy Sets

Data uncertainty and certainty are contained in fuzzy set itself, and its uncertainty and uncertainty are represented through fuzzy membership function. Mentioned uncertainty is often measured by fuzzy entropy, and the explicit fuzzy entropy formulation has also been proposed by numerous researchers [1-7]. Fuzzy entropy of fuzzy set explains that how much uncertainty fuzzy set has with respect to the corresponding crisp set.

Next, the axiomatic definition of fuzzy entropy for fuzzy set has been proposed by Liu [7]. Definition has four properties for considering data. Numerous fuzzy entropies can be also obtained satisfying Definition 2.1. However, a few explicit fuzzy entropy design was shown in literatures [1-7]. We have designed several fuzzy entropies satisfying Definition 2.1, the proposed fuzzy entropies can be found out in our literatures [8,9]. The fuzzy membership function and crisp set pair are illustrated in Fig. 1. Where, crisp set A_{crisp} is defined by the value of crisp set $A_{0.5}$. A_{crisp} represents variable range by changing as $0 \le crisp \le 1$. In Fig. 1, the value of crisp set

$A_{0.5}$ has one when $\mu_A(x) \geq 0.5$, and is zero otherwise.

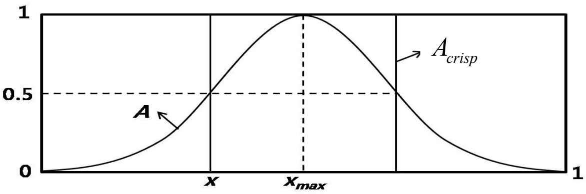

Fig. 1 Membership functions of fuzzy set A and crisp set $A_{crisp} = A_{0.5}$.

Liu suggested axiomatic definition of fuzzy entropy [7].

Definition 2.1 [7] A real function, $e : F(X) \to R^+$ is called the entropy on $F(X)$, if e has the following properties:

$(E1) : e(D) = 0, \forall D \in P(X);$

$(E2) : e([1/2]_X) = max_{A \in F(X)} e(A);$

$(E3) : e(A^*) \leq e(A),$ *for any sharpening* A^* *of* A; *and*

$(E4) : e(A) = e(A^C), \forall A \in F(X),$

where $[1/2]_X$ is the fuzzy set in which the value of the membership function is $1/2$.

Now it is possible to design fuzzy entropy that is satisfying Definition 2.1 as follows:

Theorem 2.1 A real function

$$e(A, A_{crisp}) = 2d(A, A \cap A_{crisp}) + 2d(A_{crisp}, A \cap A_{crisp}) \quad (1)$$

is the normal fuzzy entropy on $F(X)$.

Proof : To prove fuzzy entropy (1), Definition 2.1 is utilized. For all $D \in P(X)$,

$$e(D, D_{crisp}) = 2d(D, D \cap D_{crisp}) + 2d(D_{crisp}, D \cap D_{crisp}) = 0$$

is obtained. Because D_{crisp} satisfies D. Hence (E1) is satisfied naturally. For (E2), $e([1/2]_X, [1/2]_{X^{crisp}})$ satisfies maximum value. Let $[1/2]_{X^{crisp}} = [1]_X$,

$2d([1/2]_X, [1/2]_X \cap [1]_X) + 2d([1]_X, [1/2]_X \cap [1]_X) = 1$

satisfies maximum value.

(E3) is satisfied by

$e(A^*, A^*_{crisp}) = 2d(A^*, A^* \cap A^*_{crisp}) + 2d(A^*_{crisp}, A^* \cap A^*_{crisp})$
$\leq 2d(A, A \cap A_{crisp}) + 2d(A_{crisp}, A \cap A_{crisp}) = e(A, A_{crisp})$.

Where, $A^*_{crisp} = A_{crisp}$, $d(A^*, A^* \cap A^*_{crisp}) \leq d(A, A \cap A_{crisp})$, and $d(A^*_{crisp}, A^* \cap A^*_{crisp}) \leq 2d(A_{crisp}, A \cap A_{crisp})$ are also satisfied.

(E4) is obvious with graphical explanation of Fig.1. With the distance property $d(A,B) = d(A^C, B^C)$ [7], it is obtained through

$e(A^C, A^C_{crisp}) = 2d(A^C, A^C \cap A^C_{crisp}) + 2d(A^C_{crisp}, A^C \cap A^C_{crisp})$
$= 2d(A, (A^C \cap A^C_{crisp})^C) + 2d(A_{crisp}, (A^C \cap A^C_{crisp})^C)$
$= 2d(A, A \cup A_{crisp}) + 2d(A_{crisp}, A \cup A_{crisp})$.

Following equalities,
$d(A, A \cup A_{crisp}) = d(A_{crisp}, A \cap A_{crisp})$
and $d(A_{crisp}, A \cup A_{crisp}) = d(A, A \cap A_{crisp})$
are obtained from graphical explanation. Hence,
$e(A^C, A^C_{crisp}) = 2d(A, A \cup A_{crisp}) + 2d(A_{crisp}, A \cup A_{crisp})$
$= 2d(A_{crisp}, A \cap A_{crisp}) + 2d(A, A \cap A_{crisp})$
$= e(A, A_{crisp})$
is obtained. Hence (E4) is proved.

The corresponding crisp set for fuzzy set A is defined by following corollary.

Corollary 2.1 In theorem 2.1, A_{crisp} is satisfied for all $0 \leq crisp \leq 1$.

The fuzzy entropy in Eq. (1) satisfies for all value of crisp set A_{crisp}. Hence, $A_{0.1}$ and $A_{0.5}$ or some other $A_{0.X}$ can be satisfied. Now, it is interesting to search for what value of A_{crisp} satisfies the maximum or minimum entropies.

Eq. (1) is rewritten as follows:

$$e(A, A_{crisp}) = 2\int_0^x \mu_A(x)dx + 2\int_x^{x_{\max}} 1 - \mu_A(x)dx \quad (2)$$

Let $\frac{d}{dx} M_A(x) = \mu_A(x)$; $e(A, A_{crisp})$ has been shown to be

$e(A, A_{crisp}) = 2M_A(x) |_0^x + 2(x_{\max} - x) - 2M_A(x) |_x^{x_{\max}}$.

The maxima or minima are obtained by differentiating (2):

$$\frac{d}{dx} e(A, A_{crisp}) = 2\mu_A(x) - 2 + 2\mu_A(x).$$

Hence, it is clear that the point x satisfying $\frac{d}{dx} e(A, A_{crisp}) = 0$ is the critical point for the crisp set. This point is given by $\mu_A(x) = 1/2$, i.e., $A_{crisp} = A_{0.5}$. The fuzzy entropy between A and $A_{0.5}$ has a minimum value because $e(A)$ attains maxima when the corresponding crisp sets are $A_{0.0}$ and $A_{x_{\max}}$. Hence, for a nonconvex and symmetric fuzzy set, the minimum entropy of the fuzzy set is equal to that of the crisp set $A_{0.5}$. This indicates that the corresponding crisp set that has the least uncertainty or the greatest similarity with the fuzzy set is $A_{0.5}$.

In our previous results, more fuzzy entropies of fuzzy set A with respect to A_{crisp} is represented as follows [8,10]

$$e(A, A_{crisp}) = d(A \cap A_{crisp}, [1]_X) + d(A \cup A_{crisp}, [0]_X) - 1 \quad (3)$$

$$e(A, A_{crisp}) = d(A \cap A^C_{crisp}, [0]_X) + d(A \cup A^C_{crisp}, [1]_X) \quad (4)$$

$$e(A, A_{crisp}) = 1 - d(A \cap A_{crisp}, [0]_X) - d(A \cup A_{crisp}, [1]_X) \quad (5)$$

$d(A,B)$ is the Hamming distance between A and B, that is

$d(A,B) = \sum_{i=1}^{n} | \mu_A(x_i) - \mu_B(x_i) |$

$d(A \cap A_{crisp}, [1]_X) = \frac{1}{n}\sum_{i=1}^{n} | \mu_{A \cap A_{crisp}}(x_i) - 1 |$ is also satisfied. $[0]_X$ and $[1]_X$ are the fuzzy sets in which the value of the membership functions are zero and one, respectively, for the universe of discourse.

Proofs of (3), (4) and (5) are found in our previous literature [8,10]. Proposed fuzzy entropies do not give the normal fuzzy entropy. The normal fuzzy entropy can be obtained by multiplying appropriate value to the right-hand side, which satisfies maximal fuzzy entropy is one.

Next, uncertainty calculations are obtained through applying fuzzy entropy. Consider the students points for the examination. There are examination points of 65 students, 52.7 for mean and 14.5 for standard variation as Table 1.

Table 1. Examination points, mean and standard deviation of 65 students.

65 students points	82, 81.5, 76, 75, 75, 68, 67, 65.5, 65, 64.5, 64, 63.5, 63, 63, 62.5, 62, 61, 61, 60, 60, 60, 59, 59, 59, 58, 58, 58, 57.5, 57.5, 57, 56.8, 56, 55.5, 54, 53.5, 52.5, 52.5, 52.5, 52.5, 52, 51, 51, 49.5, 48, 47.5, 46.5, 46, 45.5, 45, 45, 44, 43, 41.5, 41, 40, 37, 37, 36, 33.5, 32, 31, 27, 26.5, 21, 0
Mean : 52.7, standard deviation : 14.5	

Now, let's select 5 students randomly. Two times of selections are carried out. Normally, point distribution is close to the Gaussian distribution. In Fig. 2, Gaussian distribution is considered as the fuzzy membership function, and the chosen 5 students scores are also shown in. 5 students' scores are chosen randomly. In Fig.2(a), 5 students have 50, 52, 55, 57, and 59 points. Whereas, 12, 46, 53, 55, and 91 points are illustrated in Fig. 2(b). The second trial is closer to the mean value, whereas the first one is nearer to the membership degree in the view of membership average. Hence, it is hard to determine which one is reliable data for average level student. However, we can obtain the result that the first one is reliable data for average level student in the view of heuristic point.

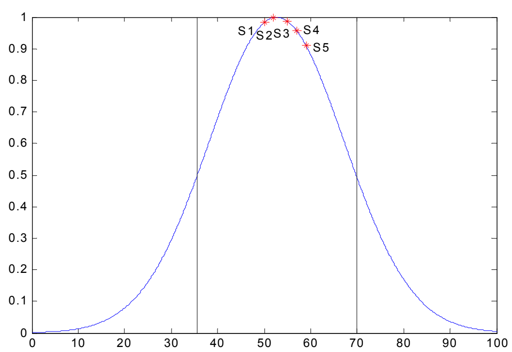

Fig.2(a) Membership function and 5 students points

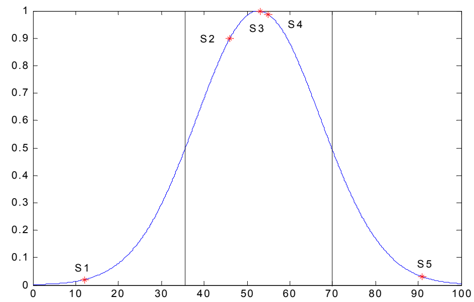

Fig.2(b) Membership function and 5 students points

The average level student's points are between 37 and 71, i.e, $\mu_{A_{0.5}}(x) = 1$ when $37 \le x \le 71$, $\mu_{A_{0.5}}(x) = 0$ otherwise. In the view of fuzzy entropy, both cases are calculated for the problem of how much they are in the average level.

Table 2. Sample, Membership value, and Fuzzy entropy for selected 5 data with Eq. (1)

	Sample	Membership value	Fuzzy entropy
Fig. 2(a)	50	0.983	0.0656
	52	0.999	
	55	0.987	
	57	0.957	
	59	0.910	
Average	**54.6**	**0.980**	**0.0656**
Fig.2(b)	12	0.019	0.0656
	46	0.899	
	53	1.000	
	55	0.987	
	91	0.031	
Average	**51.4**	**0.590**	**0.0656**

Computation results say that

$$e(A, A_{0.5}) = 2d(A, A \cap A_{0.5}) + 2d(A_{0.5}, A \cap A_{0.5})$$
$$= \frac{2}{5}(|1-0.983| + |1-0.999|$$
$$+|1-0.987| + |1-0.957| + |1-0.91|)$$
$$= 0.0656 .$$

In the above, $d(A, A \cap A_{0.5})$ has to be deleted because of distance between same points. Similarly, Fig. 2(b) shows that

$$e(A, A_{0.5}) = 2d(A, A \cap A_{0.5}) + 2d(A_{0.5}, A \cap A_{0.5})$$
$$= \frac{2}{5}|0.019-0| + |0.031-0|)$$
$$+\frac{2}{5}|1-0.899| + |1-1| + |1-0.987|)$$
$$= 0.0656 .$$

Hence, the fuzzy entropy results indicate that two trials have same degree of uncertainty. Furthermore, they show good certainty because of small uncertainty value. However, their data points are not proper to represent middle level. The reason for the same fuzzy entropy values of two trials is originated from the property of complementary, that is $e(A)=e(A^c), \forall A \in F(X)$. Now it is needed another approach to overcome this drawback of data selection problem.

3. Modified Fuzzy Entropy Formulation

Quantitative value of fuzzy entropy sometimes induces confusion like a data selection problem in Chapter 2. The result of Table 2 says that the uncertainties of two data group with respect to $A_{0.5}$ show equivalent. Main reason of that confusion is the complementary property. The shaded area of Fig. 3 represent small uncertainty for chosen data.

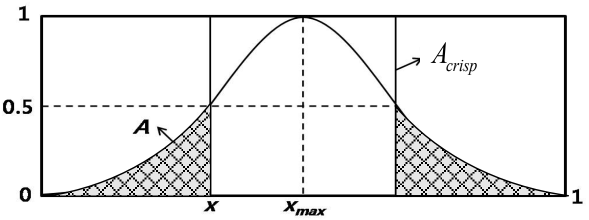

Fig. 3 Complementary characteristic Feature of Membership function

Data in shaded area has low fuzzy entropy. However, its relationship to average level is not nearly connected with. Hence it is reasonable to consider another corresponding crisp set A_{crisp} of fuzzy set A. By Corollary 2.1, A_{crisp} has to be chosen for the well matched corresponding $A_{0.1}$ and $A_{0.2}$ or some other $A_{0.X}$. Now, it is interesting to search what value of $A_{0.X}$ is proper to compute uncertainty of the membership function itself. For $crisp > 0.5$, uncertainty in the low membership function data is low. Its data uncertainty represents similar value of high membership function data. Hence, it is not appropriate to consider the corresponding crisp set as higher value of $A_{0.X}$. Whereas, the lower the membership function is the higher the uncertainty has as the fuzzy entropy for $crisp < 0.5$.

Assumption 3.1

1) Membership function has to be symmetric and non-convex with respect to both sides.

2) Area of fuzzy membership function has to be more than total half area.

Theorem 3.1 Under the assumption 3.1 a real function

$$e(A, A_{0.0}) = 2d(A, A_{0.0}) \qquad (6)$$

is the normal fuzzy entropy on $F(X)$.

Proof : Proof procedures are easily followed. For all $D \in P(X)$,

$$e(D, D_{0.0}) = d(D, D_{0.0}) = 0$$

is obtained. Because $D_{0.0}$ satisfies D itself. Hence (E1) is satisfied naturally. For (E2), $e([1/2]_X, [1/2]_{X^{0.0}})$ satisfies maximum value. Let $[1/2]_{X^{0.0}} = [1]_X$,

$$2d([1/2]_X, [1]_X) = 1$$

satisfies maximum value.

(E3) is satisfied by

$$e(A^*, A^*_{0.0}) = d(A^*, A^*_{0.0}) \le d(A, A_{0.0}) = e(A, A_{0.0}).$$

Where, $A^*_{0.0} = A_{0.0}$, inequality can be explained by Fig. 4 and definition of sharpening of fuzzy membership function.

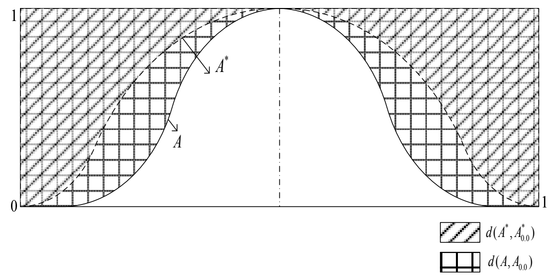

Fig. 4 Components of $d(A, A_{0.0})$ and $d(A^*, A^*_{0.0})$

(E4) is obvious using the graphical explanation. The components of $e(A, A_{0.0})$ and $e(A^C, A^C_{0.0})$ are illustrated in Fig 5.

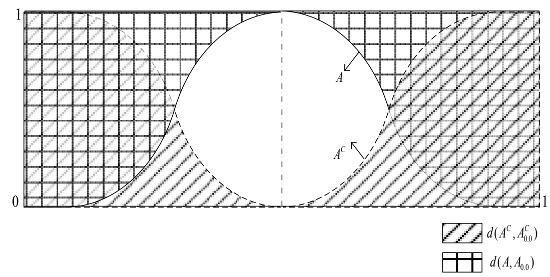

Fig. 5 Components of $d(A, A_{0.0})$ and $d(A^C, A^C_{0.0})$

Hence,

$$e(A, A_{0.0}) = d(A, A_{0.0}) = d(A^C, A^C_{0.0}) = e(A^C, A^C_{0.0})$$

is satisfied because of distance property, $d(A,B) = d(A^C, B^C)$ [7].

Corollary 3.1 With A_{crisp} approaches to $A_{0.0}$, fuzzy entropy (1) becomes (6).

In Corollary 3.1, fuzzy entropy is changed by

$$e(A, A_{0.0}) = 2d(A, A \cap A_{0.0}) + 2d(A_{0.0}, A \cap A_{0.0}) \qquad (7)$$

as $A_{near} \to A_{0.0}$. In the above, $A \cap A_{0.0} = A$ is satisfied because $A \cap A_{0.0} = \min\{\mu_A(x), \mu_{A_{0.0}}(x)\}$ and $\mu_{A_{0.0}}(x)$ is one over whole universe of discourse. Then, (7) satisfies

$$e(A, A_{0.0}) = 2d(A, A) + 2d(A_{0.0}, A) = 2d(A_{0.0}, A).$$

Hence, (6) satisfies fuzzy entropy definition.

Corollary 3.2 With A_{crisp} approaches to $A_{1.0}$ or $A_{\max}$, fuzzy entropy (1) becomes total area of fuzzy membership function itself.

From Corollary 3.2 fuzzy entropy value can be often collided with the meaning of uncertainty. It means that high uncertain data may have low fuzzy entropy value. In (6), $A_{0.0}$ can be naturally replaced into $A_{0.1}$ to calculate the fuzzy entropies of Fig.2. Their entropy calculations are equivalent each other. Because all data of Fig.2 are contained inside of $A_{0.0}$ or $A_{0.1}$.

Next, fuzzy entropy calculations are carried out with example in Chapter 2. Corresponding crisp set for fuzzy set is considered by $A_{crisp} = A_{0.0}$.

Table 3. Sample, Membership value, and Fuzzy entropy for selected 5 data with Eq. (5)

	Sample	Membership value	Fuzzy entropy
Fig. 2(a)	50	0.983	0.0656
	52	0.999	
	55	0.987	
	57	0.957	
	59	0.910	
Average	**54.6**	**0.980**	**0.0656**
Fig.2(b)	12	0.019	0.8256
	46	0.899	
	53	1.000	
	55	0.987	
	91	0.031	
Average	**51.4**	**0.590**	**0.8256**

Fuzzy entropy comparison between Fig.2(a) and Fig.2(b) shows that the proposed fuzzy entropy (6) discriminate the data group. Fig. 2(a) calculation is represented by

$$e(A, A_{0.5}) = 2d(A, A_{0.0})$$
$$= \frac{2}{5}(|1-0.983| + |1-0.999|$$
$$+ |1-0.987| + |1-0.957| + |1-0.91|)$$
$$= 0.0656 .$$

Whereas, Fig. 2(b) shows that

$$e(A, A_{0.0}) = 2d(A, A_{0.0})$$
$$= \frac{2}{5}|1-0.019| + |1-0.899|$$
$$+ |1-1| + |1-0.987| + |1-0.031|)$$
$$= 0.8256 .$$

Then, it is clear that the selection of Fig.2(a) illustrate good performance for the average level students. The difference between the result of Table 2 and that of Table 3 is originated from $A_{0.5}$ and $A_{0.1}$. Data sample of 12 and 91 are beyond of range $A_{0.5}$, hence their entropy value were small. However their entropy values are bigger when they lie inside of $A_{0.1}$ than when $A_{0.5}$ was.

4. Conclusions

Calculation of data fuzziness has been carried out by fuzzy entropy. Fuzzy entropies for fuzzy sets were developed by considering the crisp set "near" the fuzzy set. The minimum entropy of fuzzy set can be obtained when the corresponding crisp set is satisfied at $A_{crisp} = A_{0.5}$. By the fact that low entropy represents high certainty reliable data selection of average level students was done. However, obtained results are not satisfactory because of complementary characteristic of fuzzy entropy definition. To overcome this drawback another crisp set was considered by varying the A_{crisp}. Proposed fuzzy entropy is obtained by approaching A_{crisp} to $A_{0.0}$. Then heuristically reasonable performance is obtained with fuzzy entropy calculation. Data analysis is also discussed after entropy calculation was done.

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