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A Note on Age Replacement Policy of Used Item at Age t_0

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Abstract. In most of literatures of age replacement policy, the authors consider the case that a new item starts operating at time zero and is to be replaced by new one at time *T*. It is, however, often to purchase used items because of the limited budget. In this paper, we consider age replacement policy of a used item whose age is t_0. The mathematical formulas of the expected cost rate per unit time are derived for both infinite-horizon case and finite-horizon case. For each case, we show that the optimal replacement age exists and is finite and investigate the effect of the age of the used item.

Key Words : *Age replacement policy, expected cost rate per unit time, infinite-horizon case, finite-horizon case*

1. INTRODUCTION

Since Barlow and Hunter (1960) proposed an age replacement policy in which an operating item is replaced at age *T* or at failure, whichever occurs first, the age replacement policy has been extensively studied by incorporating various types of repairs at failure and cost structures for repair. Beichelt (1976), Berg, Bienvenu and Cléroux (1986), Block, Borges and Savits (1988) Sheu, Kuo and Nakagawa (1993) consider the age-replacement problem with age-dependent minimal repair and different cost structures. Cleroux, Dubuc and Tilquin (1979) and Bai and Yun, (1986) consider age replacement policy based both on the system age and the minimal repair cost. Sheu and Griffith (1996), Sheu (1998), Sheu and Chien (2004), Chien, and Sheu (2006) consider age replacement policy of system subject to shocks. Sheu (1991), Sheu, Griffith and Nakagawa (1995), Jhang and Sheu (1999) and Sheu, Yeh, Lin and Juang (1999) consider age replacement policy with age dependent replacement and random repair cost.

In most of literatures mentioned, it is noted that the authors consider only the case that a new item starts operating at time 0 and investigate the optimal age which minimizes the expected cost per unit time. In practice, it is often that a company purchases used

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equipments because of the limited budget, however. Hence, it is important to study the age replacement policy of used items.

In this paper, we consider age replacement policy of a used item in which the item whose age is t_0 starts operating at time 0 and is replaced by new one either at failure or at time *T*, whichever come first. We formulate the expected cost per unit time for both the infinite-horizon case and the finite-horizon case. We investigate the optimal age replacement policy which minimizes the expected cost rate per unit time.

The remainder of this paper is organized as follows. Section 2 describes the age replacement policy of a used item whose age is t_0 and the expected cost per unit time for both the infinite-horizon case and the finite-horizon case is formulated. In Section 3, the optimal replacement schedule is investigated. In Section 4, a numerical example is given to illustrate our results.

2. AGE REPLACEMENT POLICY OF USED ITEM AND ITS EXPECTED COST PER UNIT TIME

Let X be a random variable representing the lifetime of a unit. Let *f*, *F* and *R* be the probability density function, distribution function and reliability function of *X*, respectively and let *h* be the hazard rate of *F*.

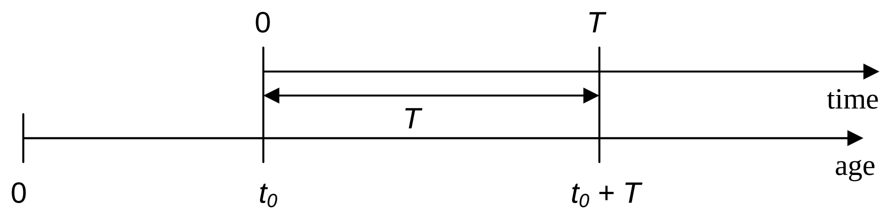

Figure 2.1. Age replacement policy of used item

We consider an age replacement policy of an used item whose age is t_0 as follows. The item whose age is t_0 starts operating at time 0 and is replaced by new one either at failure or at time *T*, whichever come first. The age replacement policy of an used item considered in this paper is the same as the age replacement policy proposed by Barlow and Hunter(1960) except that the age of the item is t_0. Figure 2.1 shows outline of age replacement policy of used item.

Let X_{t_0} be the residual life of the used item of age $t_0 \ge 0$. Let f_{t_0}, F_{t_0} and R_{t_0} be the probability density function, distribution function and reliability function of X_{t_0}. And let h_{t_0} be the hazard rate of F_{t_0}. Then for any $x \ge 0$,

$$R_{t_0}(x) = P(X_{t_0} \ge x) = \frac{P(X \ge t_0 + x)}{P(X \ge t_0)} = \frac{R(t_0 + x)}{R(t_0)},$$

$$F_{t_0}(x) = \frac{F(t_0 + x) - F(t_0)}{R(t_0)},$$

$$f_{t_0}(x) = \frac{f(t_0 + x)}{R(t_0)}$$

and

$$h_{t_0}(x) = h(t_0 + x).$$

Let C_f and C_r be cost for failure and cost for planned replacement, respectively.

2.1. Expected Cost Per Unit Time for Infinite-Horizon Case

Long run expected cost *C(T)* can be defined by using renewal reward theorem as follows. [See Ross(1992) for more details]

$$C(T) = \frac{E(\text{cost incurred during a cycle})}{E(\text{length of a cycle})} = \frac{E(R)}{E(Y)},$$

where *R* is the cost incurred during a cycle and *Y* is the length of a cycle.

It is clear from the age replacement policy that

$$R = \begin{cases} C_f & if\ X_{t_0} < T - t_0 \\ C_r & if\ X_{t_0} \geq T - t_0 \end{cases} \text{ and } Y = \begin{cases} X_{t_0} & if\ X_t < T - t_0 \\ T - t_0 & if\ X_t \geq T - t_0 \end{cases}.$$

Hence

$$E(R) = C_f P(X_{t_0} < T - t_0) + C_r P(X_{t_0} \geq T - t_0) \tag{2.1}$$

and

$$\begin{aligned} E(Y) &= t_0 + E[X_{t_0} \cdot I(X_{t_0} < T - t_0)] + (T - t_0) P(X_{t_0} \geq T - t_0) \\ &= t_0 + \int_0^{T-t_0} x_{t_0} f_{t_0}(x) dx + (T - t_0) P(X_{t_0} \geq T - t_0). \end{aligned} \tag{2.2}$$

Therefore by using the equations (2.1) and (2.2), we have the long run expected cost in age replacement policy as follows.

$$\begin{aligned} C_{t_0}(T) &= \frac{C_f P(X_{t_0} < T - t_0) + C_r P(X_{t_0} \geq T - t_0)}{t_0 + \int_0^{T-t_0} x_{t_0} f_{t_0}(x) dx + (T - t_0) P(X_{t_0} \geq T - t_0)} \\ &= \frac{C_f F_{t_0}(T) + C_r R_{t_0}(T)}{t_0 + \int_0^T x f_{t_0}(x) dx + T \cdot R_{t_0}(T)} = \frac{C_f [F(t_0 + T) - F(t_0)] + C_r R(t_0 + T)}{t_0 R(t_0) + \int_0^T x f(t_0 + x) dx + T \cdot R(t_0 + T)}. \end{aligned} \tag{2.3}$$

Since $\int_0^T xf(t_0+x)dx = xF(t_0+x)\big]_0^T - \int_0^T F(t_0+x)dx = TF(t_0+T) - \int_0^T F(t_0+x)dx$, we can simplify the equation (2.3) and obtain the following long run expected cost.

$$C_{t_0}(T) = \frac{C_f[F(t_0+T)-F(t_0)] + C_r R(t_0+T)}{t_0 R(t_0) + \int_0^T R(t_0+x)dx}\,. \tag{2.4}$$

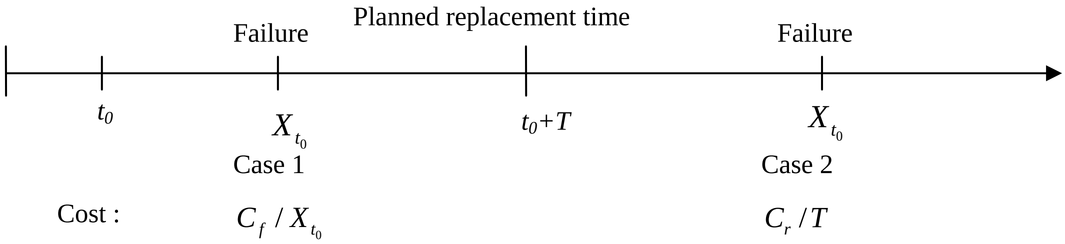

Figure 2.2. Two possible failures and related cost per unit time

2.2 Expected cost per unit time for finite-horizon case

When the planning time span is finite, we can not apply the renewal reward theorem. Sheu, Yeh, Lin and Juang (1999) consider total cost per unit time between two successive replacement when the planning time span is finite. We also utilize total cost per unit time between two successive replacements for the case that the planning time span is finite.

We can consider two possible cases of failure as shown in Figure 2.2 and can obtain total cost per unit time between two successive replacements at age t_0 as follows.

$$\left(\frac{C_f}{X_{t_0}}\right) I_{(0,T]}(X_{t_0}) + \left(\frac{C_r}{T}\right) I_{(T,\infty)}(X_{t_0})\,.$$

Let $W_{t_0}(T)$ be the expected value of the total cost per unit time between two successive replacements at age t_0. Then $W_{t_0}(T)$ becomes

$$\begin{aligned} W_{t_0}(T) &= E[I_{(0,T)}(X_{t_0}) \cdot \left(\frac{C_f}{X_{t_0}}\right)] + E[I_{(T,\infty)}(X_{t_0}) \cdot \left(\frac{C_r}{T}\right)] \\ &= C_f \cdot \int_0^T \frac{1}{x} f_{t_0}(x)dx + C_r \cdot \frac{P(X_{t_0} > T)}{T} \\ &= \frac{1}{R(t_0)}\left[C_f \cdot \int_0^T \frac{f(t_0+x)}{x}dx + C_r \cdot \frac{R(t_0+T)}{T}\right] \end{aligned} \tag{2.5}$$

3. OPTIMAL REPLACEMENT SCHEDULE

In this section, we investigate the optimal age T^* for replacement which minimizes the expected cost rate per unit time for both infinite-horizon and finite-horizon cases.

3.1 Optimal replacement schedule for infinite-horizon case

In order to find the optimal replacement age T^* which minimizes the expected long-run cost per unit time, given in (2.4), we differentiate $C_{t_0}(T)$ with respect to *T* and set it equal to 0. Then, we have

$$(C_f - C_r)[h(t_0+T)(t_0R(t_0)+\int_0^T R(t_0+x)dx)+R(t_0+T)]-C_fR(t_0)=0$$

Hence the optimal age *T** for replacement which minimizes the long-run expected cost in (2.4) is the value of *T* satisfying he following equality.

$$h(t_0+T)[t_0R(t_0)+\int_0^T R(t_0+x)dx]+R(t_0+T)=\frac{C_fR(t_0)}{C_f-C_r}. \quad (3.1)$$

Theorem 3.1. Suppose that $h(t_0)$ is strictly increasing to infinity as t_0 goes to infinity. If $t_0 \cdot h(t_0) < C_r/(C_f - C_r)$, then there exists optimal age *T** for replacement which minimizes the long-run expected cost in (2.4) and it is unique.

Proof. Let $\xi(T)$ be the left-sided term in the equation (3.1). When $T=0$, it is obvious from the assumption that $\xi(T=0)=h(t_0)t_0R(t_0)+R(t_0)=R(t_0)(t_0\cdot h(t_0)+1)<\frac{R(t_0)C_f}{C_f-C_r}$.

Taking a derivative of $\xi(T)$ with respect to *T* yields

$$\begin{aligned}\frac{d\xi(T)}{dT}&=h'(t_0+T)[t_0R(t_0)+\int_0^T R(t_0+x)dx]+h(t_0+T)R(t_0+T)-f(t_0+T)\\&=h'(t_0+T)[t_0R(t_0)+\int_0^T R(t_0+x)dx].\end{aligned}$$

Since $h(t_0)$ is strictly increasing, it is clear that $\frac{d\xi(T)}{dT}>0$ and then $\xi(T)$ is strictly increasing. Finally, we have

$$\begin{aligned}\lim_{T\to\infty}\xi(T)&=\lim_{T\to\infty}h(t_0+T)[t_0R(t_0)+\int_0^T R(t_0+x)dx]+R(t_0+T)\\&=[\lim_{T\to\infty}h(t_0+T)][t_0R(t_0)+\lim_{T\to\infty}\int_0^T R(t_0+x)dx]+\lim_{T\to\infty}R(t_0+T)\end{aligned}$$

$$= [\lim_{T\to\infty} R(t_0)h(t_0+T)][t_0+E[X_t]] = \infty \ .$$

Hence there exists optimal age *T** for replacement which minimizes the long-run expected cost in (2.4) and it is unique. ■

3.2. Optimal replacement schedule for finite-horizon case

The optimal replacement age T^* for finite-horizon case can be obtained by using similar technique in Section 3.1. Taking a derivative of $W_{t_0}(T)$ with respective to *T* and setting it equal to 0 yields $T\cdot h(t_0+T) = \dfrac{C_r}{C_f - C_r}$. Hence it is straight forward to show the existence and the uniqueness of the optimal age *T** for replacement which minimizes the expected total cost per unit time in (2.5). That is formally stated in the following theorem.

Theorem 3.2. Suppose that *h(t)* is strictly increasing to infinity as *t* goes to infinity. Then there exists optimal age *T** for replacement which minimizes the long-run expected cost in (2.5) and it is unique.

Proof. Since taking a derivative of $W_{t_0}(T)$ with respective to *T* and setting it equal to 0 yields $T\cdot h(t_0+T) = \dfrac{C_r}{C_f - C_r}$ and *h(t)* is strictly increasing to infinity as t increases, thee result directly holds. ■

4. NUMERICAL EXAMPLE

Suppose that *F* is Weibull distribution with scale parameter θ and shape parameter *m*. Then the reliability function and failure rate function are given by

$$R(t) = \exp[-(\frac{t}{\theta})^m] \ \text{ and } \ h(t) = (\frac{m}{\theta})(\frac{t}{\theta})^{m-1}, \quad \theta > 0,\ m > 0,\ t \ge 0,$$

respectively. We assume that $m > 1$ and $\theta = 1$. Then the hazard rate, $h(t)$, is strictly increasing for $t \ge 0$. For any $t_0 \ge 0$, the reliability function and the hazard rate of the used item of the age t_0 are

$$R_{t_0}(x) = \frac{\exp[-(t_0+x)^m]}{\exp[-t_0{}^m]} \ \text{ and } \ h_{t_0}(x) = m[t_0+x]^{m-1},\ x \ge 0, \text{ respectively.}$$

Cost structures are assumed to be as follows.

- Cost for failure $C_f = 1000$

- Cost for replacement $C_r = p \cdot C_f$ ($p = 0.1, 0.2, ..., 0.9$)

4.1. Infinite-horizon case

The long run expected cost in age replacement policy of the used item of the age t_0 is given by as follows.

$$C_{t_0}(T) = \frac{C_f \{\exp[-t_0{}^m] - \exp[-(t_0 + T)^m]\} + C_r \exp[-(t_0 + T)^m]}{t_0 \exp[-t_0{}^m] + \int_0^T \exp[-(t_0 + x)^m] dx}.$$

Optimal replacement age T^* is the value of T satisfying the following equation

$$m(t_0 + T)^{m-1}\{t_0 \exp[-t_0{}^m] + \int_0^T \exp[-(t_0 + x)^m] dx\} + \exp[-(t_0 + T)^m] = \frac{C_f \exp[-t_0{}^m]}{C_f - C_r}$$

Table 4.1 shows the values of the optimal replacement age, T^* , and their corresponding long run expected costs per unit time for various values of the shape parameter and various cost structures. In Table 4.1, we consider the ratio of the cost for replacement to the cost for failure greater than or equal to 0.5 since small values of the ration do not satisfy the condition for the existence of the optimal replacement age for given $t_0 = 0.5$ and 1.0 and *m*=1.5, 3.0 (0.5). It is noted from Table 4.1 that for any values of age of the used item and cost structure, the optimal replacement ages tend to decrease as the values of the shape parameter increase. That is expected since the larger shape parameter is, the more frequently failures occur. And it is also observed that for any given value of age of the used item and shape parameter, the optimal replacement ages tend to increase as the values of ratio of the cost for replacement to the cost for failure increase. That means if the cost for replacement is closed to the cost for failure, it is not necessary to replace the item before it fails. In other word, the item should be replaced before it fails if the cost for failure is higher than the cost for replacement. It is quite natural that the optimal replacement age when $t_0 = 1.0$ is shorter than the optimal replacement age when $t_0 = 0.5$ since the use item with the age of $t_0 = 1.0$ has less remaining life than the use item with the age of $t_0 = 0.5$.

Table 4.1. Optimal replacement age T^* and the long run expected cost

	C_r / C_f	*m*=1.5	*m*=2.0	*m*=2.5	*m*=3.0
$t_0 = 0.5$	0.5	0.75	0.38	0.30	0.27
		837.8252	876.8725	880.4894	872.0260
	0.6	1.54	0.67	0.47	0.40
		856.8727	930.6502	953.4644	955.8603
	0.7	3.15	1.09	0.71	0.56
		859.0690	952.3762	993.7736	1009.3802
	0.8	7.71	1.90	1.10	0.82
		859.0832	956.2860	1008.0171	1034.5517

	0.9	32.31	4.29	2.04	1.37
		859.0832	956.3432	1009.2853	39.0519
$t_0 = 1.0$	0.7	1.04	0.16	-	-
		642.0580	693.4491	-	-
	0.8	3.62	0.81	0.33	0.15
		644.4557	723.5377	766.3081	787.9535
	0.9	17.46	2.63	1.13	0.65
		644.4562	725.1880	776.2728	811.1656

4.2. Finite-horizon case

The expected value of the total cost per unit time between two successive replacements at age t_0, $W_{t_0}(T)$, is given by

$$\frac{1}{\exp[-t_0{}^m]}\left[C_f \cdot \int_0^T \frac{m(t_0+x)^{m-1}\exp[-(t_0+x)^m]}{x}dx + C_r \cdot \frac{\exp[-(t_0+T)^m]}{T}\right].$$

And the optimal replacement age T^* is the value of T satisfying the following equation

$$mT(t_0+T)^{m-1} = \frac{C_r}{C_f - C_r}.$$

Table 4.2 presents the values of the optimal replacement age, T^*, and their corresponding the expected total costs per unit time for the shape parameter *m*=1.5, 3.0 (0.5) and the ratio of the cost for replacement to the cost for failure C_r / C_f =0.1, 0.9 (0.2). Table 4.2 shows that the optimal replacement age tends to decrease as the value of the shape parameter increases. And it is also observed that for any given value of age of the used item and shape parameter, the optimal replacement ages tend to increase as the values of ratio of the cost for replacement to the cost for failure increase. That means if the cost for replacement is closed to the cost for failure, it is not necessary to replace the item before it fails. In other word, the item should be replaced before it fails if the cost for failure is higher than the cost for replacement. It is quite natural that the optimal replacement age when $t_0 = 1.0$ is shorter than the optimal replacement age when $t_0 = 0.5$ since the use item with the age of $t_0 = 1.0$ has less remaining life than the use item with the age of $t_0 = 0.5$.

Table 4.2. Optimal replacement age T^* and the expected total cost per unit time

	C_r / C_f	*m*=1.5	*m*=2.0	*m*=2.5	*m*=3.0
$t_0 = 0.5$	0.1	0.10	0.10	0.10	0.10
		8654.3687	8311.2903	7555.7249	6648.2200

	0.3	0.32	0.28	0.26	0.26
		9556.0866	9290.2940	8559.3967	7646.8724
	0.5	0.63	0.50	0.44	0.41
		9814.6484	9614.8074	8931.0510	8052.0835
	0.7	1.20	0.86	0.71	0.63
		9887.2855	9723.9641	9072.6810	8222.4109
	0.9	3.15	1.89	1.39	1.13
		9895.2383	9740.3889	9098.9900	8259.3873
$t_0 = 1.0$	0.1	0.08	0.06	0.05	0.04
		11645.1335	14984.8649	18211.1540	21306.3897
	0.3	0.26	0.19	0.15	0.12
		12800.6896	16577.6968	20229.2799	23776.4984
	0.5	0.54	0.37	0.28	0.23
		13102.1316	17016.4224	20805.3218	24494.5569
	0.7	1.08	0.70	0.51	0.40
		13178.3017	17138.6699	20975.0052	24711.7809
	0.9	3.00	1.68	1.15	0.87
		13185.5717	17152.9567	20997.2349	24742.4715

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