International Journal of Reliability and Applications
Vol. 10, No. 1, pp. 43-57, 2009

Reliability Equivalence Factors of $n-$components Series System with Non-constant Failure Rates

A. Mustafa * †
Department of Mathematics, College of Science and Humanitarian Studies, Al-Kharj, King Saud University, P.O. 83, A-Kharj 11942, Saudi Arabia.

Abstract. In this article, we study the reliability equivalence factor of a series system. The failure rates of the system components are functions of time t. we study two cases of non-constat failure rates (i) weibull distribution (ii) linear increasing failure rate distribution. There are two methods are used to improve the given system. Two types of reliability equivalence factors are discussed. Numerical examples are presented to interpret how one can utilize the obtained results.

Key Words : *Weibull distribution; hot duplication; reduction method; reliability equivalence factor; Exponential distribution; Rayleigh distribution; $\theta-$fractiles.*

1. INTRODUCTION

The concept of the reliability equivalence factors introduced by Råde (1989). Råde (1993a, 1993b) applied this concept on simple series and parallel systems consists of one and two components. Later, Sarhan (2000, 2002, 2004, 2005), Mustafa (2002), Sarhan et al. (2004), Sarhan and Mustafa (2006) and Mustafa et al.(2007b) applied the same concept on more general and complex systems. In the pervious articles authors consider that the component has constant failure rates with exponential life distribution. Mustafa (2008) studied the simple system with 2 components connected in series system with constant failure rate, he introduced new methods to improve the system reliability. Mustafa et al. (2007a), studied the series system consists n components with constant mixture failure rates.

Xia and Zhang (2007), applied the concept of the reliability equivalence on n components parallel system with non-constant failure rates, authors considered the

*Corresponding Author. *E-mail address:* abdelfatah_mustafa@yahoo.com
†*Home address:* Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

life distribution of the components are Gamma distribution. Gamma distribution can be reduced to the exponential distribution when $n = 1$, this means $GD(1, \lambda) \equiv ED(\lambda)$

In this article, we applied the concept reliability equivalence when the components have non-constant failure rates.

The main objective of this article is to calculate the REF's of a series system. The system considered in this article is a series system, which consists of n independent and identical components, with non-constant failure rate, such as Weibull failure rates and linear increasing failure rates.

We use each of the following methods to improve this system:

1. Reduction method
2. Hot duplication method

The system reliabilty function (RF) and mean time to failure (MTTF) will be used as reference of the system performances. For this reason, we obtain the RFs and MTTFs of the original and improved systems using each improving methods.

The reliability equivalence factors (REF) of the system is that factors ρ, $0 < \rho < 1$, by which the failure rates of some of the system components should be reduced to get a reliability for the system as that for a system obtained by assuming the improved methods mentioned above.

This paper is organized as follows. Section 2 presents the n-component series system with Weibull distribution with parameters $\alpha,\ \beta$ (WD($\alpha,\ \beta$)). Section 3 introduces n-component series system with linear increasing failure rate distribution α, β (LIFRD(α, β)). The RFs and MTTFs, the α-fractiles of the original and improved systems, the REFs are obtained in subsections of Sections 2 and 3. In the end of Sections 2 and 3 numerical results and conclusions are calculated, some special cases are presented.

2. SERIES SYSTEM WITH WEIBULL FAILURE RATES

The WD(α, β), in common with a small number of other distributions such as the gamma distribution (GD) and Lognormal distribution, has one very important property; the distribution has no specific characteristic shape. In fact, depending upon the values of the parameters in its RFs, it can be shaped to represent many distributions as well as shaped to fit sets of experimental data that cannot be characterized as a particular distribution other than as a WD(α, β) certain shaping parameters.

For this reason the WD(α, β) has a very important role to play in statistical analysis, Kapur and Lamberson (1977), of experimental data.

The failure density function of the weibull random variable T is defined as

$$f(t) = \frac{\beta\, t^{\beta-1}}{\alpha^{\beta}}\ \exp\left\{-(\frac{t}{\alpha})^{\beta}\right\} \tag{2.1}$$

where $t \geq 0$, the scale parameter $\alpha > 0$ and the shape parameter $\beta > 0$. The RF of T is

$$R(t) = \exp\left\{-(\frac{t}{\alpha})^{\beta}\right\} \tag{2.2}$$

The hazard (failure) rate function of T is

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\beta\, t^{\beta-1}}{\alpha^{\beta}} \tag{2.3}$$

There are two particular cases that can be deduced from the WD(α, β):

1. For $\beta = 1$, in this case WD(α, β) reduced to the exponential distribution with parameter $1/\alpha$ (ED($1/\alpha$)).
2. For $\beta = 2$, in this case WD(α, β) reduced to the Rayleigh distribution with parameter α (RD(α)).

Typical shape that can be produced for the WD(α, β) including the exponential case are shown in Figure 6.15 in Billinton and Allan (1983), for the failure rate.
It is evident from this figure that:

1. $\beta < 1$, represents a decreasing hazard rate or the debugging period,
2. $\beta = 1$, represents a constant hazard rate or the normal life period. and
3. $\beta > 1$, represents an increasing hazard rate or the wearout period.

The expected value of the weibull random variable is given by

$$E[T] = \alpha\,\Gamma\left(\frac{1}{\beta} + 1\right) \tag{2.4}$$

where Γ is the gamma function defined as

$$\Gamma(\gamma) = \int_0^{\infty} t^{\gamma-1} e^{-t}\, dt$$

which for integer values of γ, reduced to $\Gamma(\gamma) = (\gamma - 1)!$.

2.1. The Original Systems

We consider a system consists of n independent and identical components connected in series system. Let T_i be the lifetime of the component i, $i = 1, 2, \cdots, n$. It is assumed that T_i is Weibull random variable with parameter α, β.
The RF of the system, R(t), is given by

$$R_s(t) = \prod_{i=1}^{n} R_i(t) = \exp\left\{-n(\frac{t}{\alpha})^{\beta}\right\} \tag{2.5}$$

Let MTTF_s be the system MTTF, which is given by

$$\text{MTTF}_s = \int_0^\infty R_s(t)\, dt = \alpha\, n^{-\frac{1}{\beta}}\, \Gamma\left(\frac{1}{\beta} + 1\right) \tag{2.6}$$

2.2. The Improved Systems

The system reliability can be improved according to one of the following different methods:

(1) Reducing the failure rates of r components of system components by the same factors ρ, $0 < \rho < 1$.

(2) Assuming duplication method of m components of the system components by hot duplication method. It means that each component is duplicated by a hot redundant standby component.

To derive the REFs of the underlying system, we make equivalence between the improved systems that obtained by using the reduction method and the rest duplication methods.

2.2.1. Reduction Method

Let $R_{\rho,r}(t)$ be the RF of the improved system when the failure rate of the r components of the system components are reduced by the factor ρ, $0 < \rho < 1$. One can obtain the function $R_{\rho,\,r}(t)$ as follows.

$$R_{\rho,\,r}(t) = \exp\left\{-\,[n + (\rho - 1)\, r]\; (\frac{t}{\alpha})^{\beta}\right\} \tag{2.7}$$

From equation (2.7) the MTTF of the improved system, say $MTTF_{\rho,\,r}$ becomes

$$\text{MTTF}_{\rho,\,r} = \alpha\; [n + (\rho - 1)\, r]^{-\frac{1}{\beta}}\; \Gamma\left(\frac{1}{\beta} + 1\right) \tag{2.8}$$

That is, reducing the failure rate of the r components of the system components increases the system MTTF by the amount

$$\alpha\, n^{-\frac{1}{\beta}} \left\{\left[(\rho - 1)\,\frac{r}{n} + 1\right]^{-\frac{1}{\beta}} - 1\right\}\, \Gamma\left(\frac{1}{\beta} + 1\right).$$

2.2.2. Hot Duplication Method

Let $R^H_m(t)$ be the RF of the improved system assuming hot duplication of m components of the system components. The function $R^H_m(t)$ is given by

$$\begin{aligned} R^H_m(t) &= \left[2 - \exp\left\{-(\frac{t}{\alpha})^{\beta}\right\}\right]^m \exp\left\{-n(\frac{t}{\alpha})^{\beta}\right\} \\ &= \sum_{k=0}^{m} \tbinom{m}{k}\, 2^k\, (-1)^{m-k} \exp\left\{-(n + m - k)\, (\frac{t}{\alpha})^{\beta}\right\} \end{aligned} \tag{2.9}$$

Let MTTF_m^H be the MTTF of the improved system assuming hot duplication of the system components. Using Equation (2.9), one can deduce MTTF_m^H as

$$MTTF_m^H = \alpha\,\Gamma\left(\frac{1}{\beta}+1\right)\sum_{k=0}^{m} 2^k\,(-1)^{m-k}\,\tbinom{m}{k}\,(n+m-k)^{-\frac{1}{\beta}} \tag{2.10}$$

That is, hot duplication of the m components of the system components increases the system MTTF by the amount

$$\alpha\,\Gamma\left(\frac{1}{\beta}+1\right)\left[\sum_{k=0}^{m} 2^k\,(-1)^{m-k}\,\tbinom{m}{k}\,(n+m-k)^{-\frac{1}{\beta}} - n^{-\frac{1}{\beta}}\right]$$

2.3. The θ-Fractiles

This section presents the $\theta-$fractiles of the original and improved systems. Let $L(\theta)$ be the $\theta-$fractiles of the original system and $L^H(\theta)$, the θ-fractiles of the improved system by using the hot duplication method.

The $\theta-$fractiles Ł(θ) and $L_m^H(\theta)$ are defined as the solution of the following equations, respectively,

$$R\,(\alpha\,L(\theta)) = \theta,\;\; R^H\,(\alpha\,L(\theta)) = \theta. \tag{2.11}$$

It follows from Equation (2.5) and the first Equation of (2.11) that

$$L(\theta) = \left[-\frac{\ln(\theta)}{n}\right]^{\frac{1}{\beta}} \tag{2.12}$$

From Equation (2.9), the second Equation of (2.11), one can verify that $L = L_m^H(\theta)$, satisfies the following equation

$$m\ln\left[2-\exp\left\{-L^\beta\right\}\right] - n\,L^\beta - \ln(\theta) = 0 \tag{2.13}$$

Equation (2.13) has no closed form solution and can be solved using some numerical program such as Mathematica program.

2.4. Reliability Equivalence factors

In this section, we derive the survival reliability equivalence factor (SREF) and mean reliability equivalence factor (MREF) of the n components series system.

The SREF, $\rho_{m,r}^H(\theta)$, is defined as the solution ρ of the equation

$$R_{\rho,\,r}(t) = R_m^H(t) = \theta. \tag{2.14}$$

Using Equation (2.14), together with equations (2.7) and (2.9), one can verify that the factor $\rho = \rho_{m,\,r}^H(\theta)$ satisfies the following equation

$$m\;\ln\left[2-\theta^{\frac{1}{n+(\rho-1)r}}\right] - \frac{(\rho-1)r}{n+(\rho-1)r}\;\ln(\theta) = 0 \tag{2.15}$$

Equation (2.15), independent on scale and shape parameters of the weibull distribution. This equation has no closed form solutions and can be solved using some numerical program such as Mathematica Program.

Let us now explain how one can deduce the second type of reliability equivalence factor of the n components series system. This type is MREF, say $\xi^D(\theta)$. The factor $\xi^D(\theta)$ can be obtained by solving the following equation

$$\text{MTTF}_{\rho,r} = \text{MTTF}_m^H \tag{2.16}$$

Using Equation (2.16) together with equations (2.8) and (2.10), $\xi = \xi^H_{m,r}$ can be obtained as follows

$$\xi = \frac{1}{r}\left\{\left[\sum_{k=0}^{m} 2^k\,(-1)^{m-k}\binom{m}{k}\,(n+m-k)^{-\frac{1}{\beta}}\right]^{-\beta} - n\right\} + 1 \tag{2.17}$$

From Equation(2.17), the factor ξ dependent on the shape parameter β, and the values of $1 \leq m, r \leq n$ and independent on the scale parameter α.

If we put $\beta = 1$, in our article we have the ED(α), as in Råde (1993a, 1993b), Sarhan (2000) and Mustafa (2002).

2.5. Numerical Results and Conclusion

To explain how one can utilize the previously obtained theoretical results we introduce a numerical example. In such example, we calculate the two REFs of three components series system under the following assumptions:

1. The parameters of the weibull distribution, $\alpha = 5$, and $\beta > 0$, can take some values such as:

   (i) $\beta = 0.5$, ($\beta < 1$), the components have decreasing failure rates,
   (ii) $\beta = 1$, the components have constant failure rates(ED($1/\alpha$)),
   (iii) $\beta = 2$, ($\beta > 1$), the components have increasing failure rates(RD(α),
   (iv) $\beta = 3$, ($\beta > 1$), the components have increasing failure rates,

2. the system reliability will be improved according the previous methods.

For this example, we have found out that the MTTF of the original and improved system are presented in Table 2.1.

Table 2.1. The MTTF of the original and improved system.

		$MTTF_m^H$		
β	MTTF	m=1	m=2	m=3
0.5	1.111	1.597	2.344	3.511
1.0	1.667	2.083	2.667	3.500
1.5	2.169	2.549	3.059	3.759
2.0	2.558	2.901	3.353	3.961
2.5	2.859	3.169	3.573	4.109
3.0	3.096	3.379	3.743	4.223

From the above table one can conclude that: $MTTF < MTTF_m^H$, for all m, β.

The $\theta-$fractiles $L(\theta)$ and $L_m^H(\theta)$, and the REF, $\rho_{m,r}^H(\theta)$ are calculated using mathematica program system according to the previous theoretical formulae. In such calculations the level θ is chosen to be 0.1, 0.3, $\cdots$, 0.9. Table 2.2 represents the θ-fractiles of the original and improved systems that are obtained by improving the system components according to the hot duplication of m components of the system components.

Table 2.2. The $\theta-$fractlies.

	$\beta = 0.5$				$\beta = 1.0$			
θ	$L(\theta)$	m=1	m=2	m=3	$L(\theta)$	m=1	m=2	m=3
0.1	0.5891	0.8555	1.2310	1.7339	0.7675	0.9249	1.1095	1.3168
0.3	0.1611	0.2641	0.4427	0.7319	0.4013	0.5139	0.6653	0.8555
0.5	0.0534	0.0959	0.1860	0.3667	0.2311	0.3098	0.4313	6055
0.7	0.0141	0.0277	0.0643	0.1662	0.1189	0.1664	0.2537	0.4077
0.9	0.0012	0.0026	0.0082	0.0423	0.0351	.0514	0.0904	0.2055
	$\beta = 1.5$				$\beta = 2.0$			
θ	$L(\theta)$	m=1	m=2	m=3	$L(\theta)$	m=1	m=2	m=3
0.1	0.8383	0.9493	1.0717	1.2014	0.8761	0.9617	1.0533	1.1475
0.3	0.5441	0.6416	0.7621	0.9012	0.6335	0.7169	0.8157	0.9249
0.5	0.3765	0.4578	0.5708	0.7157	0.4807	0.5566	0.6567	7781
0.7	0.2418	0.3026	0.4007	0.5498	0.3448	0.4079	0.5037	0.6385
0.9	0.1072	0.1383	0.2014	0.3483	0.1874	0.2268	0.3006	0.4533
	$\beta = 2.5$				$\beta = 3.0$			
θ	$L(\theta)$	m=1	m=2	m=3	$L(\theta)$	m=1	m=2	m=3
0.1	0.8996	0.9693	1.0425	1.1164	0.9156	0.9743	1.0353	1.0961
0.3	0.6941	0.7662	0.8496	0.9395	0.7376	0.8010	0.8730	0.9493
0.5	0.5565	0.6258	0.7143	0.8182	0.6136	0.6766	0.7555	0.8460
0.7	0.4266	0.4881	0.5777	0.6984	0.4917	0.5501	0.6330	0.7415
0.9	0.2619	0.3051	0.3823	0.5311	0.3275	0.3718	0.4487	0.5901

Based on the results presented in Table 2.2, it seems that, $L(\theta) < L_m^H(\theta)$ in all studied cases.

Tables 2.3 shows the SREF and the MREF, when the m components of the system components are improved according to hot duplication method and reducing the failure rates of r components of the system components.

Table 2.3. The SREF and MREF.

	$\rho_{m,r}^H$								
	r=1			r=2			r=3		
θ	m =1	m=2	m=3	m =1	m=2	m=3	m =1	m=2	m=3
0.1	0.4895	0.0753	NA	0.7448	0.5376	0.3740	0.8298	0.6917	0.5829
0.3	0.3427	NA	NA	0.6714	0.4048	0.2036	0.7809	0.6032	0.4691
0.5	0.2376	NA	NA	0.6188	0.3036	0.0724	0.7459	0.536	0.3816
0.7	0.1429	NA	NA	0.5715	0.2030	0.0626	0.7143	0.4687	0.2916

0.9	0.0489	NA	NA	0.5245	0.0829	NA	0.6829	0.3886	0.1709

	$\xi^H_{m,r}$								
	r=1			r=2			r=3		
β	m =1	m=2	m=3	m =1	m=2	m=3	m =1	m=2	m=3
0.5	0.5022	0.0653	NA	0.7511	0.5326	0.3438	0.8341	0.6884	0.5625
1.0	0.4000	NA	NA	0.7000	0.4375	0.2143	0.800	0.6250	0.4762
1.5	0.3568	NA	NA	0.6784	0.3964	0.1578	0.7856	0.5976	0.4385
2.0	0.3329	NA	NA	0.6665	0.3734	0.1258	0.7776	0.5823	0.4172
2.5	0.3179	NA	NA	0.6589	0.3587	0.1052	0.7726	0.5725	0.4035
3.0	0.3074	NA	NA	0.6537	0.3984	0.0908	0.7691	0.5656	0.3939

According to the results presented in Tables 2.2 and 2.3, it may be observed that:

1. Hot duplication of one component, $m = 1$, of the system components

   (1) will increases $L(0.1)$, (i) from $0.5891\,\alpha$ to $0.8555\,\alpha$, when $\beta = 0.5$, (ii) from $0.7675\,\alpha$ to $0.9249\,\alpha$, when $\beta = 1.0$, (iii) from $0.8383\,\alpha$ to $0.9493\,\alpha$, when $\beta = 1.5$, (iv) from $0.8761\,\alpha$ to $0.9617\,\alpha$, when $\beta = 2.0$, (v) from $0.8996\,\alpha$ to $0.9693\,\alpha$, when $\beta = 2.5$, see Table 2.3, (vi) from $0.9156\,\alpha$ to $0.9743\,\alpha$, when $\beta = 3.0$, see Table 2.2.

   (1) The same effect on $L(0.1)$ can be occur by reducing the failure rate of (i) the one component, $r = 1$, of the system components by the survival factor $\rho = 0.4895$, (ii) the two components, $r = 2$, of the system components by the survival factor $\rho = 0.7448$, (iii) the three components, $r = 3$, of the system components by the survival factor $\rho = 0.8298$, see Table 2.3.

2. The improved system that can be obtained by improving one component, $m = 1$, of the system components according to hot duplication method, has the same mean time to failure of that system which can be obtained by doing one of the following:

   (1) reducing the failure rate of one component, $r = 1$, of the system component by the factor: (i) $\xi = 0.5022$, when $\beta = 0.5$, (ii) $\xi = 0.4$, when $\beta = 1.0$, (iii) $\xi = 0.3568$, when $beta = 1.5$, (iv) $\xi = 0.3329$, when $\beta = 2.0$,(v) $\xi = 0.3179$, when $\beta = 2.5$, (vi) $\xi = 0.3074$, when $\beta = 3.0$, see Table 2.3.

   (2) reducing the failure rate of two components, $r = 2$, of the system component by the factor: (i) $\xi = 0.7511$, when $\beta = 0.5$; (ii) $\xi = 0.7$, when $\beta = 1.0$; (iii) $\xi = 0.6784$, when $\beta = 1.5$; (iv) $\xi = 0.6665$, when $\beta = 2.0$; (v) $\xi = 0.6589$, when $\beta = 2.5$; (vi) $\xi = 0.6537$, when $\beta = 3.0$, see Table 2.3.

   (3) reducing the failure rate of three components, $r = 3$, of the system component by the factor: (i) $\xi = 0.8341$, when $\beta = 0.5$; (ii) $\xi = 0.8$, when $\beta = 1.0$; (iii) $\xi = 0.7856$, when $\beta = 1.5$; (iv) $\xi = 0.7776$, when $\beta = 2.0$;

(v) $\xi = 0.7726$, when $\beta = 2.5$; (vi) $\xi = 0.7691$, when $\beta = 3.0$, see Table 2.3

3. Similarly, one can read the rest of the results obtained assuming hot duplication method.

4. The notation NA in Table 2.3, means that the values of $\rho^H_{m,r}$ or $\xi^H_{m,r}$ is not available and therefore there is possible equivalence between the system improved by reduction method and that system improved by using the Hot duplication method.

3. SERIES SYSTEM WITH LIFR

In this section, we apply the concept of reliability equivalence to a system that consists of n independent and identical components connected in series. The failure rates of the system components are assumed to LIFR. Let T_i denotes the lifetime of component i, $i = 1, 2, \cdots, n$. It is assumed that T_i is linearly increasing distributed random variable with parameter $\lambda(t)$ which is defined as $\lambda(t) = \alpha t + \beta, \ \alpha, \beta > 0, \ t \geq 0$.

There are two particular cases that can be deduced from the linear increasing failure rate distribution:

1. For $\alpha = 0$, in this case the LIFRD(α, β) reduced to the ED(β).

2. For $\beta = 0$, in this case the LIFRD(α, β) reduced to the RD(α).

3.1. Original System

Let $R_s(t)$ be the RF of the system. The function is given as follows

$$R_s(t) = \exp\left\{-\frac{n}{2}\left(\alpha t^2 + 2\beta t\right)\right\} \tag{3.1}$$

One can deduce the MTTF as follows

$$\text{MTTF} = \sqrt{\frac{\pi}{2n\alpha}} \exp\left\{\frac{n\beta^2}{2\alpha}\right\} \left[1 - \text{erf}\left(\beta\sqrt{\frac{n}{2\alpha}}\right)\right] \tag{3.2}$$

where erf(x) is the error function that defined as follows

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\{-t^2\}\, dt$$

3.2. Improved Systems

The system reliability can be improved according to one of the following different methods.

3.2.1. Reduction Method

Let $R_{r,\rho}(t)$ denotes the RF of the improved system obtained by reducing the failure rates of r components by the factor ρ, $0 \leq \rho \leq 1$. One can obtain the function $R_{r,\rho}(t)$ to be

$$R_{r,\rho}(t) = \exp\left\{-\frac{1}{2}\left[n + r(\rho - 1)\right]\left[\alpha t^2 + 2\beta t\right]\right\} \tag{3.3}$$

From the equation (3.3), the MTTF of the improved system, say $\text{MTTF}_{r,\rho}$, becomes

$$\text{MTTF}_{r,\rho}(t) = \sqrt{\frac{\pi}{2\alpha\left[n + r(\rho - 1)\right]}}\exp\left\{\frac{\beta^2[n + r(\rho - 1)]}{2\alpha}\right\}\left[1 - \text{erf}\left(\beta\sqrt{\frac{n + r(\rho - 1)}{2\alpha}}\right)\right] \tag{3.4}$$

3.2.2. Hot Duplication Method

Let $R^H_m(t)$ be the RF of the system improved by improving m, $1 \leq m \leq n$, of its components according to hot duplication method. We can obtain $R^H_m(t)$ to be

$$R^H_m(t) = \sum_{k=0}^{m}\left\{\binom{m}{k}(-1)^{m-k}2^{m-k}\exp\left\{-\frac{1}{2}(n + m - k)(\alpha t^2 + 2\beta t)\right\}\right\}. \tag{3.5}$$

Using Equation (3.5), the MTTF of the improved system, say MTTF^H_m , is given by

$$\begin{aligned}\text{MTTF}^H_m \;=\; & \sum_{k=0}^{m}\left\{\binom{m}{k}(-1)^{m-k}2^{m-k}\sqrt{\frac{\pi}{2\alpha(m - k + n)}}\exp\left\{\frac{(n - k + m)\beta^2}{2\alpha}\right\}\times\right.\\ & \left.\left[1 - \text{erf}\left(\beta\sqrt{\frac{n - k + m}{2\alpha}}\right)\right]\right\}.\end{aligned} \tag{3.6}$$

3.3. The θ-fractiles

The θ-fractiles of the original and improved systems are given in this subsection. Let $L(\theta)$ be the θ-fractile of the original system and $L^H_m(\theta)$ be the θ-fractile of the improved system obtained assuming hot duplication method.

The θ-fractiles $L(\theta)$ and $L^H_m(\theta)$ are defined as the solutions of the following two equations, respectively

$$R\left(\frac{L(\theta)}{n(\alpha + \beta)}\right) = \theta,\; R^H_{(m)}\left(\frac{L(\theta)}{n(\alpha + \beta)}\right) = \theta. \tag{3.7}$$

It follows from Equation (3.1) and the first Equation of (3.7) that the fractile $L = L(\theta)$ can be obtained by solving

$$L^2 + \frac{2n\beta(\alpha + \beta)}{\alpha}L + \frac{2[n(\alpha + \beta)]^2}{n\alpha}\ln(\theta) = 0 \tag{3.8}$$

That is, $L(\theta)$ is given by $L = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where $a = 1$, $b = (2n\beta(\alpha+\beta))/\alpha$, $c = 2[n(\alpha+\beta)]^2/(n\alpha)$.

Using the second equation of (3.7) and equation (3.5), one can verify that the fractile $L = L_m^H(\theta)$ satisfies the following equation

$$\ln(\theta) - m \ln\left[2 - \exp\left\{-\frac{1}{2}\left(\alpha\left(\frac{L}{n(\alpha+\beta)}\right)^2 + \frac{2\beta L}{n(\alpha+\beta)}\right)\right\}\right] + \frac{n}{2}\left[\alpha\left(\frac{L}{n(\alpha+\beta)}\right)^2 + \frac{2\beta L}{n(\alpha+\beta)}\right] = 0 \tag{3.9}$$

Equation (3.9) has no closed form solutions and some numerical method should be used to calculate $L_m^H(\theta)$, such as Mathematica Program System.

3.4. Reliability Equivalence Factors

In this subsection, we derive two types of reliability equivalence factors, SREF and MREF for the n components series system.

The SREF is defined as the factor by which the failure rates of some of the system components should be reduced in order to reach equality of the reliability of another better system. Therefore, the hot reliability equivalence factor, say $\rho_{m,r}^H(\theta)$, is defined as the solution ρ of the following equation

$$m \ln\left[2 - \theta^{\frac{1}{n+r(\rho-1)}}\right] - \frac{r(\rho-1)}{n+r(\rho-1)} \ln(\theta) = 0. \tag{3.10}$$

The MREF is defined as the factor by which the failure rates of some of the system components should be reduced in order to reach equality of the MTTF of another better system. Therefore, the hot mean reliability equivalence factor, say $\xi_{m,r}^H(\theta)$, defined as the solution of the following equation

$$\mathrm{MTTF}_{r,\rho} = MTTF_m^H \tag{3.11}$$

Equations (3.10) and (3.11) have no closed form solutions and some numerical method should be used to calculate $\rho_{m,r}^H(\theta)$, $\xi_{m,r}^H(\theta)$, such as Mathematica Program System.

3.5. Numerical Results and Conclusions

In this subsection, we introduce some numerical example to illustrate how one can utilize the previously theoretical results. We assume a series system consisting of n=3 independent and identical components with failure rate $\lambda(t) = 0.09t + 0.07$. In this example, we have found out that:

1. The MTTF of the original system is MTTF=1.7957,
2. The MTTF of the improved systems obtained by assuming hot duplication method when $m = 1, 2, 3$ are given in Table 3.1.

Table 3.1. The MTTF of the improved system.

m	1	2	3
$MTTF_m^H$	2.0990	2.5017	3.0477

From the results shown in Table 3.1, we can conclude that MTTF < MTTF_m^H for all $m = 1, 2, 3$.

The θ-fractiles $L(\theta)$, $L_m^H(\theta)$ and the REF, $\rho_{m,r}^H(\theta)$ for the system studied here are calculated using Mathematica Program System according to the previous theoretical formulae. In such calculations the level θ is chosen to be 0.1, 0.3, 0.5, $\cdots$, 0.9.

Table 3.2 gives the θ-fractiles of the original and improved systems that are obtained by improving the system components according to the hot duplication method.

Table 3.2. The $\theta-$fractiles of the original and improved system.

θ	$L(\theta)$	$L_1^H(\theta)$	$L_2^H(\theta)$	$L_3^H(\theta)$
0.1	1.4639	1.8346	2.0391	2.2499
0.3	1.1079	1.2912	1.5097	1.7526
0.5	0.7766	0.9402	1.1588	1.4266
0.7	0.4916	0.6224	0.8259	1.1189
0.9	0.1916	0.2612	0.4026	0.7183

Based on the results presented in Table 3.2, it seems that $L(\theta) < L_m^H(\theta)$ for all $m = 1, 2, 3$.

Tables 3.3 shows the SREF, $\rho_{m,r}^H(\theta)$ of the improved systems obtained using hot duplication method.

Table 3.3. The SREF, $\rho_{m,r}^H(\theta)$.

	m=1			m=2			m=3		
θ	$r=1$	$r=2$	$r=3$	$r=1$	$r=2$	$r=3$	$r=1$	$r=2$	$r=3$
0.1	0.4895	0.7448	0.8298	0.0753	0.5376	0.6918	NA	0.3743	0.5829
0.3	0.3427	0.6714	0.7809	0.1904	0.4048	0.6032	NA	0.2036	0.4691
0.5	0.2376	0.6188	0.7459	NA	0.3036	0.5357	NA	0.0724	0.3816
0.7	0.1429	0.5715	0.7143	NA	0.2030	0.4687	NA	NA	0.2916
0.9	0.0489	0.5245	0.6829	NA	0.0829	0.3886	NA	NA	0.1709

According to the results presented in Table 3.3, it may be observed that:

1. Hot duplication of a one component increases L(0.1) from $\frac{1.6439}{n(\alpha+\beta)}$ to $\frac{1.8346}{n(\alpha+\beta)}$, see Table 3.2. The same increase on L(0.1) can be obtained by doing one of the following:

   (1) reducing the failure rate of one component by the factor $\rho = 0.4895$

   (2) reducing the failure rates of any two components by the factor $\rho = 0.7448$

   (3) reducing the failure rates of any three components by the factor $\rho =$ 0.8298 , see Table 3.3.

2. In the same manner, one can read the rest of results presented in Table 3.3 with different values of θ.

3. The notation NA in Table 3.3 means that there is no equivalence between the two improved systems: one obtained by reducing the failure rates of the system components and the other obtained by improving these components according to the hot duplication method.

Table 3.4, represents the MREF, $\xi^{H}_{m,r}(\theta)$ of the improved systems obtained using hot duplication method.

Table 3.4. The MREF, $\xi^{H}_{m,r}(\theta)$.

r	$m=1$	$m=2$	$m=3$
1	0.3399	NA	NA
2	0.6699	0.3793	0.1329
3	0.7799	0.5862	0.4219

Based on the results presented in Table 3.4, one can say that:

1. The improved system that can be obtained by improving one component according to hot duplication method has the same mean time to failure of that system which can be obtained by doing one of the following:
   (1) reducing the failure rate of one component by the factor $\xi = 0.3399$,
   (2) reducing the failure rates of any two components by the factor $\xi = 0.6699$,
   (3) reducing the failure rates of any three components by the factor $\xi = 0.7799$, see Table 3.4.

2. In the same manner, one can read the rest of results presented in Table 3.4 with different values of r and m.

3. The notation NA in Table 3.4 means that the mean time to failure of a design obtained from the original system by reducing the failure rates of the system components is not equal to the mean time to failure of a design obtained from the original system by assuming hot duplication method.

4. CONCLUSIONS

In this paper we introduced a series system consist of n- independent and identical components with non-constant failure rates. we assumed two general distributions Weibull and Linear increasing failure rate distribution. Two types of the reliability equivalence factors are calculated. some special cases can be obtained from the studied system such as Exponential and Rayleigh distribution. Råde(1993) and Mustafa (2002) can be obtained a special cases from our study.

REFERENCES

Billinton, R. and Allan, R.N. (1983). *Reliability Evaluation of Engineering Systems: Concepts and Techniques*, Plenum Press. New York and London.

Kapur, K.C. and Lamberson, L.R. (1977). *Reliability in Engineering Design*, John Wiley.

Lewis, E.E. (1996). *Introduction to Reliability Engineering*, John Wiley & Sons, Inc., New York, 2nd ed.

Mustafa, A. (2008). *Reliability equivalence of Two Non-identical Components. International Journal of Reliability and Applications.* **9(1)**, 79-93.

Mustafa, A.(2002). *Reliability Equivalence of Some Systems*, Master Thesis, Department of Mathematics, Faculty of Science, Mansoura University, Egypt. (http://www.mans.edu.eg/pcvs/3118/)

Mustafa, A., El-Desouky, B.S. and El-Dawoody, M.(2007). Reliability Equivalence factors of series system with mixture failure rates, *The 42th Annual Conference ISSR Cairo University 2-5 Dec., Egypt*, 190-210.

Mustafa, A., Sarhan, A.M. and Al-Ruzaiza, A. (2007). Reliability equivalence of a parallel-series system, *Pakistan Journal of Statistics*, **23(3)**, 241-254

Råde, L. (1993a). Reliability equivalence, *Microelectrons Reliability*, **33**, 323-325.

Råde, L. (1993b). Reliability survival equivalence, *Microelectrons Reliability*, **33**, 881-894.

Sarhan, A.M. (2000). Reliability Equivalence of independent and non-identical components series systems, *Reliability Engineering* & *System Safety*, **67**, 293-300.

Sarhan, A.M. (2002). Reliability Equivalence with a basic series/parallel system, *Applied Mathematics and Computation*, **132**, 115-133.

Sarhan, A.M. (2004). Reliability equivalence factors of a bridge network system, *International Journal of reliability and Applications*, **2**, 81-103.

Sarhan, A.M. (2005). Reliability equivalence factors of a parallel system, *Reliability Engineering* & *System Safety*, **87**, 405-411.

Sarhan, A.M. and Mustafa, A. (2006). Reliability equivalence of a series system consists of n independent and non-identical components, *International Journal of Reliability and Applications*, **7(2)**, 111-125.

Sarhan, A.M., Al-Ruzaiza, A.S., Alwasel, I.A. and El-Gohary, A. (2004). Reliability equivalence of a series-parallel system, *Applied Mathematics and Computation*, **154**, 257-277.

Xia, Y. and Zhang, G. (2007). Reliability equivalence factors in Gamma distribution, *Applied Mathematics and Computation*, **187**, 567-573.