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Pricing Outside Floating-Strike Lookback Options

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Abstract

A floating-strike lookback call option gives the holder the right to buy at the lowest price of the underlying asset. Similarly, a floating-strike lookback put option gives the holder the right to sell at the highest price. This paper will propose an outside floating-strike lookback call (or put) option that gives the holder the right to buy (or sell) one underlying asset at some percentage of the lowest (or highest) price of the other underlying asset. In addition, this paper will derive explicit pricing formulas for these outside floating-strike lookback options. Sections 3 and 4 assume that the underlying assets pay no dividends. In contrast, Section 5 will derive explicit pricing formulas for these options when their underlying assets pay dividends continuously at a rate proportional to their prices. Some numerical examples will be discussed.

Keywords: Outside floating-strike, lookback option, Brownian motion.

1. Introduction

Lookback options are path-dependent contingent claims whose payoffs depend on the maximum (or minimum) of the underlying asset price over a certain period. A floating-strike lookback call (or put) option gives the holder the right to buy (or sell) at the lowest (highest) price of the underlying asset. Goldman *et al.* (1979) derived explicit pricing formulas for floating-strike lookback options where the highest (or lowest) price of the underlying asset is attained during the whole life of the options. Conze and Viswanathan (1991) derived explicit pricing formulas for partial floating-strike lookback options that give the holder the right to buy (or sell) at some percentage times the lowest (or highest) price. Heynen and Kat (1994, 1997) suggest a way of reducing the price of these partial floating-strike lookback options while preserving some of their good qualities and derives explicit pricing formulas for the proposed options. Lee (2008) derives explicit pricing formulas for floating-strike lookback options whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity.

However, researches listed above concern lookback options whose payoff depends on one underlying asset. This paper proposes outside floating-strike lookback options whose payoffs depend on prices of two underlying assets: the terminal value of one asset is used for determining the payoff, and the maximum (or minimum) value of the other asset for determining the floating strike. In other words, an outside floating-strike lookback call (or put) option gives the holder the right to buy (or

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sell) one underlying asset at some percentage of the lowest (or highest) price of the other underlying asset. This paper will present explicit pricing formulas for these proposed options.

This paper is organized as follows. Section 2 will discuss some basics for pricing contingent claims and will derive some useful expectations and probabilities for pricing the proposed options. Section 3 and 4 will present explicit pricing formulas for the outside floating-strike lookback put and call options, respectively. In addition, Section 5 will derive explicit pricing formulas for these options when their underlying assets pay dividends continuously at a rate proportional to their prices. These pricing formulas are generalization of the pricing formulas in Sections 3 and 4. Some numerical examples will be discussed.

2. Esscher Transforms and Some Useful Formulas

This section discusses some basics for pricing contingent claims and derives some useful expectations for pricing the proposed options. If we assume the Black-Scholes framework, then according to the fundamental theorem of asset pricing, the prices of contingent claims such as options can be calculated as the discounted expectations of the corresponding payoffs with respect to the equivalent martingale measure.

Gerber and Shiu (1994, 1996) showed that Esscher transforms are an efficient tool for finding the equivalent martingale measure. While the Girsanov theorem used by many researchers provides us with a more general tool for changing the probability measure, the method of Esscher transforms is a more convenient and elegant tool than the Girsanov theorem if the logarithms of the prices of the underlying assets are stochastic processes with stationary and independent increments. This section briefly summarizes a special case of the method of Esscher transforms and demonstrates the factorization formula that is a main feature of this method and that can simplify many calculations.

Let $S_1(t)$ and $S_2(t)$ denote the time-t prices of two underlying assets. Assume that these assets pay no dividends. Assume that for $t \geq 0$, $i = 1$ and 2,

$$S_i(t) = S_i(0)\exp(X_i(t)), \tag{2.1}$$

where $\{\boldsymbol{X}(t) = (X_1(t), X_2(t))'\}$ is a 2-dimensional Brownian motion with drift vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, $X_i(0) = 0$ and diffusion matrix $\boldsymbol{V}$ equal to

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{2.2}$$

Thus the 2-dimensional Brownian motion is a stochastic process with independent and stationary increments and $\boldsymbol{X}(t) = (X_1(t), X_2(t))'$ has a bivariate normal distribution with mean vector $\boldsymbol{\mu}t$ and covariance matrix $\boldsymbol{V}t$.

For a nonzero real vector $\boldsymbol{h} = (h_1, h_2)'$, the moment generating function of $\boldsymbol{X}(t)$, $E(e^{\boldsymbol{h}'\boldsymbol{X}(t)})$, exists for all $t \geq 0$, because $\{\boldsymbol{X}(t)\}$ is the Brownian motion as described above. The stochastic process

$$\left\{ e^{\boldsymbol{h}'\boldsymbol{X}(t)} E\left(e^{\boldsymbol{h}'\boldsymbol{X}(1)}\right)^{-t} \right\}$$

is a positive martingale which can be used to define a new probability measure Q. In technical terms, this process is used to define the Radon-Nikodym derivative dQ/dP, where P is the original probability measure. We call Q the Esscher measure of parameter vector $\boldsymbol{h}$.

For a random variable Y that is a real-valued function of $\{\boldsymbol{X}(t), 0 \leq t \leq T\}$, the expectation of Y under the new probability measure Q is calculated as

$$E\left[Y\frac{e^{\boldsymbol{h}'\boldsymbol{X}(T)}}{E\left(e^{\boldsymbol{h}'\boldsymbol{X}(1)}\right)^{T}}\right], \tag{2.3}$$

which will be denoted by $E[Y;\boldsymbol{h}]$. The risk-neutral measure is the Esscher measure of parameter vector $\boldsymbol{h} = \boldsymbol{h}^*$ with respect to which the process $\{e^{-rt}S_i(t)\}$ is a martingale. Here, r is a risk-free rate. Thus

$$E\left[e^{-rt}S_i(t);\boldsymbol{h}^*\right] = S_i(0). \tag{2.4}$$

Therefore, $\boldsymbol{h}^*$ is the solution of

$$\boldsymbol{\mu} + \boldsymbol{V}\boldsymbol{h}^* = \left(r - \frac{\sigma_1^2}{2},\ r - \frac{\sigma_2^2}{2}\right)'. \tag{2.5}$$

For $t \geq 0$, the moment generating function of $\boldsymbol{X}(t)$ under Esscher measure of parameter vector $\boldsymbol{h}$ is

$$E\left(e^{z'X(t)};\boldsymbol{h}\right) = \exp\left[(\boldsymbol{\mu}' + \boldsymbol{h}'\boldsymbol{V})\boldsymbol{z}t + \boldsymbol{z}'\boldsymbol{V}\boldsymbol{z}\frac{t}{2}\right] \tag{2.6}$$

which implies that $\boldsymbol{X}(t)$ has a bivariate normal distribution with mean vector $(\boldsymbol{\mu}+\boldsymbol{V}\boldsymbol{h})t$ and variance $\boldsymbol{V}t$ under the Esscher measure. It can be shown that the process $\{\boldsymbol{X}(t)\}$ under the Esscher measure has independent and stationary increments. Thus, this process is a two-dimensional Brownian motion with drift vector

$$\boldsymbol{\mu} + \boldsymbol{V}\boldsymbol{h} = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} + \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\begin{pmatrix}h_1\\ h_2\end{pmatrix} = \begin{pmatrix}\mu_1 + \sigma_1^2 h_1 + \rho\sigma_1\sigma_2 h_2\\ \mu_2 + \rho\sigma_1\sigma_2 h_1 + \sigma_2^2 h_2\end{pmatrix} \tag{2.7}$$

and diffusion matrix $\boldsymbol{V}$ under the Esscher measure of parameter vector $\boldsymbol{h}$.

Let us consider a special case of the factorization formula (Gerber and Shiu, 1994, 1996). For a random variable Y that is a real-valued function of $\{\boldsymbol{X}(t),\ 0 \leq t \leq T\}$,

$$E\left[e^{g'X(T)}Y;\boldsymbol{h}\right] = E\left[e^{g'X(T)};\boldsymbol{h}\right]E\left[Y;\boldsymbol{h}+\boldsymbol{g}\right]. \tag{2.8}$$

In particular, for an event B whose condition is determined by, $\{\boldsymbol{X}(t), 0 \leq t \leq T\}$ formula (2.8) can be expressed as follows:

$$E\left[e^{g'X(T)}I(B);\boldsymbol{h}\right] = E\left[e^{g'X(T)};\boldsymbol{h}\right]\Pr\left[B;\boldsymbol{h}+\boldsymbol{g}\right], \tag{2.9}$$

where $I(\cdot)$ denotes the indicator function $\Pr(B;\boldsymbol{h})$ and denotes the probability of the event B under the Esscher measure of parameter vector $\boldsymbol{h}$.

Now, let

$$M_2(T) = \max\{X_2(\tau),\ 0 \leq \tau \leq T\} \tag{2.10}$$

and

$$m_2(T) = \min\{X_2(\tau),\ 0 \leq \tau \leq T\} \tag{2.11}$$

for $T > 0$. In Lee (2004), it can be shown that the joint distribution function of $M_2(T)$ and $X_1(T)$ is

$$\begin{aligned}
&\Pr\left[X_1(T) \le x, M_2(T) \le m\right] \\
&= \Phi_2\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{m-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) - e^{\frac{2\mu_2}{\sigma_2^2}m}\Phi_2\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}} - \frac{2\rho m}{\sigma_2\sqrt{T}}, \frac{-m-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right), \qquad (2.12)
\end{aligned}$$

where $\Phi_2(a, b; \rho)$ denotes the bivariate standard normal distribution function with correlation coefficient ρ. This probability distribution function is related to two random variables, $S_1(T)$ and $\max\{S_2(\tau),\, 0 \le \tau \le T\}$. Hence it will be used for calculating (2.13).

Next, consider some useful expectations for pricing the proposed options. Assume that $\xi = 2\mu_2/\sigma_2^2$, $\eta = 1 - 2\rho\sigma_1/\sigma_2$ and $c + \xi \ne 0$. The proof of (2.13) will be given in the Appendix.

$$\begin{aligned}
&E\left[e^{c\cdot M_2(T)} I(M_2(T) > X_1(T) + k)\right] \\
&= e^{c\mu_2 T + \frac{1}{2}c^2\sigma_2^2 T}\Phi_2\left[\frac{-k-(\mu_1 + c\rho\sigma_1\sigma_2 - \mu_2 - c\sigma_2^2)T}{\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T}}, \frac{(\mu_2 + c\sigma_2^2)T}{\sigma_2\sqrt{T}}; \frac{-\rho\sigma_1\sigma_2 + \sigma_2^2}{\sqrt{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)\sigma_2^2}}\right] \\
&\quad + \frac{\xi}{c+\xi}\Phi_2\left(\frac{-k-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) \\
&\quad + \left(\frac{\xi}{c+\xi}\frac{\eta}{|\eta|} + 2\rho\frac{\sigma_1}{\sigma_2}\frac{1}{|\eta|}\right) e^{\frac{c+\xi}{\eta}k} e^{\frac{c+\xi}{\eta}\mu_1 T + \frac{1}{2}\left(\frac{c+\xi}{\eta}\right)^2\sigma_1^2 T} \\
&\quad \times \Phi_2\left[\frac{k/\eta + \{(\mu_1 + \frac{c+\xi}{\eta}\sigma_1^2)/\eta\}T}{\sqrt{(\sigma_1^2/\eta^2)T}}, \frac{-k/\eta - \{(\mu_1 + \frac{c+\xi}{\eta}\sigma_1^2)/\eta + \mu_2 + \frac{c+\xi}{\eta}\rho\sigma_1\sigma_2\}T}{\sqrt{(\sigma_1^2/\eta^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2/\eta)T}}\right. \\
&\qquad \left.; \frac{-\sigma_1^2/\eta^2 - \rho\sigma_1\sigma_2/\eta}{\sqrt{(\sigma_1^2/\eta^2)(\sigma_1^2/\eta^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2/\eta)}}\right] \\
&\quad + \frac{c}{c+\xi}e^{-(c+\xi)\mu_2 T + \frac{1}{2}(c+\xi)^2\sigma_2^2 T} \\
&\quad \times \Phi_2\left[\frac{-k-\{\mu_1 - (c+\xi)\rho\sigma_1\sigma_2 + \eta\mu_2 - \eta(c+\xi)\sigma_2^2\}T}{\sqrt{(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2)T}}, \frac{-\{\mu_2 - (c+\xi)\sigma_2^2\}T}{\sigma_2\sqrt{T}}\right. \\
&\qquad \left.; \frac{\rho\sigma_1\sigma_2 + \eta\sigma_2^2}{\sqrt{(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2)\sigma_2^2}}\right] \\
&=: D\left(\begin{pmatrix} c \\ k \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho\right). \qquad (2.13)
\end{aligned}$$

Formula (2.13) will be useful for pricing the outside floating-strike put option. In addition, applying (2.13), the next expectation (2.14) will be easily obtained and it will be useful for pricing the outside floating-strike call option.

$$\begin{aligned}
&E\left[e^{c\cdot m_2(T)} I(m_2(T) < X_1(T) + k)\right] \\
&= E\left[e^{-c\cdot(-m_2(T))} I(-m_2(T) > -X_1(T) + (-k))\right] \\
&= E\left[e^{-c\cdot \mathrm{Max}\{-X_2(\tau),\, 0\le\tau\le T\}}\; I(\mathrm{Max}\{-X_2(\tau),\, 0 \le \tau \le T\} > -X_1(T) + (-k))\right] \\
&= D\left(-\begin{pmatrix} c \\ k \end{pmatrix}, -\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho\right). \qquad (2.14)
\end{aligned}$$

Note that the stochastic process $\{(-X_1(t), -X_2(t))'\}$ is a 2-dimensional Brownian motion with drift vector $(-\mu_1, -\mu_2)'$ and diffusion matrix $\boldsymbol{V}$.

Finally, discuss some useful probabilities for pricing the proposed options. Applying (2.13) with $c = 0$, we have

$$\begin{aligned}\Pr\left[M_2(T) > X_1(T) + k\right] &= E\left[I\left(M_2(T) > X_1(T) + k\right)\right] \\ &= E\left[e^{0\cdot M_2(T)} I\left(M_2(T) > X_1(T) + k\right)\right] \\ &= D\left(\begin{pmatrix}0\\k\end{pmatrix}, \begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1\\\sigma_2\end{pmatrix}, \rho\right).\end{aligned} \tag{2.15}$$

Similarly, applying (2.14) with $c = 0$, we have

$$\begin{aligned}\Pr[m_2(T) < X_1(T) + k] &= E\left[I(m_2(T) < X_1(T) + k)\right] \\ &= E[e^{0\cdot m_2(T)} I(m_2(T) < X_1(T) + k)] \\ &= D\left(-\begin{pmatrix}0\\k\end{pmatrix}, -\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1\\\sigma_2\end{pmatrix}, \rho\right).\end{aligned} \tag{2.16}$$

3. Outside Floating-Strike Lookback Put Option

The proposed outside floating-strike lookback put option gives the holder the right to sell one underlying asset at some percentage of the highest price of the other underlying asset attained during the whole life of the option. This section will derive an explicit pricing formula for the outside floating-strike lookback put option.

Let us take a close look at the payoff of the outside floating-strike lookback put option. Assume that $\lambda\,(> 0)$ is the percentage over the highest price. The payoff of this option can be is written as follows:

$$\left(\lambda \cdot \max(S_2(\tau),\ 0 \le \tau \le T) - S_1(T)\right)_+ . \tag{3.1}$$

To simplify writing, we define all expectations in this and next sections as taken with respect to the risk-neutral measure. In other words, under this measure, the underlying stochastic processes $\{X_i(\tau), \tau \ge 0\}$ is a Brownian motion with drift vector $(r - \sigma_1^2/2, r - \sigma_2^2/2)'$ and diffusion matrix $\boldsymbol{V}$. By the fundamental theorem of asset pricing, the time-0 value of the payoff (3.1) is

$$e^{-rT} E\left[\left(\lambda \cdot S_2(0) e^{M_2(T)} - S_1(0) e^{X_1(T)}\right)_+\right]. \tag{3.2}$$

Calculating this discounted expectation (3.2) seems to require much complicated and tedious integration, but formulas (2.13) and (2.15) can simplify and reduce many calculations.

Therefore, the time-0 value of the outside floating-strike lookback put option can be rewritten and decomposed into the sum of two expectations,

$$\begin{aligned}&e^{-rT} E\left[\left(\lambda \cdot S_2(0) e^{M_2(T)} - S_1(0) e^{X_1(T)}\right) I\left(M_2(T) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right] \\ &= \lambda e^{-rT} S_2(0) E\left[e^{M_2(T)} I\left(M_2(T) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right] \\ &\quad - e^{-rT} S_1(0) E\left[e^{X_1(T)} I\left(M_2(T) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right].\end{aligned} \tag{3.3}$$

Applying (2.13), the first expectation in the RHS(right hand side) of (3.3) can be

$$D\left(\begin{pmatrix} 1 \\ \ln\left(\dfrac{S_1(0)}{\lambda\cdot S_2(0)}\right)\end{pmatrix}, \begin{pmatrix} r-\frac{1}{2}\sigma_1^2 \\ r-\frac{1}{2}\sigma_2^2\end{pmatrix}, \begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}, \rho\right). \tag{3.4}$$

Applying the factorization formula (2.9), (2.15) and the fact that $\{e^{-rT}S_i(t)\}$ is a martingale under the risk-neutral measure, the second term in the RHS of (3.3) will be

$$\begin{aligned}
&e^{-rT}S_1(0)E\left[e^{X_1(T)}I\left(M_2(T) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda\cdot S_2(0)}\right)\right)\right] \\
&= e^{-rT}S_1(0)E\left[e^{X_1(T)}\right]\Pr\left[M_2(T) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda\cdot S_2(0)}\right); (1,0)'\right] \\
&= S_1(0)D\left(\begin{pmatrix} 0 \\ \ln\left(\dfrac{S_1(0)}{\lambda\cdot S_2(0)}\right)\end{pmatrix}, \begin{pmatrix} r+\frac{1}{2}\sigma_1^2 \\ r-\frac{1}{2}\sigma_1^2+\rho\sigma_1\sigma_2\end{pmatrix}, \begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}, \rho\right).
\end{aligned} \tag{3.5}$$

Note that the drift vector is shifted because of

$$\begin{pmatrix} r-\frac{1}{2}\sigma_1^2 \\ r-\frac{1}{2}\sigma_2^2\end{pmatrix} + \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} r+\frac{1}{2}\sigma_1^2 \\ r-\frac{1}{2}\sigma_2^2+\rho\sigma_1\sigma_2\end{pmatrix}. \tag{3.6}$$

Hence, placing (3.4) and (3.5) into (3.3), we have the time-0 value of the outside floating-strike put option

$$\begin{aligned}
&\lambda e^{-rT}S_2(0)D\left(\begin{pmatrix} 1 \\ \ln\left(\dfrac{S_1(0)}{\lambda\cdot S_2(0)}\right)\end{pmatrix}, \begin{pmatrix} r-\frac{1}{2}\sigma_1^2 \\ r-\frac{1}{2}\sigma_2^2\end{pmatrix}, \begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}, \rho\right) \\
&- S_1(0)D\left(\begin{pmatrix} 0 \\ \ln\left(\dfrac{S_1(0)}{\lambda\cdot S_2(0)}\right)\end{pmatrix}, \begin{pmatrix} r+\frac{1}{2}\sigma_1^2 \\ r-\frac{1}{2}\sigma_2^2+\rho\sigma_1\sigma_2\end{pmatrix}, \begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}, \rho\right).
\end{aligned} \tag{3.7}$$

For numerical results of pricing formula (3.7), see Table 3.1. It is observed that T and σ_1/σ_2 increase formula (3.7), but r and ρ decrease it.

4. Outside Floating-Strike Lookback Call Option

The proposed outside floating-strike lookback call option gives the holder the right to buy one underlying asset at some percentage of the lowest price of the other underlying asset attained during the whole life of the option. This section will derive an explicit pricing formula for the outside floating-strike lookback call option.

Let us take a close look at the payoff of the outside floating-strike lookback call option. Assume that $\lambda\,(>0)$ is the percentage over the lowest price. The payoff of this option is

$$(S_1(T) - \lambda\cdot\min(S_2(\tau),\, 0\le\tau\le T))_+\,. \tag{4.1}$$

By the fundamental theorem of asset pricing, the time-0 value of the payoff is

$$e^{-rT}E\left[\left(S_1(0)e^{X_1(T)} - \lambda\cdot S_2(0)e^{m_2(T)}\right)_+\right]. \tag{4.2}$$

Table 3.1. Put option prices ($S_1(0) = S_2(0) = 100, \lambda = 1, \sigma_2 = 0.2$)

			ρ				
T	σ_1/σ_2	r	−0.8	−0.4	0	0.4	0.8
0.25	0.5	0.04	8.843	8.589	8.324	8.048	7.786
		0.06	8.701	8.426	8.139	7.840	7.549
		0.08	8.570	8.273	7.964	7.641	7.323
	1.0	0.04	10.527	10.092	9.603	9.026	8.277
		0.06	10.404	9.947	9.435	8.833	8.054
		0.08	10.289	9.809	9.275	8.649	7.838
	1.5	0.04	12.383	11.848	11.236	10.503	9.525
		0.06	12.268	11.710	11.076	10.319	9.312
		0.08	12.160	11.579	10.923	10.143	9.107
0.50	0.5	0.04	12.481	12.095	11.691	11.268	10.856
		0.06	12.207	11.778	11.331	10.860	10.391
		0.08	11.960	11.491	11.001	10.482	9.954
	1.0	0.04	14.912	14.272	13.549	12.695	11.580
		0.06	14.670	13.985	13.217	12.314	11.136
		0.08	14.449	13.719	12.908	11.957	10.716
	1.5	0.04	17.571	16.795	15.900	14.824	13.381
		0.06	17.341	16.518	15.580	14.457	12.956
		0.08	17.129	16.261	15.280	14.111	12.553

Calculating this discounted expectation (4.2) seems to require much complicated and tedious integration, but formula (2.14) and (2.16) can simplify and reduce many calculations.

Therefore, the time-0 value of the outside floating-strike lookback call option can be rewritten and decomposed into the sum of two expectations,

$$\begin{aligned}
&e^{-rT}E\left[\left(S_1(0)e^{X_1(T)} - \lambda \cdot S_2(0)e^{m_2(T)}\right) I\left(m_2(T) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right] \\
&= e^{-rT}S_1(0)E\left[e^{X_1(T)} I\left(m_2(T) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right] \\
&- \lambda e^{-rT}S_2(0)E\left[e^{m_2(T)} I\left(m_2(T) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right]. \qquad (4.3)
\end{aligned}$$

Applying (2.14), we have the second expectation in the RHS of (4.3)

$$D\left(-\begin{pmatrix} 1 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, -\begin{pmatrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho\right). \qquad (4.4)$$

In addition, applying the factorization formula (2.9), (2.16) and the fact that $\{e^{-rT}S_i(t)\}$ is a martingale under the risk-neutral measure, the first term in the RHS of (4.3) will be

$$\begin{aligned}
&e^{-rT}S_1(0)E\left[e^{X_1(T)} I\left(m_2(T) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right)\right] \\
&= e^{-rT}S_1(0)E\left[e^{X_1(T)}\right] \Pr\left[m_2(T) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right); (1,0)'\right] \\
&= S_1(0)D\left(-\begin{pmatrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, -\begin{pmatrix} r + \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho\right). \qquad (4.5)
\end{aligned}$$

Table 4.1. Call option prices ($S_1(0) = S_2(0) = 100, \lambda = 1, \sigma_2 = 0.2$)

T	σ_1/σ_2	r	ρ				
			−0.8	−0.4	0	0.4	0.8
0.25	0.5	0.04	8.890	8.725	8.553	8.381	8.234
		0.06	9.053	8.907	8.754	8.601	8.473
		0.08	9.227	9.100	8.964	8.828	8.717
	1.0	0.04	10.350	10.007	9.624	9.180	8.617
		0.06	10.476	10.155	9.794	9.373	8.839
		0.08	10.609	10.311	9.971	9.572	9.066
	1.5	0.04	12.030	11.579	11.074	10.483	9.713
		0.06	12.138	11.710	11.227	10.658	9.917
		0.08	12.252	11.847	11.385	10.839	10.126
0.50	0.5	0.04	12.573	12.363	12.145	11.928	11.747
		0.06	12.905	12.732	12.548	12.366	12.219
		0.08	13.267	13.127	12.975	12.824	12.706
	1.0	0.04	14.558	14.101	13.590	13.000	12.256
		0.06	14.810	14.396	13.928	13.382	12.693
		0.08	15.083	14.711	14.283	13.781	13.145
	1.5	0.04	16.867	16.256	15.576	14.783	13.755
		0.06	17.078	16.513	15.876	15.127	14.155
		0.08	17.307	16.786	16.190	15.486	14.568

Hence, placing (4.4) and (4.5) into (4.3), we have the time-0 value of the outside floating-strike call option,

$$\begin{aligned} &S_1(0)D\left(-\begin{pmatrix} 0 \\ \ln\left(\dfrac{S_1(0)}{\lambda\cdot S_2(0)}\right)\end{pmatrix}, -\begin{pmatrix} r+\frac{1}{2}\sigma_1^2 \\ r-\frac{1}{2}\sigma_2^2+\rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix}\sigma_1 \\ \sigma_2\end{pmatrix}, \rho\right) \\ &-\lambda e^{-rT}S_2(0)D\left(-\begin{pmatrix} 1 \\ \ln\left(\dfrac{S_1(0)}{\lambda\cdot S_2(0)}\right)\end{pmatrix}, -\begin{pmatrix} r-\frac{1}{2}\sigma_1^2 \\ r-\frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix}\sigma_1 \\ \sigma_2\end{pmatrix}, \rho\right). \end{aligned} \tag{4.6}$$

For numerical results of pricing formula (4.6), see Table 4.1. This table shows that r, T and σ_1/σ_2 increase formula (4.6), but ρ decreases it. On the other hand, risk-free rate r decreases formula (3.7).

5. Continuous Constant-Yield Dividend

The previous sections have derived the explicit pricing formulas for the outside floating-strike lookback options whose underlying assets pay no dividends. The pricing formulas in Sections 3 and 4 can be extended to the case where the underlying assets pay dividends continuously at a rate proportional to their prices. This section will derive explicit pricing formulas for this case.

Let $S_i(t)$ denote the time-t price of two underlying assets for $i = 1, 2$ respectively. Assume that δ_i is the constant nonnegative dividend yield rate such that the assets pay dividends $\delta_i S_i(t)dt$ between time t and time $t + dt$. If all dividends of asset i are reinvested in the asset, each share of the asset at time 0 grows to $e^{\delta_i t}$ shares at time t. We assume that the prices of these assets follow the model (2.1). In other words, if an investor buys one share of asset i at $S_i(0)$ and reinvests all dividends in the asset, his fund value invested in asset i will be

$$e^{\delta_i t} S_i(t) = e^{\delta_i t} S_i(0) \exp(X_i(t)) \tag{5.1}$$

at time t. The risk-neutral measure is the Esscher measure of parameter vector $\boldsymbol{h} = \boldsymbol{h}^{**}$ with respect to which the process $\{e^{-(r-\delta_i)t} S_i(t)\}$ is a martingale. Therefore, $\boldsymbol{h}^{**}$ is the solution of

$$\boldsymbol{\mu} + Vh^{**} = \left(r - \delta_1 - \frac{\sigma_1^2}{2},\ r - \delta_2 - \frac{\sigma_2^2}{2}\right)'. \tag{5.2}$$

Note that the process $\{X(t)\}$ is a Brownian motion with drift vector $\boldsymbol{\mu} + Vh^{**}$ and diffusion matrix V under the risk-neutral measure. For further discussion, see Section 9 of Gerber and Shiu (1996). By the fundamental theorem of asset pricing, the time-0 values of the payoffs (3.1) and (4.1) are

$$\begin{aligned} & e^{-rT} E\left[\left(\lambda \cdot S_2(0) e^{M_2(T)} - S_1(0) e^{X_1(T)}\right)_+ ; h^{**}\right] \\ &= \lambda e^{-rT} S_2(0) E\left[e^{M_2(T)} I\left(M_2(T) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right); h^{**}\right] \\ &\quad - e^{-\delta_1 T} e^{-(r-\delta_1)T} S_1(0) E\left[e^{X_1(T)} I\left(M_2(T) > X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right); h^{**}\right] \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} & e^{-rT} E\left[\left(S_1(0) e^{X_1(T)} - \lambda \cdot S_2(0) e^{m_2(T)}\right)_+ ; h^{**}\right] \\ &= e^{-\delta_1 T} e^{-(r-\delta_1)T} S_1(0) E\left[e^{X_1(T)} I\left(m_2(T) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right); h^{**}\right] \\ &\quad - \lambda e^{-rT} S_2(0) E\left[e^{m_2(T)} I\left(m_2(T) < X_1(T) + \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right)\right); h^{**}\right] \end{aligned} \tag{5.4}$$

respectively, of which two expectations are the same as ones of (3.3) and (4.3) except that the underlying stochastic process is a Brownian motion with drift vector $\boldsymbol{\mu} + Vh^{**}$, diffusion matrix V and $e^{-(r-\delta_1)T} S_1(0) E[e^{X_1(T)}; h^{**}] = S_1(0)$. Therefore, the time-0 values of the outside floating-strike lookback put and call options are

$$\begin{aligned} & \lambda e^{-rT} S_2(0) D\left(\begin{pmatrix} 1 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, \begin{pmatrix} r - \delta_1 - \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho\right) \\ & - e^{-\delta_1 T} S_1(0) D\left(\begin{pmatrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, \begin{pmatrix} r - \delta_1 + \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho\right) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} & e^{-\delta_1 T} S_1(0) D\left(-\begin{pmatrix} 0 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, -\begin{pmatrix} r - \delta_1 + \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho\right) \\ & - \lambda e^{-rT} S_2(0) D\left(-\begin{pmatrix} 1 \\ \ln\left(\frac{S_1(0)}{\lambda \cdot S_2(0)}\right) \end{pmatrix}, -\begin{pmatrix} r - \delta_1 - \frac{1}{2}\sigma_1^2 \\ r - \delta_2 - \frac{1}{2}\sigma_2^2 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \rho\right) \end{aligned} \tag{5.6}$$

Table 5.1. Put option prices (dividend) ($S_1(0) = S_2(0) = 100, \lambda = 1, \sigma_1 = \sigma_2 = 0.2, T = 0.5, r = 0.06$)

δ_1	δ_2	ρ = −0.8	−0.4	0	0.4	0.8
0.00	0.00	14.670	13.985	13.217	12.314	11.136
0.01		14.976	14.309	13.561	12.683	11.547
0.02		15.286	14.637	13.909	13.055	11.962
0.03		15.600	14.969	14.261	13.432	12.381
0.00	0.01	14.409	13.733	12.972	12.073	10.893
0.01		14.714	14.055	13.314	12.439	11.299
0.02		15.022	14.381	13.659	12.809	11.709
0.03		15.334	14.711	14.008	13.182	12.124
0.00	0.02	14.154	13.486	12.732	11.839	10.657
0.01		14.457	13.807	13.072	12.201	11.059
0.02		14.763	14.130	13.415	12.568	11.465
0.03		15.073	14.458	13.761	12.939	11.876
0.00	0.03	13.905	13.245	12.499	11.611	10.430
0.01		14.206	13.563	12.835	11.971	10.827
0.02		14.510	13.885	13.176	12.334	11.229
0.03		14.818	14.210	13.520	12.702	11.636

Table 5.2. Call option prices (dividend) ($S_1(0) = S_2(0) = 100, \lambda = 1, \sigma_1 = \sigma_2 = 0.2, T = 0.5, r = 0.06$)

δ_1	δ_2	ρ = −0.8	−0.4	0	0.4	0.8
0.00	0.00	14.810	14.396	13.928	13.382	12.693
0.01		14.435	14.008	13.526	12.963	12.247
0.02		14.067	13.627	13.130	12.551	11.809
0.03		13.704	13.251	12.741	12.146	11.378
0.00	0.01	14.984	14.565	14.092	13.544	12.858
0.01		14.608	14.175	13.688	13.123	12.410
0.02		14.238	13.792	13.291	12.709	11.969
0.03		13.875	13.415	12.900	12.302	11.535
0.00	0.02	15.161	14.736	14.259	13.709	13.028
0.01		14.784	14.345	13.854	13.287	12.577
0.02		14.413	13.961	13.455	12.871	12.134
0.03		14.048	13.582	13.063	12.461	11.697
0.00	0.03	15.341	14.911	14.430	13.879	13.203
0.01		14.963	14.518	14.023	13.454	12.750
0.02		14.591	14.132	13.623	13.036	12.303
0.03		14.225	13.753	13.229	12.625	11.864

respectively.

Finally, let us discuss numerical results of (5.5) and (5.6). For numerical results of put pricing formula (5.5), see Table 5.1. This table shows that δ_1 increases formula (5.5), but δ_2 and ρ decrease it. In addition, for numerical results of call pricing formula (5.6), see Table 5.2. This table shows that δ_1 and ρ decrease formula (5.6), but δ_2 increases it.

6. Conclusion

This paper has derived explicit pricing formulas for the proposed outside floating-strike lookback options and discussed numerical results of the pricing formulas under either non-dividend assumption or continuous dividend assumption. More realistic assumptions in pricing outside floating-strike lookback options should be introduced in future research: stochastic interest rates, flexible monitoring periods and transaction costs.

Appendix

Proof of (2.13)

First, let us discuss the joint probability distribution function of random variables $M_2(T)$ and $X_1(T)$,

$$\begin{aligned}&\Pr\left(X_1(T)\le x, M_2(T)\le m\right)\\&=\Phi_2\left(\frac{x-\mu_1T}{\sigma_1\sqrt{T}},\frac{m-\mu_2T}{\sigma_2\sqrt{T}};\rho\right)-e^{\frac{2\mu_2}{\sigma_2^2}m}\Phi_2\left(\frac{x-\mu_1T}{\sigma_1\sqrt{T}}-\frac{2\rho m}{\sigma_2\sqrt{T}},\frac{-m-\mu_2T}{\sigma_2\sqrt{T}};\rho\right).\end{aligned}\tag{2.12}$$

Applying the fact that $(X_1(T), X_2(T))$ follows a bivariate normal distribution, the two standard bivariate normal distribution functions of (2.12) can be expressed as follows:

$$\Phi_2\left(\frac{x-\mu_1T}{\sigma_1\sqrt{T}},\frac{m-\mu_2T}{\sigma_2\sqrt{T}};\rho\right)=\Pr\left(X_1(T)\le x,\ X_2(T)\le m\right)\tag{A.1}$$

and

$$\Phi_2\left(\frac{x-\mu_1T}{\sigma_1\sqrt{T}}-\frac{2\rho m}{\sigma_2\sqrt{T}},\frac{-m-\mu_2T}{\sigma_2\sqrt{T}};\rho\right)=\Pr\left(X_1(T)\le x-2\rho\frac{\sigma_1}{\sigma_2}m,\ X_2(T)\le -m\right).\tag{A.2}$$

Hence, placing (A.1) and (A.2) into (2.12), we have

$$\begin{aligned}&\Pr\left(X_1(T)\le x,\ M_2(T)\le m\right)\\&=\Pr(X_1(T)\le x,\ X_2(T)\le m)-e^{\frac{2\mu_2}{\sigma_2^2}m}\Pr\left(X_1(T)\le x-2\rho\frac{\sigma_1}{\sigma_2}m,\ X_2(T)\le -m\right).\end{aligned}\tag{A.3}$$

Next, let us derive two double integral formulas used many times for the proof of (2.13). Applying the factorization formula (2.9), one double integral can be expressed as follows:

$$\begin{aligned}&\iint_{\substack{a\cdot x+b\cdot y<e\\ c\cdot x+d\cdot y<f}} e^{h_1\cdot x}\frac{\partial^2}{\partial y\partial x}\Pr(X_1(T)\le x, X_2(T)\le y)dxdy\\&=E\left[e^{h_1\cdot X_1(T)}I(a\cdot X_1(T)+b\cdot X_2(T)<e,\ c\cdot X_1(T)+d\cdot X_2(T)<f)\right]\\&=E\left[e^{h_1\cdot X_1(T)}\right]\Pr\left(a\cdot X_1(T)+b\cdot X_2(T)<e,\ c\cdot X_1(T)+d\cdot X_2(T)<f;(h_1,0)'\right)\\&=e^{h_1\mu_1T+\frac{1}{2}h_1^2\sigma_1^2T}\Phi_2\left(\frac{e-[a(\mu_1+h_1\sigma_1^2)+b(\mu_2+h_1\rho\sigma_1\sigma_2)]T}{\sqrt{(a^2\sigma_1^2+b^2\sigma_2^2+2ab\rho\sigma_1\sigma_2)T}},\right.\\&\qquad\left.\frac{f-[c(\mu_1+h_1\sigma_1^2)+d(\mu_2+h_1\rho\sigma_1\sigma_2)]T}{\sqrt{(c^2\sigma_1^2+d^2\sigma_2^2+2cd\rho\sigma_1\sigma_2)T}};\rho*\right),\end{aligned}\tag{A.4}$$

where $\rho*=\left\{ac\sigma_1^2+(ad+bc)\rho\sigma_1\sigma_2+bd\sigma_2^2\right\}/\left\{\sqrt{(a^2\sigma_1^2+b^2\sigma_2^2+2ab\rho\sigma_1\sigma_2)(c^2\sigma_1^2+d^2\sigma_2^2+2cd\rho\sigma_1\sigma_2)}\right\}$. Similarly, the other double integral can be calculated as follows:

$$\iint_{\substack{a\cdot x+b\cdot y<e\\ c\cdot x+d\cdot y<f}} e^{h_2\cdot y}\frac{\partial^2}{\partial y\partial x}\Pr(X_1(T)\le x, X_2(T)\le y)dxdy$$

$$= E\left[e^{h_2 \cdot X_2(T)} I(a \cdot X_1(T) + b \cdot X_2(T) < e,\ c \cdot X_1(T) + d \cdot X_2(T) < f)\right]$$
$$= E\left[e^{h_2 \cdot X_2(T)}\right] \Pr\left(a \cdot X_1(T) + b \cdot X_2(T) < e,\ c \cdot X_1(T) + d \cdot X_2(T) < f; (0, h_2)'\right)$$
$$= e^{h_2\mu_2 T + \frac{1}{2}h_2^2\sigma_2^2 T} \Phi_2\left(\frac{e - [a(\mu_1 + h_2\rho\sigma_1\sigma_2) + b(\mu_2 + h_2\sigma_2^2)]T}{\sqrt{(a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)T}},\right.$$
$$\left.\frac{f - [c(\mu_1 + h_2\rho\sigma_1\sigma_2) + d(\mu_2 + h_2\sigma_2^2)]T}{\sqrt{(c^2\sigma_1^2 + d^2\sigma_2^2 + 2cd\rho\sigma_1\sigma_2)T}}; \rho *\right). \tag{A.5}$$

Now, let us derive (2.13). The expectation of (2.13) can be expressed as

$$E\left[e^{c \cdot M_2(T)} I(M_2(T) > X_1(T) + k)\right] = \iint_{\substack{m > x+k \\ m > 0}} e^{c \cdot m} \frac{\partial^2}{\partial m \partial x} \Pr(X_1(T) \le x, M_2(T) \le m) dx dm \tag{A.6}$$

which applying (A.3), will be decomposed into the sum of three double integrals as follows:

$$\iint_{\substack{m > x+k \\ m > 0}} e^{c \cdot m} \frac{\partial^2}{\partial m \partial x} \Pr(X_1(T) \le x, X_2(T) \le m) dx dm$$
$$- \frac{2\mu_2}{\sigma_2^2} \iint_{\substack{m > x+k \\ m > 0}} e^{\left(c + \frac{2\mu_2}{\sigma_2^2}\right)m} \frac{\partial}{\partial x}\left[\Pr\left(X_1(T) \le x - 2\rho\frac{\sigma_1}{\sigma_2}m, X_2(T) \le -m\right)\right] dx dm$$
$$- \iint_{\substack{m > x+k \\ m > 0}} e^{\left(c + \frac{2\mu_2}{\sigma_2^2}\right)m} \frac{\partial^2}{\partial m \partial x}\left[\Pr\left(X_1(T) \le x - 2\rho\frac{\sigma_1}{\sigma_2}m, X_2(T) \le -m\right)\right] dx dm$$
$$:= (\mathrm{I}) - \frac{2\mu_2}{\sigma_2^2}(\mathrm{II}) - (\mathrm{III}). \tag{A.7}$$

Applying (A.5), the first double integral of (A.7) will be

$$(\mathrm{I}) = e^{c\mu_2 T + \frac{1}{2}c^2\sigma_2^2 T} \Phi_2\left(\frac{-k - (\mu_1 + c\rho\sigma_1\sigma_2 - \mu_2 - c\sigma_2^2)T}{\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)T}}, \frac{(\mu_2 + c\sigma_2^2)T}{\sigma_2\sqrt{T}}\right.$$
$$\left.; \frac{-\rho\sigma_1\sigma_2 + \sigma_2^2}{\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)\sigma_2^2}}\right). \tag{A.8}$$

Let us consider the second double integral of (A.7). Calculating the inside integral, the second double integral of (A.7) will be written as follows:

$$(\mathrm{II}) = \iint_{\substack{m > x+k \\ m > 0}} e^{(c+\xi)m} \frac{\partial}{\partial x}\left[\Pr\left(X_1(T) \le x - 2\rho\frac{\sigma_1}{\sigma_2}m,\ X_2(T) \le -m\right)\right] dx dm$$
$$= \int_{m=0}^{m=\infty} e^{(c+\xi)m} \int_{x=-\infty}^{x=m-k} \frac{\partial}{\partial x}\left[\Pr\left(X_1(T) \le x - 2\rho\frac{\sigma_1}{\sigma_2}m,\ X_2(T) \le -m\right)\right] dx dm$$
$$= \int_{m=0}^{m=\infty} e^{(c+\xi)m} \Pr\left(X_1(T) \le \left(1 - 2\rho\frac{\sigma_1}{\sigma_2}\right) m - k,\ X_2(T) \le -m\right) dm. \tag{A.9}$$

Here, assume that $\eta = 1 - 2\rho\sigma_1/\sigma_2$. If we apply integration by parts, (A.9) will be

$$\frac{1}{c+\xi}\left[e^{(c+\xi)m} \Pr(X_1(T) \le \eta m - k,\ X_2(T) \le -m)\right]_{m=0}^{m=\infty}$$
$$- \frac{1}{c+\xi} \int_{m=0}^{m=\infty} e^{(c+\xi)m} \frac{d}{dm} \Pr(X_1(T) \le \eta m - k,\ X_2(T) \le -m) dm$$
$$=: (\mathrm{II} - 1) - \frac{1}{c+\xi}(\mathrm{II} - 2). \tag{A.10}$$

The first term of (A.10) is

$$-\frac{1}{c+\xi}\Pr\left(X_1(T)\le -k, X_2(T)\le 0\right) = -\frac{1}{c+\xi}\Phi_2\left(\frac{-k-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{-\mu_2 T}{\sigma_2\sqrt{T}};\rho\right) = (\mathrm{II}-1). \qquad (\mathrm{A.11})$$

Now, we need to calculate (II-2), the last integral of (A.10). Assume that $\phi_2(x_1, x_2;\rho)$ denote the joint density function of the bivariate standard normal distribution with correlation coefficient ρ. Let $g_1 = \eta m - k$ and $g_2 = -m$. The last integral of (A.10) can be expressed in terms of $\phi_2(x_1, x_2;\rho)$ as follows:

$$\begin{aligned}(\mathrm{II}-2) &= \int_{m=0}^{m=\infty} e^{(c+\xi)m}\frac{d}{dm}\Pr(X_1(T)\le \eta m-k, X_2(T)\le -m)dm \\ &= \int_{m=0}^{m=\infty} e^{(c+\xi)m}\left[\frac{d}{dm}\int_{-\infty}^{g_1}\int_{-\infty}^{g_2}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{u-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{v-\mu_2 T}{\sigma_2\sqrt{T}};\rho\right)dvdu\right]dm, \qquad (\mathrm{A.12})\end{aligned}$$

which, differentiating the inside double integral with respect to variable m, will be

$$\begin{aligned}&\int_{m=0}^{m=\infty} e^{(c+\xi)m}\left[\frac{dg_1}{dm}\cdot\int_{-\infty}^{g_2}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{g_1-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{v-\mu_2 T}{\sigma_2\sqrt{T}};\rho\right)dv\right.\\ &\left.\quad+\frac{dg_2}{dm}\cdot\int_{-\infty}^{g_1}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{u-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{g_2-\mu_2 T}{\sigma_2\sqrt{T}};\rho\right)du\right]dm\\ &= \eta\cdot\int_{m=0}^{m=\infty}\int_{-\infty}^{g_2} e^{(c+\xi)m}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{g_1-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{v-\mu_2 T}{\sigma_2\sqrt{T}};\rho\right)dvdm\\ &\quad-\int_{m=0}^{m=\infty}\int_{-\infty}^{g_1} e^{(c+\xi)m}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{u-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{g_2-\mu_2 T}{\sigma_2\sqrt{T}};\rho\right)dudm. \qquad (\mathrm{A.13})\end{aligned}$$

Using a change of variables with $g_1 = \eta m - k$ and $g_2 = -m$ in the two double integrals of the RHS of (A.13) and applying (A.4) and (A.5), (A.13) can be calculated as follows:

$$\begin{aligned}&\frac{\eta}{|\eta|}\iint_{\substack{-\frac{1}{\eta}g_1<\frac{k}{\eta}\\ \frac{1}{\eta}g_1+v<-\frac{k}{\eta}}} e^{(c+\xi)\left(\frac{g_1+k}{\eta}\right)}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{g_1-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{v-\mu_2 T}{\sigma_2\sqrt{T}};\rho\right)dvdg_1\\ &-\iint_{\substack{u+\eta g_2<-k\\ g_2<0}} e^{(c+\xi)(-g_2)}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{u-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{g_2-\mu_2 T}{\sigma_2\sqrt{T}};\rho\right)dudg_2\\ &= \frac{\eta}{|\eta|}e^{\frac{c+\xi}{\eta}k}e^{\frac{c+\xi}{\eta}\mu_1 T+\frac{1}{2}\left(\frac{c+\xi}{\eta}\right)^2\sigma_1^2 T}\\ &\times\Phi_2\left[\frac{k/\eta+\left\{\left(\mu_1+\frac{c+\xi}{\eta}\sigma_1^2\right)/\eta\right\}T}{\sqrt{(\sigma_1^2/\eta^2)T}}, \frac{-k/\eta-\left\{\left(\mu_1+\frac{c+\xi}{\eta}\sigma_1^2\right)/\eta+\mu_2+\frac{c+\xi}{\eta}\rho\sigma_1\sigma_2\right\}T}{\sqrt{(\sigma_1^2/\eta^2+\sigma_2^2+2\rho\sigma_1\sigma_2/\eta)T}};\right.\\ &\left.\frac{-\sigma_1^2/\eta^2-\rho\sigma_1\sigma_2/\eta}{\sqrt{(\sigma_1^2/\eta^2)\left(\sigma_1^2/\eta^2+\sigma_2^2+2\rho\sigma_1\sigma_2/\eta\right)}}\right]\\ &- e^{-(c+\xi)\mu_2 T+\frac{1}{2}(c+\xi)^2\sigma_2^2 T}\Phi_2\left[\frac{-k-\{\mu_1-(c+\xi)\rho\sigma_1\sigma_2+\eta\mu_2-\eta(c+\xi)\sigma_2^2\}T}{\sqrt{(\sigma_1^2+\eta^2\sigma_2^2+2\eta\rho\sigma_1\sigma_2)T}},\right.\\ &\left.\frac{-\{\mu_2-(c+\xi)\sigma_2^2\}T}{\sigma_2 T}; \frac{\rho\sigma_1\sigma_2+\eta\sigma_2^2}{\sqrt{(\sigma_1^2+\eta^2\sigma_2^2+2\eta\rho\sigma_1\sigma_2)\sigma_2^2}}\right]\\ &= (\mathrm{II}-2). \qquad (\mathrm{A.14})\end{aligned}$$

Let us calculate the last integral of (A.7). Remind that $\xi = 2\mu_2/\sigma_2^2$, $\eta = 1 - 2\rho\sigma_1/\sigma_2$. Using a change of variables with $u = x - 2\rho\sigma_1/\sigma_2 m$ and $v = -m$, the last double integral of (A.7) will be

$$(\mathrm{III}) = \iint_{\substack{v<0 \\ u+\eta v<-k}} e^{-(c+\xi)v} \frac{\partial^2}{\partial m \partial x}[\Pr(X_1(T) \le u, X_2(T) \le v)] du dv \tag{A.15}$$

of which the second-order derivative with respect to m and x becomes

$$-\frac{\partial^2}{\partial v \partial u}[\Pr(X_1(T) \le u, X_2(T) \le v)] - 2\rho\frac{\sigma_1}{\sigma_2}\frac{\partial^2}{\partial u^2}[\Pr(X_1(T) \le u, X_2(T) \le v)], \tag{A.16}$$

if chain rule is applied with $\partial u/\partial x = 1$, $\partial u/\partial m = -2\rho\sigma_1/\sigma_2$, $\partial v/\partial x = 0$ and $\partial v/\partial m = -1$. Hence, placing (A.16) into (A.15), we have

$$\begin{aligned}(\mathrm{III}) = & -\iint_{\substack{v<0 \\ u+\eta v<-k}} e^{-(c+\xi)v} \frac{\partial^2}{\partial v \partial u}[\Pr(X_1(T) \le u, X_2(T) \le v)] du dv \\ & - 2\rho\frac{\sigma_1}{\sigma_2}\iint_{\substack{v<0 \\ u+\eta v<-k}} e^{-(c+\xi)v} \frac{\partial^2}{\partial u^2}[\Pr(X_1(T) \le u, X_2(T) \le v] du dv \\ := & -(\mathrm{III}-1) - 2\rho\frac{\sigma_1}{\sigma_2}(\mathrm{III}-2).\end{aligned} \tag{A.17}$$

Applying (A.5), the first double integral of (A.17) will be

$$\begin{aligned}(\mathrm{III}-1) = & e^{-(c+\xi)\mu_2 T + \frac{1}{2}(c+\xi)^2\sigma_2^2 T} \\ & \times \Phi_2\left[\frac{-k - \{\mu_1 - (c+\xi)\rho\sigma_1\sigma_2 + \eta\mu_2 - \eta(c+\xi)\sigma_2^2\}T}{\sqrt{(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2)T}}, \frac{-\{\mu_2 - (c+\xi)\sigma_2^2\}T}{\sigma_2\sqrt{T}};\right. \\ & \left.\frac{\rho\sigma_1\sigma_2 + \eta\sigma_2^2}{\sqrt{(\sigma_1^2 + \eta^2\sigma_2^2 + 2\eta\rho\sigma_1\sigma_2)\sigma_2^2}}\right].\end{aligned} \tag{A.18}$$

Now, let us consider (III-2), the second double integral of (A.17). The second-order derivative in the second double integral of (A.17) can be calculated as follows:

$$\begin{aligned}& \frac{\partial^2}{\partial u^2}[\Pr(X_1(T) \le u, X_2(T) \le v] \\ & = \frac{\partial^2}{\partial u^2}\left[\int_{-\infty}^{u}\int_{-\infty}^{v} \frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{z-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{w-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) dw dz\right] \\ & = \frac{\partial}{\partial u}\left[\int_{-\infty}^{v} \frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{u-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{w-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) dw\right] \\ & = \int_{-\infty}^{v} \frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\frac{\partial}{\partial u}\phi_2\left(\frac{u-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{w-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) dw.\end{aligned} \tag{A.19}$$

Placing (A.19) into the second double integral of (A.17), we have a triple integral,

$$\begin{aligned}(\mathrm{III}-2) & = \iiint_{\substack{v<0 \\ u+\eta v<-k \\ w<v}} e^{-(c+\xi)v}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\frac{\partial}{\partial u}\phi_2\left(\frac{u-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{w-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) dw du dv \\ & = \iint_{w<v<0} e^{-(c+\xi)v}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\left[\int_{u<-\eta v-k}\frac{\partial}{\partial u}\phi_2\left(\frac{u-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{w-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) du\right] dw dv \\ & = \iint_{w<v<0} e^{-(c+\xi)v}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{-\eta v-k-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{w-\mu_2 T}{\sigma_2\sqrt{T}}; \rho\right) dw dv,\end{aligned} \tag{A.20}$$

which, using a change of variables with $y = -\eta v - k$ and applying (A.4), can be calculated as follows:

$$\frac{1}{|\eta|}e^{\frac{c+\xi}{\eta}k}\iint_{\substack{-\frac{y}{\eta}<\frac{k}{\eta}\\ \frac{y}{\eta}+w<-\frac{k}{\eta}}} e^{\frac{c+\xi}{\eta}y}\frac{1}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}}\phi_2\left(\frac{y-\mu_1T}{\sigma_1\sqrt{T}},\frac{w-\mu_2T}{\sigma_2\sqrt{T}};\rho\right)dwdy$$

$$= \frac{1}{|\eta|}e^{\frac{c+\xi}{\eta}k}e^{\frac{c+\xi}{\eta}\mu_1T+\frac{1}{2}\left(\frac{c+\xi}{\eta}\right)^2\sigma_1^2T}$$

$$\times\,\Phi_2\left[\frac{k/\eta-\{-(\mu_1+(c+\xi)\sigma_1^2/\eta)/\eta\}T}{\sqrt{(\sigma_1^2/\eta^2)T}},\right.$$

$$\left.\frac{-k/\eta-\{(\mu_1+(c+\xi)\sigma_1^2/\eta)/\eta+\mu_2+(c+\xi)\rho\sigma_1\sigma_2/\eta\}T}{\sqrt{(\sigma_1^2/\eta^2+\sigma_2^2+2\rho\sigma_1\sigma_2/\eta)T}};\frac{-\sigma_1^2/\eta^2-\rho\sigma_1\sigma_2/\eta}{\sqrt{(\sigma_1^2/\eta^2)(\sigma_1^2/\eta^2+\sigma_2^2+2\rho\sigma_1\sigma_2/\eta)}}\right]$$

$$= (\mathrm{III}-2). \tag{A.21}$$

Finally, according to (A.7), (A.9) and (A.17), expectation (2.13) can be written as

$$\begin{aligned}
&(\mathrm{I})-\frac{2\mu_2}{\sigma_2^2}(\mathrm{II})-(\mathrm{III})\\
&=(\mathrm{I})-\xi\left((\mathrm{II}-1)-\frac{1}{c+\xi}(\mathrm{II}-2)\right)-\left(-(\mathrm{III}-1)-2\rho\frac{\sigma_1}{\sigma_2}(\mathrm{III}-2)\right)\\
&=(A.8)-\xi(A.11)+\frac{\xi}{c+\xi}(A.14)+(A.18)+2\rho\frac{\sigma_1}{\sigma_2}(A.21)\\
&=(2.13).
\end{aligned}$$

Here, the normal distribution function in the first term of (A.14) is the same as that in (A.21) and the distribution function in the second term of (A.14) is the same as that in (A.18). Thus, (2.13) consists of four bivariate normal distribution functions.

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