International Journal of Reliability and Applications
Vol. 10, No. 2, pp. 81-88, 2009

Nonparametric Test for Used Better Than Aged in Convex Ordering Class(UBAC) of Life Distributions with Hypothesis Testing Applications

S. E. Abu-Youssef*
Department of Statistics and O.R., King Saud University, Riyadh 11451, Saudi Arabia

Abstract. A non-parametric procedure is presented for testing exponentially against used better than aged in convex ordering class (UBAC) of life distributions based on u-test. Convergence of the proposed statistic to the normal distribution is proved. Selected critical values are tabulated for sample sizes 5(5)40. The Pitman asymptotic relative efficiency of my proposed test to tests of other classes is studied. An example of 40 patients suffering from blood cancer disease demonstrates practical application of the proposed test.

Key Words : *U-Statistics; life distributions; aging properties; hypothesis testing; asymptotic normality; efficiency.*

1. INTRODUCTION

In purchasing used items with unknown age, it may be realistic for the buyer to assume that those items have been used for a long period of time. Hence, it would be of great importance to have some criteria to compute the remaining life of the purchased item with its performance under the true age. As a criteria for comparing ages of, for instance, electrical equipment, computers, radio's or alike. Bhattacharjee(1986) discussed the tail behavior of age smooth failure distributions. Cline (1987) studied the connection between the class of age smooth distribution and the classes of life distribution with subexponential tails which have many applications in queuing theory, random walk and infinite divisibility. Well known classes of life distributions include increasing failure rate (IFR), increasing failure rate in average (IFRA), new better than used (NBU), decreasing mean residual life (DMRL) and

*Corresponding Author.
E-mail address: abuyousf@ksu.edu.sa

new better than used in expectation (NBUE). For definition and properties of these criteria we refer Deshpande et al (1986), Barlow and Proschan (1981)and Bryson and Siddique(1969).

Let X be a nonnegative continuous random variable with distribution function $F(x)$, survival function $\overline{F} = 1 - F$. At age t, we define the random remaining life by X_t with survival function $\overline{F_t} = \frac{\overline{F}(t+x)}{\overline{F}(t)}, x, t \geq 0$. Assume that X has a finite mean $\mu = E(X) = \int_0^\infty \overline{F}(u)du$. Some properties concerning the asymptotic behavior of X_t as $t \to \infty$ will be used.

Definition 1.1. (Bhattacharjee(1982)). If X is nonnegative random variable, its distribution function $\overline{F}$ is said to be finally and positively smooth if a number $\gamma \in (0, \infty)$ exists such that:

$$\lim_{t \to \infty} \frac{\overline{F}(t+x)}{\overline{F}(t)} = e^{-x\gamma}, \tag{1.1}$$

where γ is called the asymptotic decay coefficient of X. Denoting X_e be a random variable exponentially distributed by mean $\frac{1}{\gamma}$, the following definitions implies that X_t converges to X_e in distribution written as $X \xrightarrow{d} X_e$. This property is useful for description of random life times of devices of unknown age.

Definition 1.2. The distribution F is said to be used better than age UBA if for all $x, t \geq 0$

$$\overline{F}(x+t) \geq \overline{F}(t)e^{-\gamma x}, \tag{1.2}$$

where γ is called is the asymptotic decay of X. From Definition 1.2, we have the following definition:

Definition 1.3. The distribution F is said to be used better than aged in convex ordering (UBAC) if for all $x, t \geq 0$

$$\int_x^\infty \overline{F}(u+t)du \geq \overline{F}(t) \int_x^\infty e^{-\gamma u}du, \tag{1.3}$$

or

$$\nu(x+t) \geq \frac{1}{\gamma}\overline{F}(t)e^{-\gamma x}, \tag{1.4}$$

where $\nu(x+t)dt = \int_{x+t}^\infty \overline{F}(u)du$

We observe that the inequality of (1.3) is achieved when $F(x)$ has an exponential distribution with mean μ equal to the coefficient of the asymptotic decay γ, where the exponential distribution is the only which has the lack of memory property.

Alzaid (1994) showed that if F is increasing hazard rate (IHR), then F is UBA. More recently, Willmot and Cai (2000) showed that the UBA class includes the decreasing mean residual life (DMRL) class. While Al-Nachawati and Alwasel(1997) showed that UBAC class includes the UBA class of life distribution. Thus we have

$$\text{IHR} \quad \subset \quad \text{DMRL} \subset \quad \text{UBA} \subset \quad \text{UBAC} .$$

Testing exponentially against the classes of life distribution has seen a good deal of attention. For testing against IHR, we refer to Barlow and Proschan(1981) and Ahmad (1994), among others. While testing against DMRL see Ahmad(1992). Finally testing against UBA see Ahmad (2004).

The plan of the rest of this paper is as follows: In section 2 a test statistics based on a U-statistics for testing H_0: F is exponential against H_1: F is UBAC and not exponential is given. Monte carlos null distribution critical points for sample sizes $n = 5(5)40$ is investigated in section 3. The Pitman asymptotic efficiency for common alternatives is obtained in section 4. Finally an example using real data from Aboummoh et al (1994) in medical science is introduced in section 5.

2. TESTING UBAC CLASS OF LIFE DISTRIBUTION

The test presented on a sample $X_1, X_2, \ldots, X_n$ from a population with distribution F , we wish to test the null hypothesis H_0 :$\overline{F}$ is exponential with mean μ against $\overline{F}$ is UBAC and not exponential. Using the inequality (1.4), we may use the following as a measure of departure from H_0 in favor H_1:

$$\delta_U = E[\nu(x+t) - \frac{1}{\gamma}\overline{F}(t)e^{-x\gamma}], \tag{2.1}$$

which becomes as the following

$$\delta_u = \int_0^\infty \int_0^\infty \nu(x+t)dF(x)dF(t)) - \frac{1}{2\gamma}\int_0^\infty e^{-x\gamma}dF(x). \tag{2.2}$$

Note that under $H_0 : \delta_u = 0$, while under $H_1 : \delta_u > (<)0$. Thus to estimate δ_u by $\hat{\delta}_{u_n}$, let $X_1, X_2, \ldots, X_n$ be a random sample from F. Let$\overline{F}_n(x) = \frac{1}{n}\sum_{j=1}^n I(X_j > x)$ denote the empirical distribution of $\overline{F}(x)$,
$\nu_n(x) = \frac{1}{n}\sum_{j=1}^n (X_j - x)I(X_j > x)$ denote the empirical distribution of $\nu(x), dF(x) = \frac{1}{n}$, $\hat{\gamma} = \frac{n}{\sum X_i}$ is the estimate of γ and μ is estimated by $\overline{X}$, where $\overline{X} = \frac{1}{n}\sum X_i$ is the usual sample mean . Then $\hat{\delta}_{u_n}$ is given by using (2.1) as

$$\hat{\delta}_u = \int_0^\infty \int_0^\infty \nu(x+t)dF_n(x)dF_n(t) - \frac{1}{2\gamma}\int_0^\infty e^{-x\hat{\gamma}}dF_n(x), \tag{2.3}$$

i.e

$$\hat{\delta}_{u_n} = \frac{1}{2n^3}\sum_i\sum_j\sum_k \left\{2(X_j - X_i - X_k)I(X_j > X_i + X_k) - X_j e^{-X_i\hat{\gamma}}\right\}, \tag{2.4}$$

where

$$I(y > t) = \begin{cases} 1, & y > t \\ 0, & o.w. \end{cases}$$

Let us rewrite (2.4) as the following

$$\hat{\delta}_{u_n} = \frac{1}{2n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\phi(X_i, X_j, X_k). \tag{2.5}$$

where

$$\phi(X_i, X_j, X_k) = \frac{1}{2n^3}\left\{2(X_j - X_i - X_k)I(X_j > X_i + X_k) - X_j e^{-X_i\gamma}\right\}. \tag{2.6}$$

To make the test scale invariant, we take

$$\hat{\Delta}_{u_n} = \frac{\hat{\delta}_{u_n}}{\overline{X}}, \tag{2.7}$$

with measure of departure $\Delta_{u_n} = \frac{\delta_{u_n}}{\overline{X}}$.

If we define

$$\phi(X_1, X_2, X_3) = \frac{1}{2n^3}\left\{2(X_2 - X_1 - X_3)I(X_2 > X_1 + X_3) - X_2 e^{-X_1\gamma}\right\}$$

then $\hat{\Delta}_{u_n}$ in (2.7) is equivalent to the U-statistics given by

$$U_n = \frac{1}{\binom{n}{3}}\sum_{i<j<k}\psi(X_i, X_j, X_k). \tag{2.8}$$

The following theorem summarizes the large sample properties of $\hat{\Delta}_{u_n}$ or U_n .

Theorem 2.1. (i) As $n \to \infty$, $\sqrt{n}(U_n - \Delta_{u_n})$ is asymptotically normal with mean 0 and variance

$$\begin{aligned}\sigma^2 \;=\; & Var\{2\int_x^\infty\int_0^{z-X}(z-u-X)f(u)f(z)dudz - e^{\hat{\gamma}X}\int_0^\infty zf(z)dz) \\ & +2\int_0^X\int_0^{X-z}(X-u-z)f(u)f(z)dzdu - X\int_0^\infty e^{\hat{\gamma}u}f(u)du \\ & +2\int_0^X\int_0^{X-z}(X-u-z)f(u)f(z)dudz \\ & -X\int_0^\infty e^{\hat{\gamma}u}f(u)du - \int_0^\infty e^{\hat{\gamma}u}f(u)du\int_0^\infty uf(u)du\}\end{aligned} \tag{2.9}$$

(ii) Under $H_0, \Delta_u = 0$ and$\sigma_0^2 = Var[-\frac{9}{2} + \frac{3X}{2} + 5e^{-X} + 2Xe^{-X}] = \frac{19}{54}$
(iii) If F is continuous UBAC, then the test is consistent.
Proof: (i) and (ii) follow from the standard theory of U-statistics cf. Lee (1990) by direct calculation. To prove Part (iii). From(2.5), let $D(x,t) = \nu(x+t) - \frac{1}{2\gamma}\overline{F}(t)e^{x\gamma}$, Since F is UBAC and continuous , then $D(x,t) > 0$ for at least one value of x, t call it (x_o, t_0). Set $(x_1, t_1) = inf\{(x) : x \geq x_o, t \geq t_o \overline{F}(x) = \overline{F}(x_o)\}$ Thus

$$D(x_1,t_1) = \nu(x_1+t_1) - \frac{1}{2\gamma}\overline{F}(t_1)e^{-x_1\gamma} \geq \nu(x_0+t_0) - \frac{1}{2\gamma}\overline{F}(t_0)e^{-x_0\gamma} = D(x_0,t_0) > 0$$

and $F(x_1+\delta) - F(x_1) > o$. Since x_1 and t_1 are point of increase of F, thus $\Delta_u > 0$.

3. MONTE CARLO NULL DISTRIBUTION CRITICAL POINTS FOR $\hat{\Delta}_{u_n}$ TEST

In practice, simulated percentiles for small samples are commonly used by applied statisticians and reliability analyst. we have simulated the upper percentile points for 95%, 98%, 99%. Table 3.1 gives these percentile points of statistic $\hat{\Delta}_{u_n}$ in (2.7) and the calculations are based on 5000 simulated samples of sizes $n = 5(5)40$. The percentiles values change slowly as n increase. To use the above test, calculate $\sqrt{n}\hat{\Delta}_{u_n}/\sigma_0^2$ and reject H_0 if this exceeds the normal variate value $Z_{1-\alpha}$.

Table 3.1. Critical Values of $\hat{\Delta}_{u_n}$

n	95%	98%	99%
5	0.1792	0.3053	0.6773
10	0.1366	0.2218	0.4125
15	0.1204	0.1754	0.3008
20	0.1012	0.1275	0.2502
25	0.0899	0.1453	0.2107
30	0.0814	0.1172	0.1905
35	0.0781	0.1109	0.1795
40	0.073	0.0988	0.1700

4. ASYMPTOTIC RELATIVE EFFICIENCY (ARE)

Since the above test statistic $\hat{\Delta}_{u_n}$ in (2.7) is new and no other tests are known for these class UBAC. We may compare this to those of smaller classes such as DMRL and UBA . Here we choose the tests K^* and $\hat{\delta}_2$ are presented by Hollander and Proschan (1975) and Ahmad (2004) respectively for DMRL and UBA classes of life distribution. The comparisons are achived by using Pitman asymptotic relative efficiency (PARE), which is defined as follows:

Let T_{1_n} and T_{2_n} be two statistics for testing H_o: $F_\theta \epsilon \{F_{\theta_x}\}, \theta_n = \theta + \frac{c}{\sqrt{n}}$ with c an arbitrary constant, then PARE of T_{1_n} relative to T_{2_n} is defined by

$$e(T_{1_n}, T_{2_n}) = \frac{\mu_1'(\theta_o)}{\sigma_1(\theta_o)} / \frac{\mu_2'(\theta_o)}{\sigma_2(\theta_o)}$$

where $\mu_i'(\theta_o) = \lim_{n\to\infty} \frac{\partial}{\partial\theta} E(T_{in})_{\to\theta_o}$ and $\sigma_i^2(\theta_o) = \lim_{n\to\infty} VarE(T_{in}), i = 1, 2.$ Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) they are:

(i)	Linear failure rate family	: $\bar{F}_{1\theta} = e^{-x-\frac{\theta x^2}{2}}$,	$x > 0, \theta > 0$
(ii)	Makeham family	: $\bar{F}_{2\theta} = e^{-x-\theta(x+e^{-x}-1)}$,	$x > 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for linear failure rate and Makeham families respectively. Direct calculations of PAE of K^*, $\hat{\delta}_2$ and $\hat{\Delta}_{u_n}$ are summarized in Table 4.1. The efficiencies in Table 2 show clearly our U-statistic ($\hat{\Delta}_{u_n}$ perform well for F_1andF_2.

Table 4.1. PAE of $\hat{\Delta}_{u_n}$, V^* and $\hat{\delta}_2$

Distribution	K^*	$\hat{\delta}_2$	$\hat{\Delta}_{u_n}$
F_1 Linear failure rate	0.806	0.630	1.299
F_2 Makeham	0.289	0.385	0.577

In Table 4.2, we give PARE's of $\hat{\Delta}_{u_n}$ with respect to V^* and $\hat{\Delta}_n$ whose PAE are mentioned in Table 4.1.

Table 4.2. PARE of $\hat{\Delta}_{u_n}$ with respect toV^* and $\hat{\delta}_2$

Distribution	$e_{F_i}(\hat{\Delta}_{u_n}, V^*)$	$e_{F_i}(\hat{\Delta}_{u_n}, \hat{\delta}_2)$
F_1 Linear failure rate	1.61	2.06
F_2 Makham	1.996	1.49

It is clear from Table 4.2 that the statistic $\hat{\Delta}_{u_n}$ perform well for $\overline{F}_1$ and $\overline{F}_2$ and it is more efficient than both $\hat{\delta}_2$ and V^* for all cases mentioned above. Hence our test, which deals the much larger UBAC is better and also simpler.

5. NUMERICAL EXAMPLE

Example : Consider the data in Abouammoh et al (1994). These data represent 40 patients suffering from blood cancer from one of the Ministry of Health Hospital in Saudi Arabia and the ordered life times (in days) are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852.

Using equation (2.7), the value of test statistics, based on the above data is $\hat{\Delta}_{hu_n} = 0.016$. This value leads to the acceptance of H_0 at the signficance level $\alpha = 0.95$ see Table 1. Therefore the data has not UBAC Property.

6. CONCLUSIONS

Testing exponentiality against the classes of life distribution has a good deal of attention. In this study we derive a new test statistic based on a U-statistic for testing exponentiality against UBAC class of life distributions and not exponential. This test is simpler and has high relative efficiency for some commonly used alternatives. Critical values are tabulated for sample sizes 5(5)40. A set of real data is used as an example to elucidate the use of the proposed test statistic for practical reliability analysis.

ACKNOWLEDGEMENTS

The author is greatly indepted to the referees and the editor for their constructive comments. This study wos supported by Faculty of Science Reerach center project No (stat/2009/15), King Saud University.

REFERENCES

Abouammoh, A. M., Abdulghani, S. A. and Qamber, I. S. (1994). On partial orderings and testing of new better than renewal used classes.*Reliability Eng. Syst. Safety*, **43** , 37-41.

Ahmad, I. A. (2004). Some properties of classes of life distributions with unknown age. *J. Statist. Prob. Let.* **69** , 333-342.

Ahmad, I.A. (1994). A class of statistics useful in testing increasing failure rate average and new better than used life distributions. *J. Statist. Plan. Inf.* **41** , 141-149.

Ahmad, I. A. . (1997). On used better than aged convex ordering class of life distributions. *J. of Stayist. Res.*, **31**, 123-130.

Al-Nachawati, H. and Alwasel, I. A. (1997). On used better than used aged in convex ordering class of life distributions. J. of staist. Res. Vol. 31, No 1, 123-130.

Alzaid, A. A. (1994). Aging concepts for items of unknown age. *Comm. Statist. Stochastic Models*, **10**, 649-659.

Barlow, R. E. and Proschan, F. (1981). Statistical Theory of Reliability and Life Testing Probability Models. *To Begin With, Silver-Spring, MD.*.

Bhattacharjee, M. C.(1982). The class of mean residual life and some consequences. *Siam J. Alg. Disc. Math.*, **3**, 56-65.

Bhattacharjee, M. C.(1986). Tail behaviour of age smooth failure distribution and applications reliability and quality control. *Ed A. P. Basu. North Holland J.* , 69-85.

Bryson, M. C. and Siddiqui, M. M. (1969). Some criteria for aging..*J. Amer. Statist. Assoc.*, **64** , 1472-1483.

Cline, D. B. H. (1987). Convolution of distribution with exponential and subexponential tails.*J. Austeal. Math. Soc. Ser. A.*, **29** , 243-256.

Deshpande, J. V., Kochar, S. C. and Singh, H. (1986). Aspects of positive aging. *J. Appl. Prob.*, **28**, 773-479.

Hollander,M. and Prochan,F. (1972). Testing whether new is better than used. *Ann.Math.Statist.*, **43** ,1136-1146.

Hollander,M. and Prochan,F. (1975). Test for mean residual life. *Biometrika*, **62**),585-593.

Lee, A. J. U-statistics .*Marcel Dekker,New York*, (1990).

Willmot, G. E. and Cai, J. (2000). On Classes of life time distributions with unknown age.*Probab. Eng. Inform. Sci.* **14**, 473-484.