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Non-associative fuzzy-relevance logics: strong t-associative monoidal uninorm logics*†

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【Abstract】 This paper investigates generalizations of weakening-free uninorm logics not assuming associativity of intensional conjunction (so called fusion) &, as *non-associative fuzzy-relevance logics*. First, the strong t-associative monoidal uninorm logic **StAMUL** and its schematic extensions are introduced as *non-associative propositional fuzzy-relevance logics*. (Non-associativity here means that, differently from classical logic, & is no longer associative.) Then the algebraic structures corresponding to the systems are defined, and algebraic completeness results for them are provided. Next, predicate calculi corresponding to the propositional systems introduced here are considered.

【Key words】 non-associative fuzzy-relevance logic, strong t-associative monoidal uninorm logics, **RM**.

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1. Introduction

The present author [25, 26] has investigated the **R** of Relevance with mingle (**RM**) and several uninorm logics such as (I)**UL** ((Involutive) uninorm logic), and (I)**UML** ((Involutive) uninorm mingle logic) introduced by Metcalfe and Montagna [18, 19], as fuzzy-relevance logics. The aim of this paper is to introduce non-associative generalizations of such fuzzy-relevance logics. (Non-associativity here means that, differently from classical logic, intensional conjunction & is no longer associative.)

First recall that all the fuzzy-relevance systems above have associative *intensional* conjunction & and such associative logics and algebras have been studied intensively in the literature. On the other hand, logical systems with non-associative & and corresponding algebras have been very little investigated. At any rate non-associative Lambek calculi are good examples of non-associative systems (see [3, 6, 15, 20, 21]). But these systems are neither fuzzy nor relevant. Non-associative rings (e.g. Lie rings and Lie algebras) are good examples of non-associative algebras. But their non-associative operation is not logical operation, but multiplication.

Fortunately, MICA (Monotonic Identity Commutative Aggregation) operators, which do not require associativity, have been introduced (see [22, 23, 24]), and a non-associative generalization of MV-algebras is recently further introduced (see [4]). Note that Yager showed that MICA operators constitute the basic operators needed for aggregation in fuzzy system modeling. Then it is a natural question: can we introduce non-associative generalizations of **RM** and the above uninorm logics?

We in fact have further practical reasons for requiring non-associative &: first, when we think of & as intensional conjunction, some &-sentences in argument are not associative. Consider "and" as *compatibility*. Then ϕ and ψ & χ may not be compatible with each other, even though ϕ & ψ and χ are. Let ϕ, ψ, and χ be "This color changes", "This color is red", and "This color is blue", respectively, and both "This color changes and this color is A" and "This color is A and this color changes" mean that this A color changes into another one. Let the comma in the compound sentence play the role of parenthesis. Then we can think that "This color changes and this color is red, and this color is blue" means that this red color changes into blue one. But from this sentence we can not infer "This color changes, and this color is red and blue" because "this color is red" and "this color is blue" are incompatible with each other (and so there is no color to change). We are in fact considering non-associative & (which will be introduced here) as this kind of compatibility. Second, some literature have recently shown that areas such as subjective probabilities, quantum mechanics, neuroscience, etc. require non-associativity (see [2, 8, 9, 12, 13, 16]).

MICA operation is a variant of the concept of uninorm obtained by removing the associativity condition in its definition. The present author [27] have recently introduced such operations with several weak versions of associativity, and investigated their properties. In particular, he defined strong **t**-associative (sta-) uninorm as a uninorm having strong **t**-associativity in place of associativity. We here investigate logical systems (based on sta-uninorms) as non-associative generalizations of **RM** and the above uninorm logics. This will satisfy the purpose because such systems can be regarded as both fuzzy-relevant and non-associative.

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In this paper we first introduce the sta-monoidal uninorm logic **StAMUL** and its schematic extensions as *non-associative fuzzy-relevance* logics. We usually call a system *relevant* if it satisfies the *strong* relevance principle (SRP) in [1] that $\phi \to \psi$ is a theorem only if ϕ and ψ share a propositional variable, and sometimes if it satisfies the *weak* relevance principle (WRP) in [7] that $\phi \to \psi$ is a theorem only if either (i) ϕ and ψ share a propositional variable or (ii) both $\sim\phi$ and ψ are theorems. But **StAMUL**, the most basic non-associative fuzzy-relevance logic defined here, is neither strongly relevant nor weakly relevant in the above senses because it proves such formulas as (α) $(\phi \wedge (\phi \to \mathbf{f})) \to (\psi \vee (\psi \to \mathbf{f}))$, i.e., $(\phi \wedge \sim\phi) \to (\psi \vee \sim\psi)$. (Note that since **StAMUL** does not prove (EM) $\phi \vee \sim\phi$, α does not satisfy WRP in **StAMUL**.) Thus, for the relevance of **StAMUL** and its extensions, we suggest weakenings of SRP and WRP as follows:[1]

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[1] Since in StAMUL ϕ and ψ may share a constant in place of a propositional

(*Fuzzy* strong relevance principle, FSRP) $\phi \rightarrow \psi$ is a theorem only if ϕ and ψ share a propositional variable *or constant*.
(*Fuzzy* weak relevance principle, FWRP) $\phi \rightarrow \psi$ is a theorem only if either (i) ϕ and ψ share a propositional variable *or constant*, or (ii) both $\sim\phi$ and ψ are theorems.

StAMUL and its extensions instead satisfy FWRP, and so are relevant in the sense that they satisfy FWRP.

FSRP and FWRP may be regarded as fuzzy versions of SRP and WRP, respectively, in the sense that logics with the "prelinearity" axiom (PL^n_t) (see A15 below) usually prove α and axiomatizations for several fuzzy logics are obtained simply by adding (PL^n_t) to a known logic because (roughly speaking) it ensures that the logic is characterized by linearly ordered algebras. Let L be an StAMUL, i.e., a schematic extension of **StAMUL**. L is more exactly *fuzzy* in the sense that it satisfies the fuzzy condition (of a logic) of Cintula in [5] that L is complete with respect to (w.r.t.) linearly ordered **L**-algebras. After defining algebraic structures corresponding to the systems, we shall provide algebraic completeness results for the systems. This will ensure that they are all fuzzy in Cintula's sense. We next present the predicate calculi corresponding to the propositional systems considered here.

StAMUL and its extensions introduced in section 2 are not merely fuzzy-relevant, but *non-associative* in the sense that they do not prove associativity. Therefore, they all can be called *non-associative fuzzy-relevance logics*.

For brevity, by *L* (*L-algebra* resp) we shall ambiguously

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variable (see α), we add "or constant" to SRP and WRP.

express the systems ((corresponding) algebras resp) defined in section 2 (3 resp) all together, if we do not need distinguish them, but context should determine which system (algebra resp) is intended; and by ***L***-algebra (i.e. boldface L-algebra), we mean L-algebra satisfying soundness (see Definition 3.5). Also, for convenience, we shall adopt the notation and terminology similar to those in [5, 10, 11, 14], and assume being familiar with them (together with results found in them).

2. Syntax

Logical systems we shall define in this section are based on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **F**, **f**, **t**. Further definable connectives are:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$,
df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define ϕ_t as $\phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiom schemes and rules for the strong t-associative monoidal uninorm logic **StAMUL**, the basic non-associative fuzzy-relevance logic defined here.

Definition 2.1 **StAMUL** consists of the following axiom schemes and rules:

A1. $\phi \to \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi)$ ($\vee$-elimination, $\vee$-E)

A6. $(\phi \wedge (\psi \vee \chi)) \to ((\phi \wedge \psi) \vee (\phi \wedge \chi))$ ($\wedge\vee$-distributivity, $\wedge\vee$-D)

A7. $\mathbf{F} \to \phi$ (ex falsum quodlibet, EF)

A8. $\phi \to \mathbf{T}$ (Verum ex quolibet, VE)

A9. $(\phi_t \& (\psi \& \chi)) \leftrightarrow ((\phi_t \& \psi) \& \chi)$ (strong t-associativity, sAS_t)

A10. $(\phi \& \psi) \to (\psi \& \phi)$ (&-commutativity, &-C)

A11. $(\phi \& \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)

A12. $(\psi \to \chi)_t \to ((\phi \to \psi) \to (\phi \to \chi))$ (t-prefixing, PF_t)

A13. $(\phi \to \psi)_t \to ((\psi \to \chi) \to (\phi \to \chi))$ (t-suffixing, SF_t)

A14. $(\phi \to (\psi \to \chi))_t \leftrightarrow ((\phi \& \psi) \to \chi)_t$ (t-residuation, RE_t)

A15. for each n, $(\phi \to \psi)^n_t \vee (\psi \to \phi)^n_t$ ($^n{}_t$-prelinearity, PL^n_t)

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj).

Definition 2.2 A logic is a schematic extension of an arbitrary logic **L** if and only if (iff) it results from **L** by adding (finitely or infinitely many) axiom schemes. **L** is a strong t-associative monoidal uninorm logic (StAMUL) iff **L** is a schematic extension of **StAMUL**. In particular, the following are non-associative fuzzy-relevance logics extending **StAMUL**:

- Involutive StAMUL **IStAMUL** is **StAMUL** plus
  (DNE) $\sim\sim\phi \rightarrow \phi$.
- Idempotent StAMUL **StAMUIL** is **StAMUL** plus
  (ID) $\phi \leftrightarrow (\phi \ \& \ \phi)$.
- Involutive StAMUIL **IStAMUIL** is **StAMUIL** plus (DNE).

For easy reference we let Ls be a set of logical systems defined previously.

Definition 2.3 Ls = {**StAMUL**, **IStAMUL**, **StAMUIL**, **IStAMUIL**}.

In L ($\in$ Ls), **f** can be defined as $\sim$**t** and vice versa. In L with (DNE) (briefly IL), $\wedge$ is defined using $\sim$ and $\vee$.

A *theory* over L is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using the rules (mp) and (adj). T $\vdash \phi$, more exactly T $\vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. The relevant (local) deduction theorem (R(L)DT) for L is as follows:

Proposition 2.4 Let T be a theory, and ϕ, ψ formulas.

(i) (RLDT) T $\cup$ {ϕ} $\vdash_L \psi$ iff there is n such that T $\vdash_L \phi^n{}_t$

$\to \psi$.

(ii) (RDT) Let L be an StAMUL with (ID). $T \cup \{\phi\} \vdash_L \psi$ iff $T \vdash_L \phi_t \to \psi$.

Proof: Proof of (i) is as usual. (ii) is just Enthymematic Deduction Theorem (see [17]). □

A theory T is *inconsistent* if $T \vdash$ **F**; otherwise it is *consistent*. For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\to$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

Remark 2.5 **UL**, **IUL**, **UML**, **RM**, and **IUML** are the systems as follows:

- **UL** is **StAMUL** plus
  (PF) $(\psi \to \chi) \to ((\phi \to \psi) \to (\phi \to \chi))$; and
  (RE) $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$.
- **IUL** is **UL** plus (DNE).
- **UML** is **UL** plus (ID).
- **RM** is **UML** plus (DNE).
- **IUML** is **RM** plus $\mathbf{t} \leftrightarrow \mathbf{f}$ (fixed-point, FP).

Note that **UL** proves (AS) $(\phi \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi \,\&\, \psi) \,\&\, \chi)$ and so its extensions above do. Thus **StAMUL**, **IStAMUL**, **StAMUIL**, and **IStAMUIL** can be regarded as *non-associative generalizations* of **UL**, **IUL**, **UML**, and **RM** (or **IUML**), respectively.

3. Semantics

Suitable algebraic structures for Ls are obtained as varieties of isotonic commutative strong t-associative monoidal residuated lattices.

Definition 3.1 An *isotonic commutative strong t-associative monoidal residuated lattice* (icstamr-lattice) is a structure A = (A, $\top$, $\bot$, $\top_t$, $\bot_f$, $\wedge$, $\vee$, *, $\to$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded distributive lattice with top element $\top$ and bottom element $\bot$.

(II) (A, *, $\top_t$) satisfies for all x, y, z $\in$ A,

(a) $x * y = y * x$ (commutativity)

(b) $\top_t * x = x$ (identity)

(c) $x \leq y$ implies $x * z \leq y * z$ (isotonicity)

(d) $x \leq \top_t$ implies $x * (y * z) = (x * y) * z$ (strong t-associativity)

(III) $y \leq x \to z$ iff $x * y \leq z$, for all x, y, z $\in$ A (residuation).

We call (A, *, $\top_t$) satisfying (II-b, d) a *strong t-associative (sta-) monoid*. Thus (A, *, $\top_t$) satisfying (II-a, b, c, d) is an isotonic commutative sta-monoid. (A, *, $\top_t$) satisfying (II) and (ID) $x = x * x$ is an *idempotent* isotonic commutative sta-monoid. (A, *, $\top_t$) satisfying (II) and (associativity) $x * (y * z) = (x * y) * z$ on [0, 1] is a *uninorm* and this is a *t-norm* in case $\top_t = \top$.

To define an icstamr-lattice we may take in place of (II-c, d)

(c') $x * (y \vee z) = (x * y) \vee (x * z)$ and

(d') $x_t * (y * z) = (x_t * y) * z$, respectively; and

in place of (III) a family of equations as in [14].

In an icstamr-lattice * need not be associative so that (A, *, $\top_t$) does not necessarily form a commutative semigroup. But since $x * (x * x) = (x * x) * x$ by (II-a), * is still associative in case $x * y = y * x$ and $y = x * x$. This allows us to write iterated *'s without brackets w.r.t. the same element(s). By x^n, we denote $x * \cdots * x$, n factors. Using $\rightarrow$ and $\bot_f$ we can define $\top_t$ as $\bot_f \rightarrow \bot_f$, and $\sim$ as in (df1). Then, an L-algebra corresponding to L is defined as follows.

Definition 3.2 (StAMUL-algebra) An *StAMUL-algebra* is an icstamr-lattice satisfying the condition: for all x, y and for each n ($\geq$ 1), (pl^n_t) $\top_t \leq (x \rightarrow y)^n_t \vee (y \rightarrow x)^n_t$.

In an analogy to Definition 3.2, we can define several algebras corresponding to the systems mentioned in Definition 2.3.

For L ($\in$ Ls), L-algebra (defined in 3.2) is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function $v : FOR \rightarrow \mathcal{A}$ satisfying:

$v(\phi \to \psi) = v(\phi) \to v(\psi)$,
$v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$,
$v(\phi \vee \psi) = v(\phi) \vee v(\psi)$,
$v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$,
$v(\mathbf{T}) = \top$,
$v(\mathbf{F}) = \bot$,
$v(\mathbf{f}) = \bot_f$,

(and hence $v(\sim\phi) = \sim v(\phi)$ and $v(\mathbf{t}) = \top_t$).

Definition 3.4 Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and K a class of L-algebras.

(i) (Tautology) ϕ is a *$\top_t$-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq \top_t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq \top_t$ for each $\phi \in T$. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \models_K \phi$, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ $\{\phi\}$, $\mathcal{A}$) for each $\mathcal{A} \in$ K.

In the next definition, we shall use the notational convention mentioned in the last paragraph of section 1.

Definition 3.5 (L-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is an *L-algebra* iff whenever ϕ is L-provable in T (i.e. $T \vdash_L \phi$, L an L logic), it is a semantic consequence of T w.r.t.

the set $\{\mathcal{A}\}$ (i.e. $T \models_{\{A\}} \phi$, $\mathcal{A}$ a corresponding L-algebra). By $MOD^{(l)}(L)$, we denote the class of (linearly ordered) **L**-algebras. We write $T \models^{(l)}_{L} \phi$ in place of $T \models_{MOD^{(l)}_{L}} \phi$.

4. Algebraic completeness

Let **A** be an StAMUL-algebra. We first note that the nomenclature of the prelinearity condition is explained by the following subdirect representation theorem.

Proposition 4.1 Each StAMUL-algebra is a subdirect product of linearly ordered StAMUL-algebras.

Proof: Its proof is as usual. □

We next show that classes of provably equivalent formulas form an **L**-algebra. Let T be a fixed theory over L. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash_L \phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \rightarrow [\psi]_T = [\phi \rightarrow \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \,\&\, \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $\bot = [\mathbf{F}]_T$, $\top = [\mathbf{T}]_T$, $\top_t = [\mathbf{t}]_T$, and $\bot_f = [\mathbf{f}]_T$. By A_T, we denote this algebra.

Proposition 4.2 For T a theory over L, $\mathbf{A}_T$ is an **L**-algebra.

Proof: Note that A2 to A6 ensure that $\wedge$, $\vee$, and $\rightarrow$ satisfy (I) in Definition 3.1; that A9 to A11, and the theorem (IT_t) $(\phi \rightarrow \psi)_t \rightarrow ((\phi \,\&\, \chi) \rightarrow (\psi \,\&\, \chi))$ ensure that & satisfies (II) (a) - (d); that A14 ensures that (III) holds; and that A15 ensures that (pl^n_t) holds. It is obvious that $[\phi]_T \leq [\psi]_T$ iff $T \vdash_L \phi \leftrightarrow (\phi \wedge \psi)$ iff $T \vdash_L \phi \rightarrow \psi$. Finally recall that $\mathbf{A}_T$ is an **L**-algebra iff $T \vdash_L \psi$ implies $T \models_L \psi$, and observe that for ϕ in T, since $T \vdash_L \mathbf{t} \rightarrow \phi$, it follows that $[\mathbf{t}]_T \leq [\phi]_T$. Thus it is an **L**-algebra. □

Theorem 4.3 (Strong completeness) Let T be a theory, and ϕ a formula. $T \vdash_L \phi$ iff $T \models_L \phi$ iff $T \models^l_L \phi$.

Proof: (i) $T \vdash_L \phi$ iff $T \models_L \phi$. Left to right follows from definition. Right to left is as follows: from Proposition 4.2, we obtain $\mathbf{A}_T \in MOD(L)$, and for $\mathbf{A}_T$-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in Mod(T, \mathbf{A}_T)$. Thus, since from $T \models_L \phi$ we obtain that $[\phi]_T = v(\phi) \geq \top_t$, $T \vdash_L \mathbf{t} \rightarrow \phi$. Then, since $T \vdash_L \mathbf{t}$, by (mp) $T \vdash_L \phi$, as required.

(ii) $T \models_L \phi$ iff $T \models^l_L \phi$. It follows from Proposition 4.1. □

5. L$\forall$: the first order extension of L

The completeness theorems for fuzzy predicate logics presented in [11, 14] may generalize for the present situation.

A trivial generalization of those of section 6 in [11] and Chapter V in [14] gives the notions of a language, its interpretations, and formulas for $L\forall$ as follows:

Given a linearly ordered L-algebra **A**, an *A-interpretation*, i.e., an *A-structure*, of a language consisting of some predicates $P \in Pred$ and constants $c \in Const$ is a structure $\mathbf{M} = (M, (r_P)_{P\in Pred}, (m_c)_{c\in Const})$, where $M \neq \varnothing$, $r_P : M^{ar(P)} \rightarrow \mathbf{A}$, ar(P) the arity of P, and $m_c \in M$ (for each $P \in Pred$, $c \in Const$).

Let L be a predicate language and let **M** be an A-structure for L. An *M-evaluation* of object variables is a mapping e assigning to each object variable x an element $e(x) \in M$. Let e, e' be two evaluations. $e \equiv_x e'$ means that $e(y) = e'(y)$ for each variable y distinct from x.

The value of a term given by **M**,e is defined as follows: $|x|_{\mathbf{M},e} = e(x)$ and $|c|_{\mathbf{M},e} = m_c$. The *(truth) value* $|A|^{\mathbf{A}}_{\mathbf{M},e}$ of a formula (where $e(x) \in M$ for each variable x) is defined inductively: for A being $P(x, \cdots, c, \cdots)$, $|P(x, \cdots, c, \cdots)|^{\mathbf{A}}_{\mathbf{M},e} = r_P(e(x), \cdots, m_c, \cdots)$, the value commutes with connectives, and $|(\forall x)A|^{\mathbf{A}}_{\mathbf{M},e} = \inf\{|A|^{\mathbf{A}}_{\mathbf{M},e'}: e \equiv_x e'\}$ if this infimum exists, otherwise undefined, and similarly for $\exists x$ and sup. **M** is *A-safe* if all infs and sups needed for definition of the value of any formula exist in **A**, i.e., $|A|^{\mathbf{A}}_{\mathbf{M},e}$ is defined for all A, e.

Let A be a formula of a language L and let **M** be a safe A-structure for L. The *truth value* of A in **M** is $|A|^{\mathbf{A}}_{\mathbf{M}} = \inf\{|A|^{\mathbf{A}}_{\mathbf{M},e}: e\ \mathbf{M}\text{-evaluation}\}$.

A formula A of a language L is an *A-tautology* if $|A|_{\mathbf{M}} \geq \top_{t\mathbf{A}}$ for each safe **A**-structure **M**, i.e., $|A|^{\mathbf{A}}_{\mathbf{M},e} \geq \top_t$ for each safe **A**-structure **M** and each **M**-evaluation of object variables.

The axioms of $L\forall$ are those of L plus the following set of axioms for quantifiers (taken by Hájek [14] as those of the basic predicate logic BL$\forall$):

($\forall$1) $(\forall x)A(x) \rightarrow A(t)$ (t substitutable for x in A(x))
($\exists$1) $A(t) \rightarrow (\exists x)A(x)$ (t substitutable for x in A(x))
($\forall$2) $(\forall x)(A \rightarrow B) \rightarrow (A \rightarrow (\forall x)B)$ (x not free in A)
($\exists$2) $(\forall x)(A \rightarrow B) \rightarrow ((\exists x)A \rightarrow B)$ (x not free in B)
($\forall$3) $(\forall x)(A \vee B) \rightarrow ((\forall x)A \vee B)$ (x not free in B)

Rules of inference for L$\forall$ are MP, AD, and generalization (GN), i.e., from A infer $(\forall x)A$. (Note that if L$\forall$ has involutive negation (i.e. L$\forall$ is IL$\forall$), one quantifier is definable from the other one and the negation $\sim$, for instance, $(\exists x)A := \sim(\forall x)\sim A$. Thus the above set of axioms for quantifiers could be simplified, i.e., ($\forall$3), ($\exists$1), and ($\exists$2) become provable as in the Łukasiewicz predicate logic Ł$\forall$ (cf. see Remark 5.4.2 in [14]).

Proposition 4.1 (i) The axioms ($\forall$1), ($\forall$2), ($\forall$3), ($\exists$1), and ($\exists$2) are **A**-tautologies for each linearly ordered L-algebra **A**. (ii) The rules MP, AD, and GN preserve **A**-tautologyhood.

Proof (i) By Lemmas 5.1.9 in [14].

(ii) MP and GN are by Lemma 5.1.10 in [14]. Thus, for L$\forall$ we need just to consider that the rule AD preserves **A**-tautologyhood. For AD, we show that

(1) for any formulas A, B, safe **A**-structure **M**, and evaluation e, $|A|^{A}_{M,e} \wedge |B|^{A}_{M,e} \leq |A \wedge B|^{A}_{M,e}$; thus, if $|A|^{A}_{M,e}, |B|^{A}_{M,e} \geq \top_{tA}$, then $|A \wedge B|^{A}_{M,e} \geq \top_{tA}$, and

(2) consequently, $|A|^{A}_{M} \wedge |B|^{A}_{M} \leq |A \wedge B|^{A}_{M}$; thus if A, B

are $\geq \top_{tA}$-true in **M**, then $A \wedge B$ is.

(1) is as in propositional calculus. To prove (2) put $|A|_w = a_w$, $\inf_w a_w = a$, $|B|_w = b_w$, and $\inf_w b_w = b$. We have to show that $\inf_w a_w \wedge \inf_w b_w \leq \inf_w(a_w \wedge b_w)$ (indices **A**, **M** deleted, w runs over all evaluations $\equiv_x e$). Since L$\forall$ proves $(\forall x)(A \wedge B) \leftrightarrow ((\forall x)A \wedge (\forall x)B)$ (see Corollary 5.1.22 (17) in [14]) and thus $\inf_w(a_w \wedge b_w) = \inf_w a_w \wedge \inf_w b_w$, it is immediate. □

Definitions of a theory T over L$\forall$ are almost the same as L. We need just to consider such definitions in **M**. Let **A** be a linearly ordered L-algebra and let **M** be a safe **A**-structure for the language of T. **M** is an *A-model* of T if $|A|^{\mathbf{A}}_{\mathbf{M}} \geq \top_{tA}$ in each $A \in T$. T is *linear* if for each pair A, B of formulas, $T \vdash A \rightarrow B$ or $T \vdash B \rightarrow A$. Then, Proposition 4.1 ensures that L$\forall$ is sound w.r.t. linearly ordered **L**-algebras.

Proposition 4.2 (Soundness) Let T be a theory in the language of T over L$\forall$ and let A be a formula of T. If $T \vdash_{L\forall} A$, then $T \models^{l}_{L} A$, i.e., $|A|^{\mathbf{A}}_{\mathbf{M}} \geq \top_{tA}$ for each linearly ordered L-algebra **A** and each **A**-model **M** of T.

Proof By induction on the length of a proof. □

To investigate completeness for L$\forall$, we have the same definition on "consistency" of a theory T as in L. We moreover define the Henkinness of T (over L$\forall$) as follows: T is *Henkin* if for each closed formula of the form $(\forall x)A(x)$ unprovable in T, i.e., T $\nvdash$ $(\forall x)A(x)$, there is a constant c in the language of T such that A(c) is unprovable in T, i.e., T $\nvdash$ A(c).

For each theory T over L$\forall$, let $\mathbf{A}_T$ be the algebra of classes of T-equivalent closed formulas with the usual operations. It is clear that $\mathbf{A}_T$ is an **L**-algebra. Let T be Henkin. Then the canonical $\mathbf{A}_T$-structure is safe and we have $[\phi]_T = |\phi|^{\mathbf{AT}}_{\mathbf{MT}}$ and so $\mathbf{M}_T$ is an $\mathbf{A}_T$-model of T. Hence, since each theory can be extended into linear Henkin theory, the completeness for L$\forall$ below is straightforward.

Lemma 4.3 For each theory T and each closed formula A, if T $\nvdash$ A, then there is a linear Henkin supertheory T' of T such that T' $\nvdash$ A.

Proof See Lemma 5.2.7 in [14]. □

Lemma 4.4 For each linear Henkin theory T and each closed formula A, if T $\nvdash$ A, then there is a linearly **L**-algebra **A** and **A**-model **M** of T such that $|A|^{\mathbf{A}}_{\mathbf{M}} < \top_{\mathrm{fT}}$.

Proof By Lemma 5.2.8 in [14]. □

By using Lemmas 4.3 and 4.4, we can show the completeness for L$\forall$ as follows.

Theorem 4.5 (Completeness) Let T be a theory over L$\forall$ and let A be a formula. T $\vdash_{L\forall}$ A iff T $\models^{l}_{L}$ A.

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