The Korean Journal of Applied Statistics (2009)
22(1), 205–220

New Family of the Exponential Distributions for Modeling Skewed Semicircular Data

Hyoung-Moon Kim[1]

[1]Dept. of Applied Statistics, Konkuk University

(Received November 2008; accepted December 2008)

Abstract

For modeling skewed semicircular data, we derive new family of the exponential distributions. We extend it to the l-axial exponential distribution by a transformation for modeling any arc of arbitrary length. It is straightforward to generate samples from the l-axial exponential distribution. Asymptotic result reveals two things. The first is that linear exponential distribution can be used to approximate the l-axial exponential distribution. The second is that the l-axial exponential distribution has the asymptotic memoryless property though it doesn't have strict memoryless property. Some trigonometric moments are also derived in closed forms. Maximum likelihood estimation is adopted to estimate model parameters. Some hypotheses tests and confidence intervals are also developed. The Kolmogorov-Smirnov test is adopted for goodness of fit test of the l-axial exponential distribution. We finally obtain a bivariate version of two kinds of the l-axial exponential distributions.

Keywords: l-axial data, skewed angular data, projection, test, confidence interval.

1. Introduction

Many useful circular models may be generated from known probability distributions on the real line or on the plane, by a variety of mechanisms. A few general methods include that (1) a wrapping method by wrapping a linear distribution around the unit circle, (2) a method through characterizing properties such as maximum entropy, *etc.*, (3) an offset method and (4) a stereographic projection method that identifies points on the real line with those on the circumference of the circle. Using these methods, circular models are prevalent at most textbooks (Fisher, 1993; Jammalamadaka and SenGupta, 2001; Mardia and Jupp, 2000).

However none of those methods and models concentrate on the semicircular or the axial data. Sometimes angular data are given as modulo π. Some examples are as follows: (1) the long axis of particles in sediments or the optical axis of a crystal, rather than a direction, (2) orientations of core samples, (3) a sea turtle example that a sea turtle emerges from the ocean in search of a nesting site on dry land, *etc.* Therefore we do not need full circular model in these data. Guardiola

This work was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD, Basic Research Promotion Fund)(KRF-2007-331-C00059).
[1]Associate Professor, Dept. of Applied Statistics, Konkuk University, 1 Hwayang-Dong, Gwangjin-Gu, Seoul 143-701, Korea. E-mail: hmkim@konkuk.ac.kr

(2004) and Jones (1968) noted this fact. Guardiola (2004) proposes a simple projection method to obtain the semicircular normal distribution.

Furthermore most of current models are symmetric. Even recent models appearing at Jones and Pewsey (2005), Pewsey *et al.* (2007) are symmetric. Pewsey (2002, 2004) considers testing problems that the underlying distribution is reflectively symmetric about an unknown central direction and about a median axis, respectively. Recently some skewed circular models are developed using a wrapping method by Pewsey (2000, 2006, 2008) and Jammalamadaka and Kozubowski (2003, 2004). As noted before none of these models concentrate on semicircular data either. In this sense, we need to develop a model that can handle skewed semicircular data. The exponential distribution which is analytically very simple plays a prominent role in physics. It holds for distances in time, especially between the happening of rare events. As a model for life testing, it does serve as a first approach. Reliability theory and reliability engineering also make extensive use of the exponential distribution. Because of the memoryless property of this distribution, it is well-suited to model the constant hazard rate portion of the bathtub curve used in reliability theory. Obviously the exponential distribution is skewed to the right.

Hence we develop a new family of distributions based on the exponential distribution and the projection method. We project the exponential distribution over a quarter-circular segment to obtain the 4-axial exponential(4AE) distribution. Extension to the l-axial exponential(LAE) distribution is obtained from simple transformation of 4AE random variable. Some special cases are the semicircular exponential(SCE) distribution and the circular exponential(CE) distribution. Asymptotic result reveals that linear exponential distribution can be used to approximate the LAE distribution. Furthermore we obtain the asymptotic memoryless property of the LAE distribution though it doesn't have strict memoryless property. We derive some trigonometric moments of the 4AE distribution and the SCE distribution. As a result, we find that some of the trigonometric moments are the same as those of semicircular Laplace distribution(SCL) (Ahn and Kim, 2008) by the stochastic relationship between the SCL distribution and the 4AE distribution. We derive the maximum likelihood estimator of model parameters in three different situations. Furthermore statistical tests and two-sided confidence intervals are also derived when a location parameter is known. The Kolmogorov-Smirnov test is adopted for goodness of fit test of the l-axial exponential distribution. Bivariate l-axial exponential distributions are derived using the bivariate exponential distribution (Gumbel, 1960) and a bivariate transformation. Each marginal density from the bivariate LAE distributions follows the LAE distribution.

This article is organized as follows. Section 2 defines the 4AE distribution and extend it to the LAE distribution. SCE and CE distributions are obtained as special cases of the LAE distributions. We get the trigonometric moments of the 4AE distribution and the SCE distribution. We estimate parameters of the LAE distribution by a maximum likelihood method in Section 3. Some hypothesis tests and confidence intervals are also developed in the same section. A bivariate extension of two kinds of the LAE distributions is considered in Section 4. Conclusion is drawn in Section 5.

2. New Family of the Exponential Distributions

2.1. Definition

The exponential distribution occurs naturally when describing the lengths of the inter-arrival times in a homogeneous Poisson process. In queuing theory, the service times of agents in a system are often modeled as exponentially distributed variables. In physics, if you observe a gas at a fixed temperature and pressure in a uniform gravitational field, the heights of the various molecules also follow an approximate exponential distribution.

Let X have an exponential distribution with a parameter σ, *i.e.*, the density of X is

$$\frac{1}{\sigma}e^{-\frac{x}{\sigma}}, \qquad x > 0,\ \sigma > 0.$$

For brevity, we shall say that X follows $\exp(\sigma)$. For a positive real number r, define the angle θ by $\theta = \tan^{-1}(x/r)$. Hence, $x = r\tan(\theta)$. Obviously, the probability density function(pdf) of θ is given by

$$\frac{1}{\varphi}\sec^2(\theta)\exp\left(-\frac{\tan(\theta)}{\varphi}\right), \quad \varphi = \frac{\sigma}{r},\ 0 < \theta < \frac{\pi}{2}. \tag{2.1}$$

This distribution can be used only for modeling any angular data having a range, $0 < \theta < \pi/2$. So it is desirable to extend it to a model which can handle any angular data having a range, $-\pi/2 < \theta < \pi/2$. This type of data is called axial or semicircular data. Furthermore we need to extend the suggested model to the l-axial distribution, which is applicable to any arc of arbitrary length say $2\pi/l$ for $l = 1, 2, \ldots$. Occasionally, measurements result in any arc of arbitrary length, say $2\pi/l,\ l = 1, 2, \ldots$, so it is desirable to extend the derived distribution.

Let $\theta^* = 4\theta/l,\ l = 1, 2, \ldots$, then the *pdf* of θ^* is given by

$$\frac{l}{4\varphi}\sec^2(l\theta^*/4)\exp\left(-\frac{\tan(l\theta^*/4)}{\varphi}\right), \quad \varphi = \frac{\sigma}{r},\ 0 < \theta^* < \frac{2\pi}{l}.$$

We denote this distribution as l-axial exponential(LAE) distribution. Note that $l = 1$ gives us the circular exponential(CE) distribution, $l = 2$ suggests the semicircular exponential(SCE) distribution or the 2-axial exponential distribution(2AE) and $l = 4$ is the derived distribution, (2.1), which we call it as the 4-axial exponential(4AE) distribution. We will use the notation SCE instead of 2AE for easy understanding.

More generally, we introduce the parameter μ as the location parameter for the LAE distribution and define it as LAE(μ, φ).

Definition 2.1. *The pdf of $LAE(\mu, \varphi)$ is defined as*

$$\frac{l}{4\varphi}\sec^2(l(\theta^* - \mu)/4)\exp\left(-\frac{\tan(l(\theta^* - \mu)/4)}{\varphi}\right), \quad \varphi = \frac{\sigma}{r},\ \mu < \theta^* < \frac{2\pi}{l} + \mu,\ \ -\pi < \mu < \pi. \tag{2.2}$$

Geometrically r is the distance between the radius and the support of the exponential density. The closer the support is to the radius, the larger φ. Hence it is not a parameter. So without loss of generality we can assume that $r = 1$. For the following three figures of the circular plots we use the same values of φ $(= 1, 1/2, 1/5)$ with $\mu = 0$ for comparison purpose. Figure 5.1, 5.2 and 5.3 show the shapes of the *pdfs* of the 4AE distributions, the SCE distribution and the CE distribution, respectively. All *pdfs* are skewed to the right. The degree of skewness increases as φ gets smaller. Hence we can handle skewed angular data.

2.2. Some basic properties

We begin with a Lemma (Gradshteyn and Ryzhik, 2007) which will be used continuously.

Lemma 2.1.

$$\tan(x) = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!}|B_{2k}|x^{2k-1}, \quad x^2 < \frac{\pi^2}{4},$$
$$\sec(x) = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!}x^{2k}, \quad x^2 < \frac{\pi^2}{4},$$

where the number B_n, representing the coefficients of $t^n/n!$ in the expansion of the function

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad 0 < |t| < 2\pi$$

are called Bernoulli numbers. And the numbers E_n, representing the coefficients of $t^n/n!$ in the expansion of the function

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \frac{\pi}{2}$$

are known as the Euler numbers.

We consider the properties of the LAE distribution. If a location parameter μ is known, then it is obvious that the LAE distribution is a member of one parameter exponential family. However it is not a member of two parameter exponential family when both parameters are unknown. It is straightforward to generate samples from the LAE distribution from the following property.

Property 1. First, generate samples from $\exp(\sigma)$, and then use the inverse transformation, $\theta^* = \mu + 4/l\tan^{-1}(x/r)$, $r = \sigma/\varphi$.

Similarly, the cumulative distribution function(cdf) of the LAE distribution is $F(\theta^*;\mu,\varphi) = F_X(r\tan(l(\theta^*-\mu)/4))$, where the function $F_X(\cdot)$ is the *cdf* of the exponential distribution, $\exp(\sigma)$.

Property 2. The *cdf* of LAE(μ,φ) is

$$F(\theta^*;\mu,\varphi) = 1 - \exp\left(-\frac{\tan(l(\theta^*-\mu)/4)}{\varphi}\right).$$

By inverting the *cdf* of the LAE distribution, we get the following quantile function.

Property 3. The quantile function for LAE(μ,φ) is

$$F^{-1}(p;\mu,\varphi) = \mu + \frac{4}{l}\tan^{-1}(\varphi\log(1-p)).$$

Let $X_1,\ldots,X_n$ be independent exponentially distributed random variables with a parameter σ. Then $\min\{X_1,\ldots,X_n\}$ is also exponentially distributed, with a parameter σ/n. Similar relationship also occurs at the LAE distribution.

Property 4. Let $\theta_i^* \sim$ LAE$(0,\varphi)$, $i = 1,2,\ldots,n$, independently, then $\min\{\theta_1^*,\ldots,\theta_n^*\}$ follows LAE$(0,\varphi/n)$.

However $\max\{X_1,\ldots,X_n\}$ and $\max\{\theta_1^*,\ldots,\theta_n^*\}$ are not exponentially and l-axial exponentially distributed, respectively. Suppose that $\theta_X \sim$ SCL$(0,\varphi)$ and $\theta_Y \sim$ 4AE$(0,\varphi)$, then there is an important stochastic relationship between the SCL distribution and the 4AE distribution.

Property 5. $\theta_Y = \tan^{-1}|\tan(\theta_X)|$.

Since in 'linear' statistics, $Y = |X| \sim \exp(\sigma)$, where $X \sim \text{Laplace}(0, \sigma)$. We consider the asymptotic behavior of an LAE(μ, φ) when φ goes to 0.

Property 6. Let $Y = l(\theta^* - \mu)/(4\varphi)$, then $Y \sim \exp(1)$ asymptotically for sufficiently small φ.

For the density (2.2), suppose that $Y = l(\theta^* - \mu)/(4\varphi)$, then the *pdf* of Y is given by

$$\sec^2(y\varphi)\exp\left(-\frac{\tan(y\varphi)}{\varphi}\right).$$

By Lemma 2.1, if we use only up to the first order terms, then the density of Y is approximately exp(1). Hence the LAE distribution can be approximated by 'linear' exponential distribution for sufficiently small φ.

One more useful property of the LAE distribution is the asymptotic memoryless property. The LAE distribution doesn't have memoryless property strictly, but it has the property asymptotically.

Property 7. If $\theta^* \sim \text{LAE}(0, \varphi)$, then for sufficiently small θ_0^* and θ_1^* such that $0 < \theta_0^* < \theta_1^* < 2\pi/l$,

$$P(\theta^* > \theta_1^* | \theta^* > \theta_0^*) \approx P(\theta^* > \theta_1^* - \theta_0^*).$$

Since $P(\theta^* > \theta_1^*|\theta^* > \theta_0^*) = P(\theta^* > \theta_1^*)/P(\theta^* > \theta_0^*) = \exp\left[-\{\tan(l\theta_1^*/4) - \tan(l\theta_0^*/4)\}/\varphi\right]$ and the last equation becomes $\exp\{-l(\theta_1^* - \theta_0^*)/(4\varphi)\} \approx P(\theta^* > \theta_1^* - \theta_0^*)$ asymptotically by Lemma 2.1 if we use only up to the first order terms.

2.3. Trigonometric moments

In this section, we first concentrate on the 4AE distribution, and then move to the SCE distribution because of simple explanation. Similar to those of any circular density, trigonometric moments of the LAE distribution are defined as follows: $\phi_p = Ee^{ip\theta^*} = \alpha_p + i\beta_p = E\cos(p\theta^*) + iE\sin(p\theta^*)$, $p = 0, \pm1, \pm2, \ldots$. For the explanation purpose, we use the notations θ for the 4AE distribution and θ^* for the LAE distribution if nothing is commented even though the 4AE distribution is a special case of the LAE distribution. We assumed that the location parameter $\mu = 0$ without loss of generality. To get the sine moments and the cosine moments, we need to change $\cos(p\theta^*)$ and $\sin(p\theta^*)$ to the functions of x using the given transformation $x = \tan(\theta^*)$. It can be done by the multiple-angle formulas, that is,

$$\begin{aligned}
\cos(p\theta^*) &= \sum_{k=0}^{p}\binom{p}{k}\cos^k(\theta^*)\sin^{p-k}(\theta^*)\cos\left\{(p-k)\frac{\pi}{2}\right\},\\
\sin(p\theta^*) &= \sum_{k=0}^{p}\binom{p}{k}\cos^k(\theta^*)\sin^{p-k}(\theta^*)\sin\left\{(p-k)\frac{\pi}{2}\right\},
\end{aligned}$$

when p is a positive integer. This multiple-angle formulas established only using the Euler formula and binomial theorem.

Lemma 2.2. *Using $x = \tan(\theta^*)$, above multiple-angle formulas are given in terms of x by*

$$\begin{aligned}
\cos(p\theta^*) &= \sum_{k=0}^{p}\binom{p}{k}c_{p-k}^{1}x^{p-k}(1+x^2)^{-\frac{p}{2}},\\
\sin(p\theta^*) &= \sum_{k=0}^{p}\binom{p}{k}c_{p-k}^{2}x^{p-k}(1+x^2)^{-\frac{p}{2}},
\end{aligned}$$

where $\sin(\theta^*) = x/\sqrt{1+x^2}$, $\cos(\theta^*) = 1/\sqrt{1+x^2}$,

$$\cos\left\{(p-k)\frac{\pi}{2}\right\} = c^1_{p-k} = \begin{cases} 1, & \text{if } p-k=4m, \\ 0, & \text{if } p-k=2m+1, \\ -1, & \text{if } p-k=4m+2 \end{cases}$$

and

$$\sin\left\{(p-k)\frac{\pi}{2}\right\} = c^2_{p-k} = \begin{cases} 1, & \text{if } p-k=4m+1, \\ 0, & \text{if } p-k=2m, \\ -1, & \text{if } p-k=4m+3, \end{cases}$$

where $m = 0, 1, 2, \ldots$.

2.3.1. Trigonometric moments of the 4AE distribution

Theorem 2.1. *The first four* $\alpha_p = E\cos(p\theta)$, $p = 1, 2, 3, 4$ *for* $4AE(0, \varphi)$ *are given as follows:*

$$\begin{aligned}
\alpha_1 &= \frac{\pi}{2\varphi}\left\{H_0\left(\frac{1}{\varphi}\right) - Y_0\left(\frac{1}{\varphi}\right)\right\}, \\
\alpha_2 &= \frac{\sqrt{\pi}}{\sqrt{2}\varphi^{\frac{3}{2}}}\left\{H_{-\frac{1}{2}}\left(\frac{1}{\varphi}\right) - Y_{-\frac{1}{2}}\left(\frac{1}{\varphi}\right)\right\} - 1, \\
\alpha_3 &= \frac{\pi}{2\varphi^2}\left\{Y_{-1}\left(\frac{1}{\varphi}\right) - H_{-1}\left(\frac{1}{\varphi}\right)\right\} - \frac{3}{\pi\varphi}G^{31}_{13}\left(\frac{1}{4\varphi^2}\middle|\begin{matrix}-1/2\\0,0,1/2\end{matrix}\right), \\
\alpha_4 &= 1 - \frac{4}{\sqrt{\pi}\varphi}G^{31}_{13}\left(\frac{1}{4\varphi^2}\middle|\begin{matrix}-1/2\\1/2,0,1/2\end{matrix}\right),
\end{aligned}$$

where $H_\nu(z)$ *is the Struve function,* $Y_\nu(z)$ *is the Bessel function of the second kind (Abramowitz and Stegun, 1972) and* $G^{mn}_{pq}\left(x\middle|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right)$ *is called as Meijer's G-function (Gradshteyn and Ryzhik, 2007).*

Proof. These moments are exactly the same as those of SCL$(0, \varphi)$ (Ahn and Kim, 2008). Because of Property 5, the result follows immediately by this stochastic relationship between the 4AE distribution and the SCL distribution. □

Since the 4AE distribution is not symmetric, we need to derive the sine moments, $\beta_p = E\sin(p\theta)$.

Theorem 2.2. *The first four* β_p, $p = 1, 2, 3, 4$ *for* $4AE(0, \varphi)$ *are given as follows:*

$$\begin{aligned}
\beta_1 &= \frac{1}{\varphi}\left[\frac{\pi}{2}\left\{H_1\left(\frac{1}{\varphi}\right) - Y_1\left(\frac{1}{\varphi}\right)\right\} - 1\right], \\
\beta_2 &= \frac{2}{\varphi}\left\{-ci\left(\frac{1}{\varphi}\right)\cos\left(\frac{1}{\varphi}\right) - si\left(\frac{1}{\varphi}\right)\sin\left(\frac{1}{\varphi}\right)\right\}, \\
\beta_3 &= \frac{1}{\pi\varphi}\left\{3\,G^{31}_{13}\left(\frac{1}{4\varphi^2}\middle|\begin{matrix}0\\1/2,0,1/2\end{matrix}\right) - G^{31}_{13}\left(\frac{1}{4\varphi^2}\middle|\begin{matrix}-1\\-1/2,0,1/2\end{matrix}\right)\right\}, \\
\beta_4 &= \frac{2}{\sqrt{\pi}\varphi}\left\{G^{31}_{13}\left(\frac{1}{4\varphi^2}\middle|\begin{matrix}0\\1,0,1/2\end{matrix}\right) - G^{31}_{13}\left(\frac{1}{4\varphi^2}\middle|\begin{matrix}-1\\0,0,1/2\end{matrix}\right)\right\},
\end{aligned} \tag{2.3}$$

where ci(x) is the cosine integral and si(x) is the sine integral defined as follows:

$$ci(x) = -\int_x^\infty \frac{\cos(t)}{t}dt = C + \log(x) + \int_0^x \frac{\cos(t)-1}{t}dt,$$
$$si(x) = -\int_x^\infty \frac{\sin(t)}{t}dt = -\frac{\pi}{2} + Si(x),$$

C is the Euler constant and $Si(x) = \int_0^x \sin(t)/tdt$.

Proof. The proof is the process of using some transformations. For all sine moments, we first use the transformation, $x = \tan(\theta)$, so we skip this part for the following proof. For the first sine moment, after the transformation,

$$\beta_1 = \frac{1}{\varphi}\int_0^\infty x\left(1+x^2\right)^{-\frac{1}{2}} \exp\left(-\frac{x}{\varphi}\right) dx.$$

Result follows by the integral formula 3.366.3 (Gradshteyn and Ryzhik, 2007). To obtain β_2, after the transformation,

$$\beta_2 = \frac{2}{\varphi}\int_0^\infty x\left(1+x^2\right)^{-1} \exp\left(-\frac{x}{\varphi}\right) dx.$$

We use the integral formula 3.354.2 (Gradshteyn and Ryzhik, 2007) to get the result. The third sine moment is as follows after the transformation and simple algebra:

$$\beta_3 = \frac{1}{\varphi}\left\{3\int_0^\infty x\left(1+x^2\right)^{-\frac{3}{2}} \exp\left(-\frac{x}{\varphi}\right) dx - \int_0^\infty x^3\left(1+x^2\right)^{-\frac{3}{2}} \exp\left(-\frac{x}{\varphi}\right) dx\right\}.$$

For two integrals of right hand side, we use the integral formula 3.389.2 (Gradshteyn and Ryzhik, 2007). The fourth sine moment β_4, after the transformation and simple algebra:

$$\beta_4 = \frac{4}{\varphi}\left\{\int_0^\infty x(1+x^2)^{-2} \exp\left(-\frac{x}{\varphi}\right) dx - \int_0^\infty x^3(1+x^2)^{-2} \exp\left(-\frac{x}{\varphi}\right) dx\right\}.$$

We use the same integral formula of β_3, the integral formula 3.389.2 to get the result. □

2.3.2. Trigonometric moments of the SCE distribution

Theorem 2.3. *The first two $\alpha_p = E\cos(p\theta^*)$, $p = 1, 2$ for $SCE(0, \varphi)$ are given as follows:*

$$\alpha_1 = \frac{\sqrt{\pi}}{\sqrt{2}\varphi^{\frac{3}{2}}}\left\{H_{-\frac{1}{2}}\left(\frac{1}{\varphi}\right) - Y_{-\frac{1}{2}}\left(\frac{1}{\varphi}\right)\right\} - 1,$$
$$\alpha_2 = 1 - \frac{4}{\sqrt{\pi}\varphi}G_{13}^{31}\left(\frac{1}{4\varphi^2}\middle| \begin{matrix} -1/2 \\ 1/2, 0, 1/2 \end{matrix}\right).$$

Proof. The first cosine moment, α_1, is the same as α_2 of the SCL distribution and the second cosine moment is the same as α_4 of the SCL distribution (Ahn and Kim, 2008) by Property 5 and the property of an even function. □

In general we find that the k^{th} cosine moment, $\alpha_k = E\cos(k\theta^*)$, of the SCE distribution is the same as the $2k^{th}$ cosine moment, $\alpha_{2k} = E\cos(2k\theta^*)$, of the SCL distribution by Property 5. We also need to calculate the sine moments, $\beta_p = E\sin(p\theta^*)$, since the SCE distribution is not symmetric.

Theorem 2.4. *The first two sine moments of $SCE(0, \varphi)$ are given by*

$$\beta_1 = \frac{2}{\varphi}\left\{-ci\left(\frac{1}{\varphi}\right)\cos\left(\frac{1}{\varphi}\right) - si\left(\frac{1}{\varphi}\right)\sin\left(\frac{1}{\varphi}\right)\right\},$$

$$\beta_2 = \frac{2}{\sqrt{\pi}\varphi}\left\{G_{13}^{31}\left(\frac{1}{4\varphi^2}\middle| \begin{matrix}0\\1,0,1/2\end{matrix}\right) - G_{13}^{31}\left(\frac{1}{4\varphi^2}\middle| \begin{matrix}-1\\0,0,1/2\end{matrix}\right)\right\}.$$

Proof. Note that the first sine moment of the SCE distribution is the same as β_2 given at (2.3) of the 4AE distribution. And the second sine moment of the SCE distribution is the same as β_4 given at (2.3) of the 4AE distribution since we are using the transformation, $\theta^* = 4\theta/l$, $l = 1, 2, \ldots$, mentioned in Section 2.1. □

Note that, in general, the k^{th} sine moment, $\alpha_k = E\sin(k\theta^*)$, of the SCE distribution is the same as the $2k^{th}$ sine moment, $\alpha_{2k} = E\sin(2k\theta)$, of the 4AE distribution because of the transformation, $\theta^* = 4\theta/l$, $l = 1, 2, \ldots$.

3. Statistical Inference

The log-likelihood for a random sample of size n, $\theta^* = (\theta_1^*, \ldots, \theta_n^*)$, from LAE$(\mu, \varphi)$ is given by

$$l(\mu, \varphi; \theta^*) = n\log\left(\frac{l}{4\varphi}\right) + \sum_{i=1}^{n}\log\{\sec^2(l(\theta_i^* - \mu)/4)\} - \frac{\sum_{i=1}^{n}\tan(l(\theta_i^* - \mu)/4)}{\varphi}. \tag{3.1}$$

We consider the maximum likelihood estimators of φ and μ in three different situations. First we consider the case of μ known and φ unknown.

3.1. Assuming μ known and φ unknown

We first derive the maximum likelihood estimator of φ when $\theta^* \sim$ LAE(μ, φ), where μ is known. And then discuss how to do hypotheses tests and how to develop confidence intervals.

Theorem 3.1. *Let $\theta^* \sim LAE(\mu, \varphi)$, where μ is known, then the maximum likelihood estmator of φ can be drawn in closed form as follows:*

$$\hat{\varphi} = \frac{\sum_{i=1}^{n}\tan(l(\theta_i^* - \mu)/4)}{n}. \tag{3.2}$$

Proof. The first derivative of (3.1) is given by

$$\frac{dl(\mu, \varphi; \theta^*)}{d\,\varphi} = -\frac{n}{\varphi} + \frac{\sum_{i=1}^{n}\tan(l(\theta_i^* - \mu)/4)}{\varphi^2}.$$

Set it to 0, the result follows. The second derivative with respect to φ after plugging $\hat{\varphi}$ instead of φ is negative. Furthermore (3.1) is $-\infty$ at the boundaries of φ. So (3.2) is a global maximum and hence it is the maximum likelihood estimator of φ. □

Theorem 3.2. *Let $\theta^* \sim LAE(\mu, \varphi)$, where μ is known, then the mle of φ given at (3.2) is UMVUE of φ too.*

Proof. By the Factorization theorem, $\sum_{i=1}^{n} \tan(l(\theta_i^* - \mu)/4)$ is a sufficient statistic for φ. This family of distribution assuming μ known is a member of one parameter exponential family so it is complete. Hence $\sum_{i=1}^{n} \tan(l(\theta_i^* - \mu)/4)$ is a complete sufficient statistic for φ. Expectation of this complete sufficient statistic becomes $n\varphi$ after transforming $x_i = \tan(l(\theta_i^* - \mu)/4)$, $i = 1, \dots, n$ and by simple algebra. Hence the maximum likelihood estimator of φ is also UMVUE(Uniformly Minimum Variance Unbiased Estimator) by the Rao-Blackwell-Lehmann-Scheffé theorem (Lehmann and Casella, 1998). □

Theorem 3.3. *Let $\theta^* \sim LAE(\mu, \varphi)$, where μ is known, then the exact distribution of the mle of φ given at (3.2) follows $1/n\Gamma(n, \varphi)$.*

Proof. Let $X_i = \tan(l(\theta_i^* - \mu)/4)$, then the distribution of X_i follows $\exp(\varphi)$ independently. So the exact distribution of the mle of φ is $1/n\Gamma(n, \varphi)$ by the property of gamma distribution. □

We can find that mle converges in probability and converges almost surely as following:

Theorem 3.4. *Let $\theta^* \sim LAE(\mu, \varphi)$, where μ is known, then the mle satisfies*

$$\lim_{n\to\infty} P(|\hat{\varphi} - \varphi| \geq \epsilon) = 0 \text{ and } P\left(\lim_{n\to\infty} |\hat{\varphi} - \varphi| \geq \epsilon\right) = 0.$$

Proof. Let $X_i = \tan(l(\theta_i^* - \mu)/4)$, then the mle of φ is the sample mean of X_i. Hence, by the weak (strong) law of large numbers, the result follows immediately since $\operatorname{Var}\tan(l(\theta_i^* - \mu)/4) = \varphi^2 < \infty$. □

Furthermore we derive an asymptotic distribution of the mle of φ.

Theorem 3.5. *Let $\theta^* \sim LAE(\mu, \varphi)$, where μ is known, then the asymptotic distribution of the mle of φ given at (3.2) is given by*

$$\sqrt{n}(\hat{\varphi} - \varphi) \xrightarrow{\mathcal{L}} N(0, \varphi^2).$$

Proof. Let $X_i = \tan(l(\theta_i^* - \mu)/4)$, then the mle of φ is the sample mean of X_i. Hence, by the central limit theorem(CLT), the result follows immediately since $\operatorname{Var}\tan(l(\theta_i^* - \mu)/4) = \varphi^2 < \infty$. □

We can do hypotheses test based on the exact distribution. The test statistic based on the exact distribution is now

$$TS_0 = \frac{2n\hat{\varphi}}{\varphi_0}.$$

The distribution of the test statistic is $\chi^2(2n)$ under $H_0 : \varphi = \varphi_0$ by the property of gamma distribution. So we reject H_0 if $TS_0 > \chi^2_{2n,\alpha/2}$ or $TS_0 < \chi^2_{2n,1-\alpha/2}$. Under $H_0 : \varphi \geq \varphi_0$, we reject H_0 if $TS_0 < \chi^2_{2n,1-\alpha}$. Similarly we reject $H_0 : \varphi \leq \varphi_0$ if $TS_0 > \chi^2_{2n,\alpha}$. Since, under the given two one-sided H_0's, the test statistic's distribution is $\chi^2(2n)$. Exact $(1-\alpha)100\%$ two-sided confidence interval for φ is given by

$$\left(\frac{2n\hat{\varphi}}{\chi^2_{2n,\frac{\alpha}{2}}}, \frac{2n\hat{\varphi}}{\chi^2_{2n,1-\frac{\alpha}{2}}}\right).$$

Similarly we can do hypotheses test based on an approximate distribution. The test statistic based on an approximate distribution is now

$$TS_0 = \frac{\hat{\varphi}-\varphi_0}{\varphi_0/\sqrt{n}}.$$

Since the distribution of the test statistic is the standard normal distribution under $H_0 : \varphi = \varphi_0$, we reject H_0 if $|TS_0| > z_{\alpha/2}$. Under $H_0 : \varphi \geq \varphi_0$, we reject H_0 if $TS_0 < -z_\alpha$. Similarly we reject $H_0 : \varphi \leq \varphi_0$ if $TS_0 > z_\alpha$. Since, under the given two H_0's, the test statistic's distribution is the standard normal distribution for the one-sided tests. Approximate $(1-\alpha)100\%$ two-sided confidence interval for φ is given by

$$\hat{\varphi} \pm z_{\frac{\alpha}{2}}\frac{\hat{\varphi}}{\sqrt{n}},$$

since $\hat{\varphi}$ is a consistent estimator of φ by Theorem 3.4 and by the following Slutsky's Theorem (Lehmann and Casella, 1998).

Lemma 3.1. *(Slutsky's Theorem) If $Y_n \xrightarrow{\mathcal{L}} Y$, and A_n and B_n tend in probability to a and b, respectively, then $A_n + B_nY_n \xrightarrow{\mathcal{L}} a + bY$.*

To validate all these theoretical results, an ad hoc approach is that first get a good estimate of a location parameter μ. Then the data is now free of location so all above theoretical results can be applied which is fruitful. For example, circular mean (Jammalamadaka and SenGupta, 2001) is a good candidate for removing location effect which is defined by

$$\overline{\mu}_0 = \begin{cases} \tan^{-1}\left(\frac{S}{C}\right), & \text{if } C > 0,\ S \geq 0, \\ \frac{\pi}{2}, & \text{if } C = 0,\ S > 0, \\ \tan^{-1}\left(\frac{S}{C}\right) + \pi, & \text{if } C < 0, \\ \tan^{-1}\left(\frac{S}{C}\right) + 2\pi, & \text{if } C \geq 0,\ S < 0, \\ \text{undefined}, & \text{if } C = 0,\ S = 0, \end{cases}$$

where $(C, S) = \left(\sum_{i=1}^n \cos(\theta_i^*), \sum_{i=1}^n \sin(\theta_i^*)\right)$.

3.2. Assuming μ unknown, φ known or both μ and φ unknown

In these two situations it is impossible to get the maximum likelihood estimators in closed forms. So we first calculate the minus log-likelihood for a random sample of size n, $\theta^* = (\theta_1^*, \ldots, \theta_n^*)$, from the LAE distribution and then, for the following example, the corresponding estimates have been computed by direct minimization (Byrd *et al.*, 1995) of the minus log-likelihood itself. Byrd's method allows box constraints, that is, each variable can be given a lower and/or an upper bound. For our example, to improve estimation process we can use ranges of μ and φ of the likelihood as box constraints which satisfy $(\theta^*_{(n)} - 2\pi/l < \mu < \theta^*_{(1)})$ and $\varphi > 0$, where $\theta^*_{(1)}$ is the minimum sample and $\theta^*_{(n)}$ is the maximum sample.

EXAMPLE 3.1. We simulated a data set of size 100 from SCE $(\mu = 0, \varphi = 1)$ using the stochastic relationship, $\theta^* = \mu + 4/l \tan^{-1}(x/r)$, $r = \sigma/\varphi$ by Property 1. By direct minimization of the minus log-likelihood, we get estimates $\hat{\mu} = 0.001$ and $\hat{\varphi} = 1.002$. Circular data plot with *pdfs* and Healy's plot (Healy, 1968) are shown in Figure 5.4. We can visually note that a satisfactory fit of the density to the data by Figure 5.4. Two *pdfs*(the solid line corresponds the original *pdf* and the dashed line represents to the fitted line) are almost the same. Healy's plot is based on

$$d_i = r\tan(l(\theta^*_i - \mu)/4), \quad (i = 1, \dots, n). \tag{3.3}$$

And d_i is sampled from the exponential distribution with a parameter σ if the fitted model is appropriate. Practically the exact parameter values in Equation (3.3) need to be replaced by estimates. Above d_i then sorted and plotted against the $\exp(\sigma)$ percentage points. Similarly, the cumulative $\exp(\sigma)$ probabilities can be plotted against their nominal values $1/n, 2/n, \dots, 1$; the points should lie on the bisection line of the quadrant.

Above approach is a graphical method, whereas the following one is a theoretical one. Suppose $\theta^* \sim \mathrm{LAE}(\mu, \varphi)$ and we wish to test the hypothesis,

$$H_0 : F_{\theta^*}(\theta^*) = F_0(\theta^*)\ \forall\ x \quad vs. \quad H_1 : \exists\ x \text{ such that } F_{\theta^*}(\theta^*) \neq F_0(\theta^*),$$

where $F_0(\theta^*)$ is given by Property 2. Then the Kolmogorov-Smirnov test (Lehmann and Romano, 2005) can be adopted. Given a random sample of size n, $\theta^* = (\theta^*_1, \dots, \theta^*_n)$, from the LAE distribution, we first arrange those in increasing order of magnitude. The empirical distribution function is defined by

$$\hat{F}_n(\theta^*) = \begin{cases} 0, & \text{if } \theta^* < \theta^*_{(1)}, \\ \dfrac{i}{n}, & \text{if } \theta^*_{(i)} \le \theta^* < \theta^*_{(i+1)}, \\ 1, & \text{if } \theta^*_{(n)} \le \theta^*. \end{cases}$$

The value of Kolmogorov-Smirnov statistic is defined by

$$D_n = \sup_{\theta^*} \sqrt{n}\,\left|\hat{F}_n(\theta^*) - F_0(\theta^*)\right|.$$

The Kolmogorov-Smirnov test rejects the null hypothesis if $D_n > s_{n,1-\alpha}$, where $s_{n,1-\alpha}$ is the $1-\alpha$ quantile of the null distribution of D_n when F_0 is the uniform $U(0,1)$ distribution (Smirnov, 1948). The finite sampling distribution of D_n under F_0 is the same for all continuous F_0, but its exact form is difficult to express. By the duality of tests and confidence regions, the Kolmogorov-Smirnov test can be inverted to yield uniform confidence bands for F, given by

$$R_{n,1-\alpha} = \left\{ F : \sup_{\theta^*} \sqrt{n} \left| \hat{F}_n(\theta^*) - F(\theta^*) \right| \le s_{n,1-\alpha} \right\}.$$

By construction, $P_F\{F \in R_{n,1-\alpha}\} = 1 - \alpha$ if F is continuous. So the confidence band is then

$$\max\{0, \hat{F}_n(\theta^*) - s_{n,1-\alpha}\} \le F(\theta^*) \le \min\{1, \hat{F}_n(\theta^*) + s_{n,1-\alpha}\}.$$

4. Bivariate Extension

There have been several versions of the bivariate exponential distributions (Johnson and Kotz, 1972). Among them, we use Gumbel's bivariate exponential distribution (Gumbel, 1960) because of the simple nature of the density function. Similar approach can be applied to the different versions of the bivariate exponential distributions, for example, Freund (1961). Gumbel (1960) presents two types of bivariate exponential density. The first type *pdf* is given as follows:

$$f(x_1,x_2)=e^{-x_1-(1+\delta x_1)x_2}\{(1+\delta x_1)(1+\delta x_2)-\delta\},\quad x_i>0,\ i=1,2,\ 0\le\delta\le 1.$$

The second type *pdf* is the following:

$$f(x_1,x_2)=e^{-x_1-x_2}\{1+\alpha(2e^{-x_1}-1)(2e^{-x_2}-1)\},\quad x_i>0,\ i=1,2,\ -1\le\alpha\le 1. \tag{4.1}$$

The marginal density function is the exponential function at both types, *i.e.* $X_i\sim\exp(1)$, $i=1,2$. We can construct a bivariate LAE distribution in a manner similar to the construction of (univariate) LAE distribution. We shall use the same transformation applied in a bivariate context, *i.e.*, $x_i = r\tan(\theta_i)$, $i=1,2$ and then use $\theta_i^*=4\theta_i/l$, $i=1,2$ and $l=1,2,\ldots$. After simple algebra, the density function of the first type bivariate LAE distribution is defined as following.

Definition 4.1. *The first type bivariate LAE pdf is defined as*

$$\frac{l^2r^2}{16}\left\{\prod_{i=1}^{2}\sec^2\left(\frac{l\theta_i^*}{4}\right)\right\}\exp\left[-r\tan\left(\frac{l\theta_1^*}{4}\right)-\left\{1+\delta r\tan\left(\frac{l\theta_1^*}{4}\right)\right\}r\tan\left(\frac{l\theta_2^*}{4}\right)\right]\times$$
$$\left[\prod_{i=1}^{2}\left\{1+\delta r\tan\left(\frac{l\theta_i^*}{4}\right)\right\}-\delta\right],\quad 0<\theta_i^*<\frac{2\pi}{l},\ i=1,2,\ l=1,2,\ldots,\ \text{and } 0\le\delta\le 1. \tag{4.2}$$

Applying the same approach to the second type *pdf*, (4.1), we get the second type bivariate LAE density as following.

Definition 4.2. *The second type bivariate LAE pdf is given by*

$$\frac{l^2r^2}{16}\left\{\prod_{i=1}^{2}\sec^2\left(\frac{l\theta_i^*}{4}\right)\right\}\exp\left\{-\sum_{i=1}^{2}r\tan\left(\frac{l\theta_i^*}{4}\right)\right\}\left[1+\alpha\prod_{i=1}^{2}\left\{2\exp\left(-r\tan\left(\frac{l\theta_i^*}{4}\right)\right)-1\right\}\right],$$
$$0<\theta_i^*<\frac{2\pi}{l},\ i=1,2,\ l=1,2,\ldots,\ \text{and } -1\le\alpha\le 1. \tag{4.3}$$

Similar to (univariate) LAE distribution, we may also introduce the location parameters, $\mu_i, i=1,2$ to the above two *pdfs*, (4.2) and (4.3). So the densities with the location parameters can be obtained plugging in $\theta_i^*-\mu_i, i=1,2$ instead of $\theta_i^*, i=1,2$ at the above two density functions. Note that $l=2$ gives the bivariate version of SCE distribution and $l=1$ suggests bivariate version of CE distribution.

Note that each marginal density of θ_i^*, $i=1,2$ from the above two *pdfs* follows LAE$(0,\varphi)$, $\varphi=1/r$, after integrating out the other angular random variable,

$$\frac{l}{4\varphi}\sec^2(l\theta_i^*/4)\exp\left(-\frac{\tan(l\theta_i^*/4)}{\varphi}\right),\quad \varphi=\frac{1}{r},\ 0<\theta_i^*<\frac{2\pi}{l}\ \text{and } i=1,2.$$

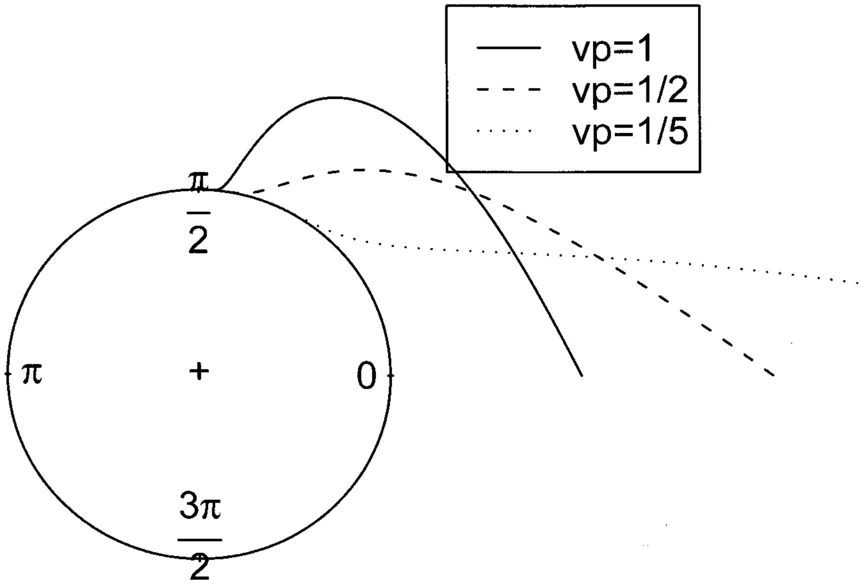

Figure 5.1. Circular plot of the *pdfs* of 4AE(0, φ)

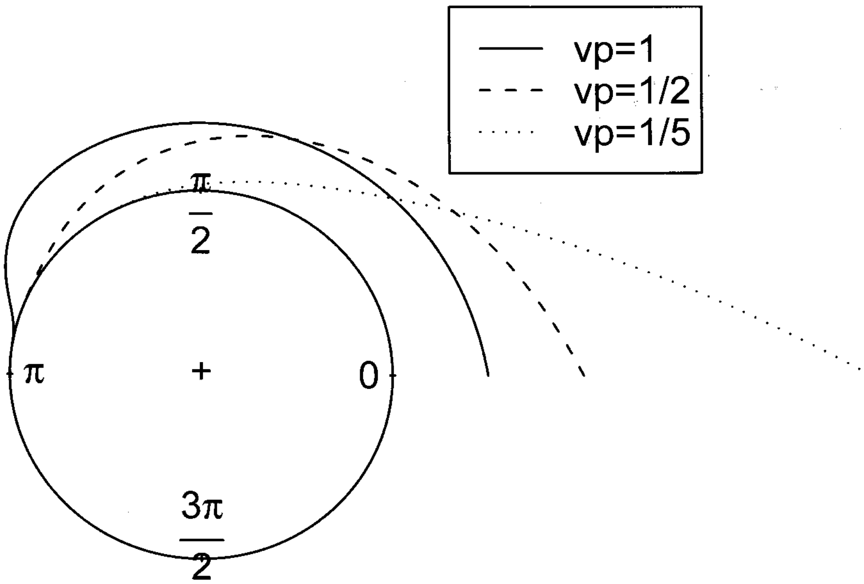

Figure 5.2. Circular plot of the *pdfs* of SCE(0, φ)

5. Conclusion

We derived the 4AE distribution from the exponential distribution via the projection of an exponential distribution over a quarter-circular segment. Then we extended it to the LAE distribution using the transformation, $\theta^* = 4\theta/l,\ l = 1, 2, \ldots$, for modeling any arc of arbitrary length say $2\pi/l$

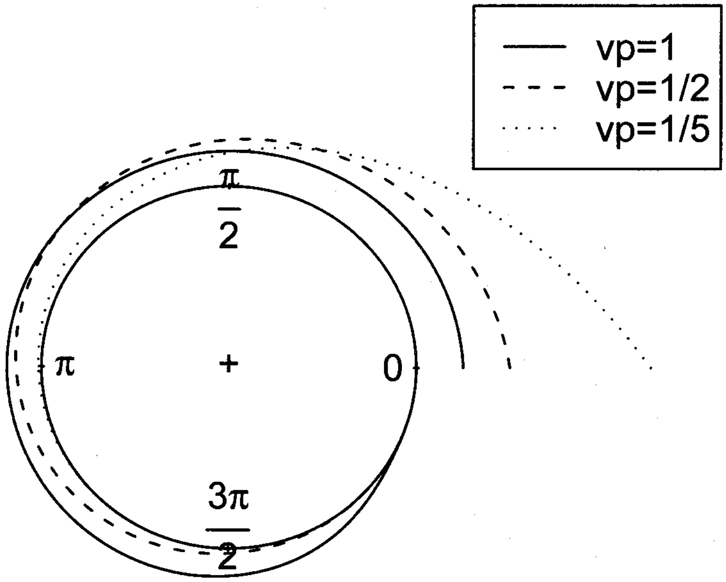

Figure 5.3. Circular plot of the *pdf*s of CE(0, φ)

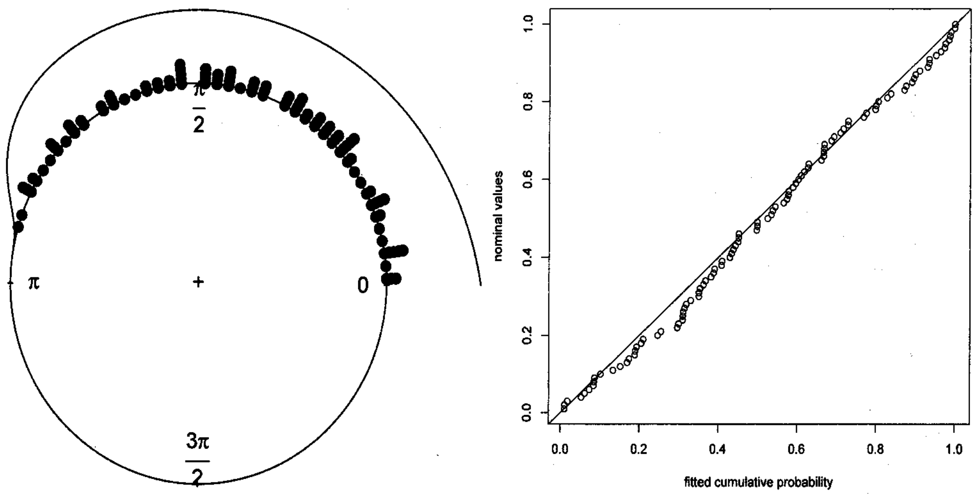

Figure 5.4. Circular data plot with *pdf*s (the solid line represents the original *pdf* and the dashed line corresponds to the fitted *pdf*) and Healy's plot

for $l = 1, 2, \ldots$. Occasionally, measurements result in any arc of arbitrary length. Derived new family of the distributions can be used to model skewed angular data. Asymptotic results reveal that linear exponential distribution can be used to approximate the LAE distribution and the LAE distribution has the approximate memoryless property though it doesn't have strict memoryless property as the exponential distribution.

Trigonometric moments are derived for the 4AE distribution and the SCE distribution. We find that, by the stochastic relationship between the SCL distribution and the 4AE distribution, some cosine moments of the SCL distribution and the SCE distribution are the same. Furthermore, by a simple transformation, some sine moments of the 4AE distribution and the SCE distribution are the same. When a location parameter is known, we derive mle of φ in closed form. Based on exact and asymptotic distributions of φ, we suggest hypotheses tests and confidence intervals. To find estimates of the LAE distribution when both parameters are unknown, we use the direct minimization of minus log likelihood which result in maximum likelihood estimates. To check validity of estimation process, we simulated a data set from the SCE distribution. By the density plots and the Healy's plot, the result is satisfactory. The Kolmogorov-Smirnov test is adopted for goodness of fit test of the l-axial exponential distribution. Finally a bivariate version of two kinds of the LAE distributions is obtained using the bivariate exponential distributions. Each marginal density of $\theta_i^*, i = 1, 2$ from two types of a bivariate LAE distribution follows $\text{LAE}(0, \varphi), \varphi = 1/r$.

References

Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions*, Dover Publications, New York.

Ahn, B. J. and Kim, H. M. (2008). A new family of semicircular models: The semicircular Laplace distributions, *Communications of the Korean Statistical Society*, **15**, 775–781.

Byrd, R. H., Lu, P., Nocedal, J. and Zhu, C. (1995). A limited memory algorithm for bound constrained optimization, *SIAM Journal Scientific Computing*, **16**, 1190–1208.

Fisher, N. I. (1993). *Statistical Analysis of Circular Data*, Cambridge University Press, Cambridge.

Freund, J. E. (1961). A bivariate extension of the exponential distribution, *Journal of the American Statistical Association*, **56**, 971–977.

Gradshteyn, I. S. and Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products*, 7th ed., Academic Press, San Diego.

Guardiola, J. H. (2004). *The Semicircular Normal Distribution*, Baylor University, Institute of Statistics.

Gumbel, E. J. (1960). Bivariate exponential distributions, *Journal of the American Statistical Association*, **55**, 698–707.

Healy, M. J. R. (1968). Multivariate normal plotting, *Applied Statistics*, **17**, 157–161.

Jammalamadaka, S. R. and Kozubowski, T. J. (2003). A new family of circular models: The wrapped Laplace distributions, *Advances and Applications in Statistics*, **3**, 77–103.

Jammalamadaka, S. R. and Kozubowski, T. J. (2004). New families of wrapped distributions for modeling skew circular data, *Communications in Statistics-Theory and Methods*, **33**, 2059–2074.

Jammalamadaka, S. R. and SenGupta, A. (2001). *Topics in Circular Statistics*, World Scientific Publishing, Singapore.

Johnson, N. L. and Kotz, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions*, John Wiley & Sons, New York.

Jones, T. A. (1968). Statistical analysis of orientation data, *Journal of Sedimentary Research*, **38**, 61–67.

Jones, M. C. and Pewsey, A. (2005). A family of symmetric distributions on the circle, *Journal of the American Statistical Association*, **100**, 1422–1428.

Lehmann, E. L. and Casella, G. (1998). *Theory of Point Estimation*, 2nd Ed., Springer-Verlag, New York.

Lehmann, E. L. and Romano, J. P. (2005). *Testing Statistical Hypotheses*, 3rd Ed., Springer-Verlag, New York.

Mardia, K. V. and Jupp, P. E. (2000). *Directional Statistics*, John Wiley & Sons, Chichester.

Pewsey, A. (2000). The wrapped skew-normal distribution on the circle, *Communications in Statistics-Theory and Methods*, **29**, 2459–2472.

Pewsey, A. (2002). Testing circular symmetry, *The Canadian Journal of Statistics*, **30**, 591–600.

Pewsey, A. (2004). Testing for circular reflective symmetry about a known median axis, *Journal of Applied Statistics*, **31**, 575–585.

Pewsey, A. (2006). Modelling asymmetrically distributed circular data using the wrapped skew-normal distribution, *Environmental and Ecological Statistics*, **13**, 257–269.

Pewsey, A. (2008). The wrapped stable family of distributions as a flexible model for circular data, *Computational Statistics & Data Analysis*, **52**, 1516–1523.

Pewsey, A., Lewis, T. and Jones, M. C. (2007). The wrapped t family of circular distributions, *Australian & New Zealand Journal of Statistics*, **49**, 79–91.

Smirnov, N. V. (1948). Tables for estimating the goodness of fit of empirical distributions, *The Annals of Mathematical Statistics*, **19**, 279–281.