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Reliability Equivalence Factors of Non-identical Components Series System with Mixture Failure Rates

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Abstract. The aim of this work is to generalize reliability equivalence technique to apply it to a system consists of n independent and non-identical components connected in series system, that have mixing constant failure rates. We shall improve the system by using some reliability techniques: (i) reducing some failure rates; (ii) add hot redundancy components; (iii) add cold redundancy components; (iv) add cold redundancy components with imperfect switches. We start by establishing two different types of reliability equivalence factors, the survival equivalence (SRE), and mean reliability equivalence (MRE) factors. Also, we introduced some numerical results.

Key Words : *Mixture Distributions, Reliability Equivalence, Improving System, Exponential Distribution, and α-fractiles.*

1. INTRODUCTION

The concept of reliability equivalence factors has been introduced by Råde (1989). Råde (1990, 1991 and 1993) has applied such concepts to various reliability systems. Later Sarhan (2000, 2002, 2004 and 2005), Mustafa (2002), Sarhan et al. (2004), Sarhan and Mustafa (2006), Mustafa et al. (2007) and Mustafa (2008) applied the same concept on more general and complex systems.

Generally, there are two basic methods to improve a given system, see Sarhan(2000). These methods are: (1) reduction method, (2) redundancy method. The redundancy

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methods includes three possible methods: hot duplication method (HDM), cold duplication method (CDM) and imperfect duplication method (IDM).

In spacecraft, for example: satellites or other space applications, in well-logging equipment and in pacemakers and similar biomedical applications and in engineering applications, the redundancy method may not be an appropriate method to be us for a system in which the minimum size and weight are overriding considerations, see Lewis (1996).

In such applications, space or weight limitations may indicate an increase in component reliability rather than redundancy. Therefore, more emphasis must be placed on robust design, manufacturing quality control, and on controlling the operating environment. Thus, the concept of reliability equivalence take place. In this concept, the improved design of the system, which obtained by following the reduction method, should be equivalent to that improved design of the system which obtained by using one of the redundancy methods.

The previous articles (1989-2008) in reliability equivalence technique assumed that the system components have one type of constant failure rate.

In this paper, we study the concept of reliability equivalence of an n-independent and non-identical components series system when the failure rate of each component is presented as a mixture of constant failure rates. Let T_i be the lifetime of the component i, $i = 1, 2, ..., n$. It is assumed that T_i is exponentially distributed random variable with parameterλ_i which is defined as $\lambda_i = \alpha_{i1}\lambda_{i1} + \alpha_{i2}\lambda_{i2} + \alpha_{i3}\lambda_{i3}$, $\alpha_{ij} \geq 0$, $\sum_{j=1}^{3} \alpha_{ij} = 1, i = 1, 2, \cdots, n$ where λ_{i1}, λ_{i2} and λ_{i3} are the industry, shock and human failure rates of component, see Everitt and Hand (1981).

The main objective of this paper is to calculate two types of reliabilty equivalence factors (REF) of the studied system. These types are the survival reliability equivalence factor (SREF) and mean reliability equivalence factor (MREF). In obtaining such types of REFs, the reliability fuction (RF) and mean time to failure (MTTF) are used respectively as performance measures of the system reliability.

This paper is organized as follows. Section 2, derives the RF and MTTF of the original system. Section 3, presents the RFs and MTTFs of the improved systems. The α-fractiles of the original and improved systems are presented in Section 4. Two different types of REFs of the system are derived in Section 5. Finally, some numerical results and conclusions are listed in Section 6.

2. THE ORIGINAL SYSTEM

We consider a series system that consists of n-components, the failure rates of system components are assumed to be constant. Let $R(t)$ be the RF of the system. The function $R(t)$ is given by

$$R(t) = \prod_{i=1}^{n} \exp\{-\lambda_i t\} = \exp\{-\Lambda t\}. \tag{2.1}$$

Where, $\Lambda = \sum_{i=1}^{n} \lambda_i$, $\lambda_i = \sum_{j=1}^{3} \alpha_{ij}\lambda_{ij}$.

From equation (2.1), one can easily obtain the MTTF as follows

$$\text{MTTF} = \int_0^\infty R(t)dt = \frac{1}{\Lambda} \tag{2.2}$$

3. THE IMPROVED SYSTEMS

The quality of the system reliability can be improved using four different methods of the system improvements.

3.1. Reduction Method

Let $R_{B,\rho}(t)$ denotes the RF of the improved system when the mixture failure rate of the set of B components are reduced by the same factor ρ, $0 < \rho < 1$. One can obtain the function $R^H_{B,\rho}(t)$, as follows

$$R(t) = \left[\prod_{i\in B} \exp\{-\rho\lambda_i t\}\right] \left[\prod_{i\in \bar{B}} \exp\{-\lambda_i t\}\right] = \exp\left\{-\left[\Lambda - (1-\rho)\Lambda_B\right] t\right\} \tag{3.1}$$

Where $\Lambda_B = \sum_{i\in B} \lambda_i$, $B \subseteq N$, $\bar{B} = N\backslash B$ and $N = \{1, 2, \cdots, n\}$.

From equation (3.1), the MTTF of the improved system, say $\text{MTTFB}_{B,\rho}$, becomes

$$\text{MTTF}_{B,\rho} = \int_0^\infty R_{B,\rho}(t)dt = \frac{1}{\Lambda - (1-\rho)\Lambda_B}. \tag{3.2}$$

That is, reducing the mixture failure rates of the set of B components increases the mean time to system failure by the amount $\frac{(1-\rho)\Lambda_B}{\Lambda[\Lambda-(1-\rho)\Lambda_B]}$.

In addition, we assume that any component has three types of failures, $\lambda_{i1}, \lambda_{i2}$ and λ_{i3}, $i = 1, 2, ..., n$. We can reduce some types of failure rate say, the set $C \subseteq \{1, 2, 3\}$. Let $R_{B,\rho_C}(t)$ denotes the RF of the improved system when the set of C failure rates from the set of B components are reduced by the factor ρ_C, $0 < \rho_C < 1$. The function $R_{B,\rho_C}(t)$ can be obtained as follows

$$\begin{aligned} R_{B,\rho_C} &= \left[\prod_{i\in B} \left\{-\left(\sum_{j\in C} \rho_j\alpha_{ij}\lambda_{ij} + \sum_{j\in \bar{C}} \alpha_{ij}\lambda_{ij}\right) t\right\}\right] \left[\prod_{i\in \bar{B}} \exp\{-\lambda_i t\}\right] \\ &= \exp\left\{-\left[\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}\right] t\right\}, \end{aligned} \tag{3.3}$$

where

$$\Lambda_{\rho_{B_C}} = \sum_{i\in B}\sum_{j\in C} \rho_j\alpha_{ij}\lambda_{ij},\ \Lambda_{B_C} = \sum_{i\in B}\sum_{j\in C} \alpha_{ij}\lambda_{ij},\ B \subseteq N,\ \bar{B} = N\backslash B,\ N = \{1, 2, \cdots, n\},$$

and $C \subseteq \{1, 2, 3\}$, in set C, that is 1: industry failure, 2: shock failure, 3: human failure. From equation (3.3), the MTTF of the improved system, say MTTF_{B,ρ_C}, becomes

$$\mathrm{MTTF}_{B,\rho_C} = \int_0^\infty R_{B,\rho_C}(t)dt = \frac{1}{\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}}. \tag{3.4}$$

That is, reducing the set of C failure rates of the set of B components increases the mean time to system failure by the amount $\frac{\Lambda_{B_C} - \Lambda_{\rho_{B_C}}}{\Lambda[\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}]}$.

3.2. Hot Duplication Method

Let $R_A^H(t)$ be the RF of the improved system obtained by assuming hot duplications of a set of A components, $A \subseteq \{1, 2, \cdots, n\}$. The function $R_A^H(t)$ is given by

$$R_A^H(t) = \left[\prod_{i \in A} R_i^H(t)\right] \left[\prod_{i \in \bar{A}} R_i(t)\right],$$

where $R_i^H(t)$ denotes the RF of component i after modification using the HDM. The function $R_i^H(t)$ is given as

$$R_i^H(t) \left[2 - \exp\{-\lambda_i t\}\right] \exp\{-\lambda_i t\}$$

Thus, $R_A^H(t)$ becomes

$$\begin{aligned} R_A^H(t) &= \left[\prod_{i \in A} (2 - \exp\{-\lambda_i t\}) \exp\{-\lambda_i t\}\right] \prod_{i \in \bar{A}} \exp\{-\lambda_i t\} \\ &= 2^m \exp\{-\Lambda t\} \left[\prod_{i \in A} \left(1 - \frac{1}{2} \exp\{-\lambda_i t\}\right)\right], \; m = |A| \end{aligned} \tag{3.5}$$

Sarhan (2000), writes the following relation

$$\prod_{i \in A} \left(1 - \frac{1}{2} \exp\{-\lambda_i t\}\right) = \sum_{l=0}^{m} \left[(-1)^{-l} 2^{-l} \sum_{i=1}^{\binom{m}{l}} \exp\left\{-\gamma_{i(l)}^{(m)} t\right\}\right].$$

where $\gamma_{i(l)}^{(m)} = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_l}$, $i_1 < i_2 < \cdots < i_l \in A$, $\gamma_{(0)}^{(m)} = 0$, $\gamma_{i(l)}^{(m)} \neq \gamma_{j(l)}^{(m)}$ for $i \neq j$ and $1 \leq i, j \leq \binom{m}{l}$.

Substituting from the above relation into equation (3.5), one can verify that

$$R_A^H(t) = 2^m \exp\{-\Lambda t\} \sum_{l=0}^{m} \left[(-1)^l 2^{-l} \sum_{i=1}^{\binom{m}{l}} \exp\left\{-\gamma_{i(l)}^{(m)} t\right\}\right]. \tag{3.6}$$

Let MTTF_A^H be the MTTF of improved system assuming hot duplication of the set of A components. Using equation (3.6), one can deduce MTTF_A^H as

$$\text{MTTF}_A^H = 2^m \sum_{l=0}^{m} \left[(-1)^l 2^{-l} \sum_{i=1}^{\binom{m}{l}} \left\{ \Lambda + \gamma_{i(l)}^{(m)} t \right\}^{-1} \right]. \tag{3.7}$$

That is, hot duplication of the set of A components increases the mean time to system failure by the amount $\frac{2^m-1}{\Lambda} + 2^m \sum_{l=1}^{m} \left[(-1)^l 2^{-l} \sum_{i=1}^{\binom{m}{l}} \left\{ \Lambda + \gamma_{i(l)}^{(m)} t \right\}^{-1} \right]$.

3.3. Cold Duplication Method

Let $R_A^C(t)$ be the RF of the improved system obtained by assuming cold duplications of the set of A components, $A \subseteq \{1, 2, \cdots, n\}$. The function $R_A^C(t)$ is given by

$$R_A^C(t) = \left[\prod_{i \in A} R_i^C(t) \right] \left[\prod_{i \in \bar{A}} R_i(t) \right],$$

where $R_i^C(t)$ denotes the RF of component i after modification using the cold duplication method. The function $R_i^C(t)$ is given as

$$R_i^C(t) = (1 + \lambda_i t) \exp\{-\lambda_i t\}.$$

Thus, $R_A^C(t)$ becomes

$$\begin{aligned} R_A^C(t) &= \left[\prod_{i \in A} (1 + \lambda_i t) \exp\{-\lambda_i t\} \right] \left[\prod_{i \in \bar{A}} \exp\{-\lambda_i t\} \right] \\ &= \exp\{-\Lambda t\} \left[\prod_{i \in A} (1 + \lambda_i t) \right]. \end{aligned} \tag{3.8}$$

Further, we have

$$\prod_{i \in A} (1 + \lambda_i t) = \sum_{l=0}^{m} a_l t^l$$

where $a_l = \sum_{i_1 < i_2 < \cdots < i_l \in A} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_l}$, $a_0 = 1$, $m = |A|$, see Sarhan (2000). substituting from the above relation into equation (3.8), it follows that

$$R_A^C(t) = \exp\{-\Lambda t\} \left[\sum_{l=0}^{m} a_l t^l \right] \tag{3.9}$$

From equation (3.9), the MTTF of the improved system, say MTTF_A^C, assuming cold duplications of the set of A components is given as

$$\text{MTTF}_A^C = \sum_{l=0}^{m} \frac{a_l \Gamma(l+1)}{\Lambda^{l+1}}, \; m = |A|. \tag{3.10}$$

That is, cold duplication of the set of A components increases the mean time to system failure by the amount $\sum_{l=1}^{m} \frac{a_l \Gamma(l+1)}{\Lambda^{l+1}}$.

3.4. Imperfect Switching Duplication Method

Let us consider now that, the system reliability can be improved assuming cold duplication method with imperfect switch of m, $1 \leq m \leq n$, components. Let A denotes the index set of the components which will be improved according to this method and $\bar{A} = N \backslash A$, $|A| = m$. In such method, it is assumed that the component $i \in A$ is connected by a cold redundant standby component via a random switch having a constant failure rate, say β_i.

Let $R_A^I(t)$ be the RF of the improved system when the set of A components is improved according to the cold duplication method with imperfect switch. The function is given as

$$R_A^I(t) = \left[\prod_{i \in A} R_i^I(t)\right] \left[\prod_{i \in \bar{A}} R_i(t)\right]$$

where $R_i^I(t)$ denotes the RF of component i after modification according to cold duplication method with imperfect switch, The function is given as

$$R_i^I(t) = \frac{1}{\phi_i} \exp\{-\lambda_i t\} \; [1 + \phi_i - \exp\{-\beta_i t\}] \, , \phi_i = \frac{\beta_i}{\lambda_i}, i \in A$$

Thus, $R_A^I(t)$ is given as

$$\begin{aligned} R_A^I(t) &= \left[\prod_{i \in A} \frac{1}{\phi_i} \exp\{-\lambda_i t\} \; [1 + \phi_i - \exp\{-\beta_i t\}]\right] \left[\prod_{i \in \bar{A}} \exp\{-\lambda_i t\}\right] \\ &= \frac{\exp\{-\Lambda t\}}{\prod_{i \in A} \phi_i} \prod_{i \in A} [1 + \phi_i - \exp\{-\beta_i t\}] \end{aligned} \tag{3.11}$$

But, we have

$$\begin{aligned} \prod_{i \in A} [1 + \phi_i - \exp\{-\beta_i t\}] &= \prod_{i \in A} [\psi_i - \exp\{-\beta_i t\}] \\ &= \sum_{l=0}^{m} \left[(-1)^l \sum_{i=1}^{\binom{m}{l}} \psi_{i(m-l)}^{(m)} \exp\left\{ -(\beta_{(m)} - \beta_{i(m-l)}^{(m)}) t \right\} \right] \end{aligned} \tag{3.12}$$

Where
$\psi_i = 1 + \phi_i, \psi_{i(l)}^{(m)} = \psi_{i_1} \psi_{i_2} \cdots \psi_{i_l}, i_1 < i_2 < \cdots < i_l \in A$, $\psi_{i(l)}^{(m)} \neq \psi_{j(l)}^{(m)}, \forall i \neq j$, $\psi_{i(0)}^{(m)} = 1$, $\beta_{(m)} = \sum_{i \in A} \beta_i$, $\beta_{i(l)}^{(m)} = \beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_l}$, $\beta_{i(l)}^{(m)} \neq \beta_{j(l)}, \forall i \neq j$, $\beta_{i(0)}^{(m)} = 0, 1 \leq i, j \leq \binom{m}{l}$, see Sarhan (2000).

Substituting from equation (3.12) into (3.11), we get

$$R_A^I(t) = \frac{1}{\prod_{i\in A}\phi_i}\sum_{l=0}^{m}\left[(-1)^l\sum_{i=1}^{\binom{m}{l}}\psi_{i(m-l)}^{(m)}\exp\left\{-(\Lambda+\beta_{(m)}-\beta_{i(m-l)}^{(m)})t\right\}\right] \tag{3.13}$$

From equation (3.13), the MTTF of the improved system, say MTTF_A^I is given by

$$\mathrm{MTTF}_A^I = \frac{1}{\prod_{i\in A}\phi_i}\sum_{l=0}^{m}\left[(-1)^l\sum_{i=1}^{\binom{m}{l}}\frac{\psi_{i(m-l)}^{(m)}}{\Lambda+\beta_{(m)}-\beta_{i(m-l)}^{(m)}}\right] \tag{3.14}$$

That is, cold duplication with imperfect switch of the set of A components increases the mean time to system failure by the amount

$$\frac{1}{\prod_{i\in A}\phi_i}\sum_{l=0}^{m}\left[(-1)^l\sum_{i=1}^{\binom{m}{l}}\frac{\psi_{i(m-l)}^{(m)}}{\Lambda+\beta_{(m)}-\beta_{i(m-l)}^{(m)}}\right]-\frac{1}{\Lambda}.$$

4. THE α-FRACTILES

This section presents the α-fractiles of the original and improved systems. Let $L(\alpha)$ be the α-fractile of the original system and $L_A^D(\alpha)$, $D = H, C$ and I, $A \subseteq \{1, 2, 3, \cdots, n\}$ the α-fractiles of the improved systems. The α-fractiles $L(\alpha)$ and $L_A^D(\alpha)$ are defined as the solution of the following equations, respectively,

$$R\left(\frac{L(\alpha)}{\Lambda}\right) = \alpha,\ R_A^D\left(\frac{L_A^D(\alpha)}{\Lambda}\right) = \alpha. \tag{4.1}$$

It follows from equations (2.1) and the first equation (4.1) that

$$L(\alpha) = -\ln(\alpha) \tag{4.2}$$

From the second equation of (4.1), when D=H, and equation (3.5), one can verify that $L = L_A^H(\alpha)$ satisfies the following equation

$$L+\ln(\alpha)-\left[m\ln(2)+\sum_{i\in A}\ln\left(1-\frac{1}{2}\exp\left\{-\frac{\lambda_i L}{\Lambda}\right\}\right)\right] = 0. \tag{4.3}$$

Similarly, from equation (3.8) and the second equation of (4.1), when D=C,$L = L_A^D(\alpha)$ can be obtained by solving the following equation

$$L+\ln(\alpha)-\sum_{i\in A}\ln\left(1+\frac{\lambda_i L}{\Lambda}\right) = 0. \tag{4.4}$$

Finally, from equation (3.8) and the second equation of (4.1), when D=I, $L = L_A^I(\alpha)$ satisfies the following equation

$$L + \ln(\alpha) - \sum_{i\in A}\left[\ln\left(1+\phi_i - \exp\left\{-\frac{\beta_i L}{\Lambda}\right\}\right) - \ln(\phi_i)\right] = 0. \tag{4.5}$$

Equations (4.3)-(4.5) have no closed form solutions and can be solved using some numerical program such as Mathematica Program System.

5. RELIABILITY EQUIVALENCE FACTORS

In this section, we derive SREF and MREF of the n components series system. Where A is the set of components improved according to one of the duplication methods (HDM, CDM and IDM) and B is the set of components improved according to a reduction method.

5.1. The SREF

In this subsection, we shall derive the SREF in three different methods. When the mixture failure rate of the set of B components are reduced by the same factor ρ, these factors will be denoted by $\rho^D_{A,B}(\alpha)$, $D = H, C, I$ and $A, B \subseteq \{1, 2, 3, \cdots, n\}$. The factor $\rho^D_{A,B}(\alpha)$ is defined as the solution ρ of the equation

$$R^D_A(t) = R_{B,\rho}(t) = \alpha. \tag{5.1}$$

Using equation (5.1), when $D = H$, together with equations (3.1) and (3.5), one can verify that the factor $\rho = \rho^H_{A,B}(\alpha)$ satisfies the following equation

$$m\ln(2) + \frac{(1-\rho)\Lambda_B}{\Lambda-(1-\rho)\Lambda_B}\ \ln(\alpha) + \sum_{i\in A}\ln\left[1-\frac{1}{2}\alpha^{\frac{\lambda_i}{\Lambda-(1-\rho)\Lambda_B}}\right] = 0. \tag{5.2}$$

The factor $\rho = \rho^H_{A,B}(\alpha)$ can be obtained by solving the above equation with respect to ρ.

Similarly, using equation (5.1), when $D = C$, together with equations (3.1) and (3.8), one can deduce the following equation

$$\left[\frac{(1-\rho)\Lambda_B}{\Lambda-(1-\rho)\Lambda_B}\right]\ln(\alpha) + \sum_{i\in A}\ln\left[1-\frac{\lambda_i\ln(\alpha)}{\Lambda-(1-\rho)\Lambda_B}\right] = 0. \tag{5.3}$$

By solving the above equation with respect to ρ, one can obtain $\rho = \rho^C_{A,B}$.

Finally, one can use equation (5.1), when $D = I$, together with equations (3.1) and (3.11) to verify that the factor $\rho = \rho^I_{A,B}(\alpha)$ satisfies the equation

$$\left[\frac{(1-\rho)\Lambda_B}{\Lambda-(1-\rho)\Lambda_B}\right]\ln(\alpha) + \sum_{i\in A}\left\{\ln\left[1+\phi_i-\alpha^{\frac{\beta_i}{\Lambda-(1-\rho)\Lambda_B}}\right] - \ln(\phi_i)\right\} = 0. \tag{5.4}$$

Equations (5.2)-(5.4) have no closed form solutions and can be solved using some numerical program such as Mathematica Program System.

Another REFs, say ρ_{B_C}, that obtained when the set of type failures C of B component system are reducing. These factors will be denoted by $\Lambda^D_{\rho_{B_C}}(\alpha), D = H, C, I$, $B \subseteq \{1, 2, \cdots, n\}$ and $C \subseteq \{1, 2, 3\}$. The factor $\rho_{B_C} = \Lambda^D_{\rho_{B_C}}(\alpha)$ is defined as the solution of the equation

$$R^D_A(t) = R_{B,\rho_C}(t) = \alpha. \tag{5.5}$$

Using equation (5.5), when $D = H$, together with equations (3.3) and (3.5), one can verify that the factor $\Lambda_{\rho_{B_C}} = \Lambda^H_{\rho_{B_C}}(\alpha)$ satisfies the following equation

$$m\ln(2) + \left[\frac{\Lambda_{B_C} - \Lambda_{\rho_{B_C}}}{\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}}\right]\ln(\alpha) + \sum_{i\in A}\ln\left[1 - \frac{1}{2}\alpha^{\frac{\lambda_i}{\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}}}\right] = 0. \tag{5.6}$$

Similarly, using equation (5.5), when $D = C$, together with equations (3.3) and (3.8), one can obtain $\Lambda_{\rho_{B_C}} = \Lambda^C_{\rho_{B_C}}$ as the solution of the following equation

$$\left[\frac{\Lambda_{B_C} - \Lambda_{\rho_{B_C}}}{\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}}\right]\ln(\alpha) + \sum_{i\in A}\ln\left[1 - \frac{\lambda_i\ln(\alpha)}{\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}}\right] = 0. \tag{5.7}$$

Finally, one can use equation (5.5), when $D = I$, together with equations (3.3) and (3.11) to verify that the factor $\Lambda_{\rho_{B_C}} = \Lambda^I_{\rho_{B_C}}$ satisfies the equation

$$\left[\frac{\Lambda_{B_C} - \Lambda_{\rho_{B_C}}}{\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}}\right]\ln(\alpha) + \sum_{i\in A}\left\{\ln\left[1 + \phi_i - \alpha^{\frac{\beta_i}{\Lambda - \Lambda_{B_C} + \Lambda_{\rho_{B_C}}}}\right] - \ln(\phi_i)\right\} = 0. \tag{5.8}$$

Equations (5.6)-(5.8) have no closed form solutions and may be solved numerically by using Mathematica Program System.

5.2. The MREF

The MREF, say $\xi^D_{A,B}$, for $D = H, C$ and I can be obtained by solving the following equation

$$\mathrm{MTTF}_{B,\rho} = \mathrm{MTTF}^D_A. \tag{5.9}$$

Using equation (5.9) together with equation (3.2), one can verify that $\xi^D_{A,B}$ satisfies the equation

$$\xi^D_{A,B} = 1 + \frac{1}{\Lambda_B}\left[\frac{1}{\mathrm{MTTF}^D_A} - \Lambda\right]. \tag{5.10}$$

Also, the factor $\Lambda^D_{\xi_{B_C}}$ can be obtained by solving the following equation

$$\mathrm{MTTF}_{B,\rho_C} = \mathrm{MTTF}^D_A. \tag{5.11}$$

Using equation (5.11) together with equation (3.4), one can deduce the following equation

$$\Lambda^{D}_{\xi_{B_C}} = \frac{1}{\mathrm{MTTF}^{D}_{A}} + \Lambda_{B_C} - \Lambda \tag{5.12}$$

Equations (5.10) and (5.12) may be solved numerically by using Mathematica program System, to get $\xi^{D}_{A,B}$ and $\Lambda^{D}_{\xi_{B_C}}$ for given A, B, n and λ_i. The MTTF^{D}_{A} are given, for $D = H, C$ and I, by solving equations (3.7), (3.10) and (3.14) respectively.

6. NUMERICAL RESULTS AND CONCLUSIONS

To explain how one can utilize the previously obtained theoretical results, we introduce a numerical example. In such example, we calculate the two different reliability equivalence factors of a three-components series system under the following assumptions:

1. The failure rate of the component i, is $\lambda_i = \sum_{j=1}^{3} \alpha_{ij}\lambda_{ij}$, $i = 1, 2, 3$, $\sum_{j=1}^{3} \alpha_{ij} = 1, \alpha_{ij} \geq 1, \lambda_{ij} > 0$.
2. The industry, shock and human failure rates of the three component are given, respectively, as
   (i) the first component has $\lambda_{11} = 0.07$, $\lambda_{12} = 0.06$, $\lambda_{13} = 0.055$ with $\alpha_{11} = 0.4$, $\alpha_{12} = 0.35$, $\alpha_{13} = 0.25$,
   (ii) the second component has $\lambda_{21} = 0.08$, $\lambda_{22} = 0.075$, $\lambda_{23} = 0.07$ with $\alpha_{21} = 0.5$, $\alpha_{22} = 0.3$, $\alpha_{23} = 0.2$,
   (iii) the third component has $\lambda_{31} = 0.09$, $\lambda_{32} = 0.088$, $\lambda_{33} = 0.078$ with $\alpha_{31} = 0.52$, $\alpha_{32} = 0.26$, $\alpha_{33} = 0.22$,
3. The system reliability will be improved when two or three components are improved according to one of the previous duplication methods, when $|A| = |B| = 2, 3$.
4. In the imperfect switch duplication method $\beta_1 = 0.01$, $\beta_2 = 0.02$, $\beta_3 = 0.03$.

For this example, we have found that:
The mean time to failure of the original system is MTTF=4.423. The MTTF of the improved systems assuming HDM, CDM, IDM are presented in Table 6.1.

Table 6.1. The MTTF of the improved systems.

A	MTTF^H	MTTF^I	MTTF^C
{1,2 }	6.895	7.662	7.978
{1, 3 }	7.038	7.836	8.293
{2, 3 }	7.459	8.146	8.768
{1, 2, 3}	9.256	11.339	12.727

From the above table, one can conclude that:

$$\mathrm{MTTF} < \mathrm{MTTF}_A^H < \mathrm{MTTF}_A^I < \mathrm{MTTF}_A^C, \text{ for all } A \subseteq \{1,2,3\}$$

The α-fractiles $L(\alpha)$, $L_A^D(\alpha)$ and the reliability equivalence factors $\rho_{A,B}^D(\alpha)$, $D = H, C, I$, and $A, B \subseteq \{1,2,3\}$ are calculated using Mathematica Program System according to the previous theoretical formulae. In such calculations the level α is chosen to be $0.1, 0.2, \cdots, 0.5$.

Table 6.2 represents the α-fractiles of the original and improved systems that are obtained by improving two or three components according to the previously mentioned methods.

Table 6.2. The α-fractiles of the original and improved system.

		A={1,2}			A={1,3}			A={2,3}		
α	$L(\alpha)$	L^H	L^I	L^C	L^H	L^I	L^C	L^H	L^I	L^C
0.1	2.303	3.285	3.694	3.869	3.316	3.728	3.972	3.367	3.806	4.126
0.2	1.609	2.459	2.754	2.876	2.495	2.799	2.973	2.551	2.883	3.117
0.3	1.204	1.946	2.169	2.258	1.985	2.218	2.349	2.043	2.305	2.484
0.4	0.916	1.562	1.729	1.795	1.601	1.781	1.879	1.660	1.867	2.005
0.5	0.693	1.245	1.369	1.416	1.284	1.420	1.493	1.343	1.504	1.609

Based on the results presented in Table 6.2, it seems that:
$L(\alpha) < L^H(\alpha) < L^I(\alpha) < L^C(\alpha)$ in all studied cases. This is confirmed by the results obtained for MTTF.

Tables 6.3 and 6.4 show the SREF of the improved systems using each duplication method for some A, B and C. According to the results presented in Tables 6.3 and 6.4, it may be observed that:

1. Hot duplication of the components 1 and 2, $A = \{1,2\}$, will increase $L(0.1)$ from $\frac{2.303}{\Lambda}$ to $\frac{3.285}{\Lambda}$, see Table 6.2. The same effect on $L(0.1)$ can occur by:

   (1) reducing the mixture failure rate of: (i) the component 1 and 2, $B = \{1,2\}$, by the factor $\rho = 0.515$, (ii) the component 1 and 3, $B = \{1,3\}$, by the factor $\rho = 0.548$, (iii) the component 2 and 3, $B = \{2,3\}$, by the factor $\rho = 0.586$, (iv) the component 1,2 and 3, $B = \{1,2,3\}$, by the factor $\rho = 0.701$, see Table 6.3,

   (1) reducing some types of the mixture failure rate as follows: (i) types 1 and 2, $C = \{1,2\}$ of the mixture of component 1 and 2, $B = \{1,2\}$, by the factor $\Lambda_{\rho_{B_C}} = 0.044$, in this case, $\Lambda_{\rho_{B_C}} = \sum_{i \in B} \sum_{j \in C} \rho_j \alpha_{ij} \lambda_{ij} = 0.044$, so $0.068\rho_1 + 0.0435\rho_2 = 0.044$. Then $\rho_1 \in (0,1)$ and $\rho_2 = \frac{0.044 - 0.068\rho_1}{0.0435}$, (ii) types 1 and 3, $C = \{1,3\}$ of the mixture of component 1 and 2, $B = \{1,2\}$, by the factor $\Lambda_{\rho_{B_C}} = 0.028$, in this case, $0.068\lambda_1 + 0.02775\lambda_3 = 0.028$. Then $\rho_1 \in (0,1)$ and $\rho_3 = \frac{0.0288 - 0.068\rho_1}{0.02775}$, (iii) types 2 and 3, $C = \{2,3\}$

of the mixture of component 1 and 2, $B = \{1, 2\}$, by the factor $\Lambda_{\rho_{B_C}} = 0.004$, in this case, $0.0435\lambda_2 + 0.02775\lambda_3 = 0.004$, then $\rho_2 \in (0, 1)$ and $\rho_3 = \frac{0.004-0.0435\rho_2}{0.02775}$, see Table 6.4.

2. In the same manner, one can read the rest of results presented in Tables 6.3 and 6.4, when the other duplication methods are used with different A, B and C.

3. The notation NA, means that there is no equivalence between the two improved systems: one obtained by reducing the failure rates or some failure rates, C of the set B of the system components and the other obtained by improving the set of A components according to the duplication methods.

Table 6.3. The SREF $\rho^D_{A,B}(\alpha)$.

		A={1,2}			A={1,3}			A={2,3}		
α	B	ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C
0.1	{1,2}	0.515	0.388	0.343	0.504	0.379	0.318	0.487	0.359	0.282
	{1,3}	0.548	0.431	0.388	0.538	0.422	0.365	0.522	0.403	0.332
	{2,3}	0.586	0.479	0.439	0.577	0.471	0.418	0.562	0.453	0.388
	{1, 2,3}	0.701	0.623	0.595	0.694	0.618	0.579	0.684	0.605	0.558
0.2	{1,2}	0.439	0.325	0.285	0.424	0.310	0.255	0.401	0.283	0.215
	{1,3}	0.478	0.372	0.334	0.464	0.358	0.307	0.442	0.332	0.269
	{2,3}	0.522	0.425	0.390	0.509	0.412	0.365	0.489	0.389	0.331
	{1, 2,3}	0.654	0.584	0.559	0.645	0.575	0.541	0.631	0.558	0.516
0.3	{1,2}	0.381	0.278	0.242	0.361	0.257	0.209	0.333	0.225	0.163
	{1,3}	0.423	0.327	0.294	0.406	0.309	0.263	0.379	0.278	0.221
	{2,3}	0.472	0.384	0.354	0.456	0.367	0.325	0.432	0.339	0.287
	{1,2, 3}	0.619	0.555	0.533	0.607	0.543	0.513	0.589	0.522	0.485
0.4	{1,2}	0.329	0.236	0.205	0.306	0.212	0.168	0.273	0.173	0.118
	{1,3}	0.375	0.289	0.260	0.354	0.266	0.225	0.323	0.230	0.179
	{2,3}	0.428	0.349	0.322	0.408	0.328	0.291	0.379	0.295	0.248
	{1,2, 3}	0.587	0.529	0.510	0.572	0.515	0.488	0.552	0.491	0.457
0.5	{1,2}	0.280	0.198	0.171	0.253	0.169	0.130	0.214	0.125	0.076
	{1,3}	0.330	0.254	0.228	0.304	0.226	0.190	0.269	0.185	0.139
	{2,3}	0.387	0.317	0.293	0.363	0.291	0.258	0.330	0.254	0.212
	{1, 2,3}	0.557	0.506	0.489	0.537	0.488	0.464	0.516	0.461	0.431

Table 6.4. The SREF $\Lambda_{\rho^D_{B_C}}(\alpha)$.

			A={1,2}			A={1,3}			A={2,3}		
α	B	C	ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C
0.1	{1,2}	{1,2}	0.044	0.026	0.019	0.042	0.025	0.016	0.040	0.022	0.012
		{1,3}	0.028	0.011	0.004	0.027	0.009	0.001	0.024	0.006	NA
		{2,3}	0.004	NA	NA	0.002	NA	NA	NA	NA	NA
	{1,3}	{1,2}	0.051	0.034	0.027	0.049	0.032	0.024	0.047	0.029	0.019
		{1,3}	0.038	0.022	0.014	0.037	0.019	0.011	0.034	0.016	0.006
		{2,3}	0.007	NA	NA	0.006	NA	NA	0.003	NA	NA
	{2,3}	{1,2}	0.065	0.047	0.041	0.063	0.046	0.037	0.061	0.043	0.032
		{1,3}	0.050	0.033	0.026	0.049	0.031	0.023	0.046	0.029	0.018
		{2,3}	0.009	NA	NA	0.007	NA	NA	0.005	NA	NA
	{1,2,3}	{1,2}	0.114	0.096	0.089	0.112	0.095	0.086	0.109	0.092	0.081
		{1,3}	0.092	0.075	0.068	0.091	0.073	0.065	0.088	0.070	0.059
		{2,3}	0.044	0.026	0.019	0.042	0.025	0.016	0.039	0.022	0.011
0.2	{1,2}	{1,2}	0.033	0.018	0.012	0.031	0.015	0.008	0.028	0.012	0.002
		{1,3}	0.018	0.002	NA	0.016	NA	NA	0.012	NA	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{1,3}	{1,2}	0.041	0.025	0.019	0.038	0.023	0.015	0.035	0.019	0.009
		{1,3}	0.028	0.014	0.006	0.025	0.009	0.002	0.022	0.006	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{2,3}	{1,2}	0.054	0.038	0.036	0.052	0.036	0.028	0.049	0.032	0.023
		{1,3}	0.039	0.024	0.018	0.038	0.022	0.014	0.035	0.018	0.009
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{1,2,3}	{1,2}	0.103	0.087	0.082	0.101	0.085	0.077	0.098	0.081	0.072
		{1,3}	0.082	0.066	0.060	0.079	0.064	0.056	0.076	0.059	0.050
		{2,3}	0.033	0.017	0.012	0.031	0.015	0.008	0.028	0.011	0.002
0.3	{1,2}	{1,2}	0.025	0.011	0.006	0.023	0.008	0.001	0.019	0.004	NA
		{1,3}	0.009	NA	NA	0.007	NA	NA	0.03	NA	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{1,3}	{1,2}	0.032	0.018	0.013	0.029	0.015	0.008	0.026	0.011	0.002
		{1,3}	0.019	0.007	0.001	0.017	0.002	NA	0.013	NA	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{2,3}	{1,2}	0.046	0.032	0.027	0.043	0.029	0.022	0.039	0.024	0.016
		{1,3}	0.032	0.017	0.012	0.029	0.015	0.008	0.025	0.009	0.001
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{1,2,3}	{1,2}	0.095	0.081	0.076	0.092	0.078	0.071	0.088	0.073	0.065
		{1,3}	0.073	0.059	0.054	0.071	0.056	0.049	0.067	0.052	0.043
		{2,3}	0.025	0.011	0.006	0.022	0.008	0.001	0.018	0.003	NA
0.4	{1,2}	{1,2}	0.018	0.005	0.001	0.015	0.002	NA	0.010	NA	NA
		{1,3}	0.002	NA	NA	NA	NA	NA	NA	NA	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{1,3}	{1,2}	0.025	0.012	0.008	0.022	0.009	0.003	0.017	0.004	
		{1,3}	0.012	0.001	NA	0.009	NA	NA	0.004	NA	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{2,3}	{1,2}	0.039	0.026	0.021	0.036	0.022	0.016	0.031	0.017	0.009

		{1,3}	0.025	0.012	0.007	0.022	0.008	0.002	0.017	0.003	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{1,2,3}	{1,2}	0.088	0.075	0.070	0.085	0.071	0.068	0.079	0.066	0.058
		{1,3}	0.067	0.053	0.049	0.063	0.049	0.044	0.058	0.045	0.037
		{2,3}	0.018	0.005	0.001	0.015	0.002	NA	0.009	NA	NA
0.5	{1,2}	{1,2}	0.011	NA	NA	0.007	NA	NA	0.002	NA	NA
		{1,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{1,3}	{1,2}	0.018	0.007	0.003	0.015	0.003	NA	0.009	NA	NA
		{1,3}	0.006	NA	NA	0.002	NA	NA	NA	NA	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{2,3}	{1,2}	0.032	0.021	0.017	0.028	0.016	0.01	0.023	0.010	NA
		{1,3}	0.018	0.006	0.003	0.014	0.002	NA	0.009	NA	NA
		{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
	{1,2,3}	{1,2}	0.089	0.069	0.066	0.077	0.065	0.060	0.072	0.059	0.052
		{1,3}	0.059	0.048	0.044	0.056	0.044	0.039	0.050	0.038	0.031
		{2,3}	0.011	NA	NA	0.007	NA	NA	0.002	NA	NA

Tables 6.5 and 6.6 show the MREF of the improved systems using each duplication method for some A, B and C.

Table 6.5. The MREF, $\xi^D_{A,B}$.

	A={1,2}			A={1,3}			={2,3}		
B	ξ^H	ξ^I	ξ^C	ξ^H	ξ^I	ξ^C	xi^H	ξ^I	ξ^C
{1,2}	0.418	0.314	0.277	0.397	0.293	0.242	0.339	0.258	0.195
{1,3}	0.458	0.361	0.327	0.438	0.342	0.295	0.385	0.309	0.251
{ 2,3,}	0.504	0.415	0.383	0.486	0.397	0.354	0.437	0.367	0.314
{1,2,3}	0.642	0.577	0.554	0.628	0.564	0.533	0.593	0.543	0.504

Table 6.6. The MREF, $\Lambda^D_{\xi_{B_C}}$.

		A={1,2}			A={1,3}			A={2,3}		
B	C	$\Lambda^H_{\xi_{B_C}}$	$\Lambda^I_{\xi_{B_C}}$	$\Lambda^C_{\xi_{B_C}}$	$\Lambda^H_{\xi_{B_C}}$	$\Lambda^I_{\xi_{B_C}}$	$\Lambda^C_{\xi_{B_C}}$	$\Lambda^H_{\xi_{B_C}}$	$\Lambda^I_{\xi_{B_C}}$	$\Lambda^C_{\xi_{B_C}}$
{1,2}	{1,2}	0.030	0.016	0.011	0.027	0.013	0.006	0.019	0.008	NA
	{1,3}	0.015	0.001	NA	0.012	NA	NA	0.004	NA	NA
	{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
{1,3}	{1,2}	0.038	0.023	0.018	0.035	0.020	0.013	0.027	0.015	0.007
	{1,3}	0.025	0.010	0.005	0.022	0.007	0.001	0.014	0.002	NA
	{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
{2,3}	{1,2}	0.051	0.037	0.031	0.048	0.034	0.027	0.040	0.029	0.020
	{1,3}	0.037	0.022	0.017	0.034	0.019	0.012	0.026	0.015	0.006
	{2,3}	NA	NA	NA	NA	NA	NA	NA	NA	NA
{1,2,3}	{1,2}	0.100	0.086	0.080	0.097	0.083	0.076	0.082	0.078	0.069
	{1,3}	0.079	0.064	0.059	0.076	0.062	0.054	0.068	0.056	0.048
	{2,3}	0.030	0.016	0.011	0.027	0.013	0.006	0.019	0.008	NA

Based on the results presented in Tables 6.5 and 6.6, one can conclude that:

1. The improved system that can be obtained by improving components 1 and 2, $A = \{1, 2\}$, according to hot duplication method, has the same mean time to failure of that system which can be obtained by doing one of the following:

   (1) reducing the mixture failure rates of: (i) components 1 and 2, $B = \{1, 2\}$, by the same factor $\xi = 0.418$, (ii) components 1 and 3, $B = \{1, 3\}$, by the same factor $\xi = 0.458$, (iii) components 2 and 3, $B = \{2, 3\}$, by the same factor $\xi = 0.504$, (iv) components 1,2 and 3, $B = \{1, 2, 3\}$, by the same factor $\xi = 0.642$, see Table 6.5.

   (2) reducing some types of the mixture failure rate as follows:
   (i) types 1 (industry), 2 (shock), $C = \{1, 2\}$ of the mixture of component 1 and 2, $B = \{1, 2\}$, by the factor $\Lambda_{\xi_{B_C}} = 0.030$, in this case, $\Lambda_{\xi_{B_C}} = \sum_{i \in B} \sum_{j \in C} \xi_j \alpha_{ij} \lambda_{ij}$, so $0.068\xi_1 + 0.0435\xi_2 = 0.030$. Then $\xi_1 \in (0, 1)$ and $\xi_2 = \frac{0.030 - 0.068\xi_1}{0.0435}$, (ii) types 1 (industry), 3 (human), $C = \{1, 3\}$ of the mixture of component 1 and 2, $B = \{1, 2\}$, by the factor $\Lambda_{\xi_{B_C}} = 0.015$, in this case, $0.068\xi_1 + 0.02775\xi_3 = 0.015$. Then $\xi_1 \in (0, 1)$ and $\xi_3 = \frac{0.015 - 0.068\xi_1}{0.02775}$, see Table 6.6.

2. The Notation NA in Tables 6.5 and 6.6 MTTF of a design obtained from the original system by reducing the set of failure rates is not equal to the mean time to failure of a design obtained from the original system by assuming duplication methods.

3. In the same manner, one can read the rest of results presented in Tables 6.5 and 6.6 when the other duplication methods are used with different A, B and C.

REFERENCES

Billinton, R and Allan, R., R (1983). *Reliability Evaluation of Engineering Systems: Concepts and Techniques,* Plenum Press. New York and London.

Everitt, B. S. And Hand, D. J. (1981). *Finite Mixture Distribution,* London New York.

Lewis, E.E. (1996). *Introduction to Reliability Engineering,* John Wiley & Sons, Inc., New York, 2nd ed.

Mustafa, A.(2002), *Reliability Equivalence of Some Systems,* Master Thesis, Department of Mathematics, Faculty of Science, Mansoura University, Egypt. (http://www.mans.edu.eg/pcvs/3118)

Mustafa A. (2008), Reliability equivalence of Two Non-identical Components, *International Journal of Reliability and Applications,* **9(1)**, 79-93.

Mustafa,A. Sarhan, A.M. and Al-Ruzaiza, A. S. (2007), Reliability equivalence of a parallel-series system, *Pakistan Journal of Statistics*, **23(3)**, 241-254.

Sarhan, A.M., Tadj, L. Al-Khodari A. and Mustafa, A.(2008), Equivalence factors of a parallel- series system, *Applied Sciences*, **10**, 219-230.

Råde, L. (1989), Reliability equivalence, studies in statistical quality control and reliability,*Mathematical Statistics, Chalmers University of Technology, S41296, Gothenburg, Sweden.*

Råde, L. (1990), Reliability Systems of 3-state components, studies in statistical quality control and reliability, *Mathematical Statistics, Chalmers University of Technology, S41296, Gothenburg, Sweden.*

Råde, L. (1991), Performance measures of reliabilty system with a cold standby with a random switvh, studies in statistical quality control and reliability, *Mathematical Statistics, Chalmers University of Technology, S41296, Gothenburg, Sweden.*

Råde, L. (1993a), Reliability equivalence, *Microelectrons Reliability*, **33**, 323-325.

Råde, L. (1993b). Reliability survival equivalence. *Microelectrons Reliability*, **33**, 881-894.

Sarhan, A.M. (2000). Reliability Equivalence of independent and non-identical components series systems. *Reliability Engineering & System Safety*, **67**, 293-300.

Sarhan, A.M. (2002). Reliability Equivalence with a basic series/parallel system. *Applied Mathematics and Computation*, **132**, 115-133.

Sarhan A.M. (2004), Reliability equivalence factors of a bridge network system, *International Journal of reliability and Applications*, **2**, 81-103.

Sarhan A.M. (2005), Reliability equivalence factors of a parallel system, *Reliability Engineering & System Safety*, **87**, 405-411.

Sarhan A.M. and Mustafa, A. (2006), Reliability equivalence of a series system consists of n independent and non-identical components, *International Journal of Reliability and Applications*, **7(2)**, 111-125.

Sarhan, A.M., Al-Ruzaiza, A.S., Awasel, I.A. and El-Gohary, A. (2004). Reliability equivalence of a series-parallel system, *Applied Mathematics and Computation*, **154**, 257-277.