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STATIONARY GLOBAL DYNAMICS OF LOCAL MARKETS WITH QUADRATIC SUPPLIES

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ABSTRACT. The method of Lattice Dynamical System is used to establish a global model on an infinite chain of many local markets interacting each other through a diffusion of prices between them. This global model extends the Walrasian evolutionary cobweb model in an independent single local market to the global market evolution.

We assume that each local market has linear decreasing demands and quadratic supplies with naive predictors, and investigate the stationary behaviors of global price dynamics and show that their dynamics are conjugate to those of Hénon maps and hence can exhibit complicated behaviors such as period-doubling bifurcations, chaos, and homoclic orbits etc.

1. INTRODUCTION

The Cobweb model for the local market dynamics has been well introduced and studied by many researchers (e.g., [6],[7],[8],[14]). The Cobweb model describes the dynamics of equilibrium prices in a single independent local market for a non-storable good that takes one time period to produce, so that producers must form price expectations one period ahead using the past history of prices.

Let $p_n^e = H(\mathbf{P}_{n-1})$, where p_n^e is the expected price by the producers at time n and $\mathbf{P}_{n-1} = (p_{n-1}, p_{n-2}, \cdots, p_{n-L})$ is a vector of past prices of lag-length L and $H(\cdot) : \mathbf{R}^L \to \mathbf{R}$ is a real-valued function, so called a *predictor*. Let p_n be the actual market price at time n by the consumers, and let $D(p_n)$ be the consumer demand and $S(p_n^e)$ be the producer supply for the goods. The supply $S(p_n^e)$ is derived from producer's maximizing expected profit with a cost function $c(q)$, i.e.,

$$S(p_n^e) = arg \max_{q_n} \{p_n^e q_n - c(q_n)\}. \tag{1.1}$$

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The demand function $D(\cdot)$ depends on the current market price p_n and is assumed to be strictly decreasing in the price p_n to ensure that its inverse D^{-1} is well-defined. The supply function $S(\cdot)$ depends on the expected price p_n^e and will be assumed to be quadratic in our paper. The intersection point p^* of the demand and supply curve such that $D(p^*) = S(p^*)$ is called the steady state equilibrium price.

If the beliefs of producers are homogeneous, i.e., all producers use the same predictor H, then the market equilibrium price dynamics in the cobweb model is described by

$$D(p_n) = S(H(\mathbf{P}_{n-1})), \quad \text{or,} \quad p_n = D^{-1}(S(H(\mathbf{P}_{n-1}))). \tag{1.2}$$

Thus, the actual equilibrium price dynamics in a local market depends on the demand D, the supply S, and the predictor H used by the producers.

2. The Model for the Global Market Dynamics

Over the last decade, a new class of infinite dimensional dynamical systems, so called Lattice Dynamical Systems(LDS) have been introduced and studied by many researchers (e.g., [1],[9],[10]). These LDS's have been proved to be one of the most efficient tools to analyze space-time behaviors of the extended systems.

To begin with, we define the phase space (or state space) of the LDS. Suppose that at each site j of a d-dimensional lattice $\mathbf{Z}^d$, we have a finite dimensional local dynamical system which is defined by some map $f_j : M_j \longrightarrow M_j$, where M_j is a local phase space at the site j. For simplicity and applicability to our model, we will confine our attention to an infinite chain ($d = 1$) and the identical local map, i.e., $f_j = f, M_j = \mathbf{R}^1 \, \forall j \in \mathbf{Z}$, where $\mathbf{R}^1$ is a 1-dimensional real Euclidean space with ordinary inner product $(\cdot, \cdot)$ and the norm $|\cdot| = \sqrt{(\cdot, \cdot)}$. Then we have an infinite dimensional dynamical system on a space

$$M = \prod_{j \in \mathbf{Z}} M_j = \{p = \{p_j\} | p_j \in \mathbf{R}, j \in \mathbf{Z}\} \tag{2.1}$$

where M is obviously a linear space with respect to componentwise addition and scalar multiplication. A point (or, a state) $p = \{p_j\} \in M$ can be thought of as a bi-infinite sequence of real numbers. To make the linear space M to be a Hilbert space, we equip M with the inner product defined by

$$\langle p, q \rangle_\rho = \sum_{j \in \mathbf{Z}} \frac{(p_j, q_j)}{\rho^{|j|}} \quad \forall p, q \in M, \tag{2.2}$$

where $\rho > 1$ is some fixed number depending on the particular problem. Then the norm $\|\cdot\|_\rho$ is induced by

$$\|\cdot\|_\rho = \sqrt{\langle\cdot,\cdot\rangle_\rho}$$

and now we can define the phase space of our LDS by

$$B_\rho = \{p \in M | \, \|p\|_\rho < \infty\}. \tag{2.3}$$

Then it can be easily shown that B_ρ is a Hilbert space (e.g., [1]). Next, we define the evolution operator on B_ρ in the following.

Definition 2.1. Define the evolution operator $\Phi : B_\rho \to B_\rho$ by

$$(\Phi p)_j = F(\{p_j\}^s), \forall j \in \mathbf{Z}, \tag{2.4}$$

where $\{p_j\}^s = \{p_i | \, |i-j| \le s, \; s \ge 1 \text{ integer}\}$ for each $j \in \mathbf{Z}$, i.e., $\{p_j\}^s$ is the set of values p_i at the site i which are within the distance of radius s from the site j, and $F : \mathbf{R}^{2s+1} \to \mathbf{R}$ is a differentiable map of class C^2 such that

$$\left|\frac{\partial F}{\partial p_i}\right| \le K, \quad \left|\frac{\partial^2 F}{\partial p_i \partial p_k}\right| \le K, \tag{2.5}$$

for any collection $\{p_j\}^s$ and some constant $K > 0$.

Then it is easy to verify that under the condition (2.5), $\Phi(B_\rho) \subset B_\rho$ and Φ is Lipschitz continuous with the constant $L = C(2s+1)^{\frac{3}{2}}\rho^{\frac{s}{2}}$ (e.g., [1]).

Definition 2.2. Given a state $p(n) = \{p_j(n)\}_{j=-\infty}^{\infty} \in B_\rho$ at the moment n, we can obtain via (2.4) the next state $p(n+1)$, that is,

$$p(n+1) = \Phi(p(n)), \quad \text{or,} \tag{2.6}$$
$$p_j(n+1) = (\Phi(p(n)))_j = F(\{p_j(n)\}^s).$$

The dynamical system $(\Phi^n, B_\rho)_{n \in \mathbf{Z}^+}$ is called a *Lattice Dynamical System*(LDS).

Formula (2.6) implies that given a state $p(n) \in B_\rho$, we can calculate its next state $p(n+1)$, so we can obtain the forward orbit of the evolution operator Φ, i.e.,

$$p(0), p(1) = \Phi(p(0)), p(2) = \Phi(p(1)) = \Phi^2(p(0)), \cdots.$$

Before ending this section, let us consider several kinds of basic motions (or solutions) in the LDS (2.6).

Definition 2.3. (i) A state (or solution) $p(n) = \{p_j(n)\}$ for the LDS (2.6) is *spatially homogeneous* if $p_j(n) = \psi(n) \, \forall j \in \mathbf{Z}$, i.e., a spatially homogeneous solution $\{\psi(n)\}$ does not depend on the space coordinates j and so has the same value at each site j.

(ii) A solution $p(n) = \{p_j(n)\}$ is *static* (or *stationary, steady state,standing wave*) if $p_j(n) = \phi_j \,\forall n \in \mathbf{Z}^+$, i.e., a static solution $\{\phi_j\}$ does not depend on time n, and is standing there along the space coordinates j at all times n.
(iii) A solution $p(n) = \{p_j(n)\}$ is a *traveling wave with wave velocity* m/l if $p_j(n) = \xi(lj+mn)$, where $l > 0, m \in \mathbf{Z}$ and $(l,m) = 1$(i.e., relatively prime). Here, the ratio m/l is called the *wave velocity* of the traveling wave.

For instance, suppose that the local system $f : M_j \to M_j$ has a fixed point p^*. Then the state $p = \{p_j\}, p_j = p^* \forall j \in \mathbf{Z}$ is a spatially homogeneous static solution, i.e., a fixed point of the evolution map Φ and also can be thought of as a travelling wave with arbitrary velocity.

Now, as our LDS model for the global market dynamics, we will take the following form:

$$
\begin{aligned}
(2.7) \qquad p_j(n+1) &= (\Phi(p(n)))_j \\
&= (1-\alpha)p_j(n) + \alpha f(p_j(n)) + \varepsilon(p_{j-1}(n) - 2p_j(n) + p_{j+1}(n)),
\end{aligned}
$$

where a solution $p_j(n), j \in \mathbf{Z}, n \in \mathbf{Z}^+$ represents the price of a good at the site (or local market)j at the time n, and $f : \mathbf{R} \to \mathbf{R}$ is a Walrasian local market price dynamics at each site j, and $\alpha \in [0,1]$ is a parameter denoting the weighted average between $p_j(n)$ and $f(p_j(n))$, and the parameter ε is a diffusion coefficient measuring the intensity of interaction between the neighboring local markets. Thus, in this global market model, the price $p_j(n+1)$ at site j and at time $n+1$ is determined by several factors, i.e., the previous price $p_j(n)$, the local market dynamics f, the weight $\alpha \in [0,1]$ of the average between them, and the diffusion coefficient $\varepsilon > 0$. Notice that the parameter α plays a role of controlling each local market in such a way that if $\alpha = 1$ then $p_j(n+1)$ is determined completely by the local market dynamics f together with diffusion term and if $\alpha = 0$ then the local market dynamics is suppressed completely and $p_j(n+1)$ depends only on the present price and diffusion term.

Remark 2.1. For a solution $p_j(n)$ of our model (2.7) to have a meaning in economic sense, we impose a boundary condition at infinity that $p_j(n)$ must be bounded, i.e., $|p_j(n)| \le C \;\forall j \in \mathbf{Z}, n \in \mathbf{Z}^+$ for some $C > 0$. Also, we require that a solution $p_j(n)$ must have nonnegative value for all $j \in \mathbf{Z}, n \in \mathbf{Z}^+$. If a solution of (2.7) does not satisfy these conditions, then it would not be an admissible solution for our model.

Remark 2.2. Besides the solutions given in the Definition 2.3, there can also be many other solutions, e.g., spatially and/or temporally periodic solutions, spatially and/or temporally chaotic bounded solutions, and so on. In this paper, we restrict our attention only to those periodic solutions or bounded chaotic solutions which are the basic solutions mentioned in the Definition 2.3, e.g., spatially periodic static solutions, temporally periodic spatially homogeneous solutions, spatially and temporally periodic traveling wave solutions, etc.

3. Stationary Global Price Dynamics

We assume that the predictor H is naive, the demand D is linear decreasing, and the supply S is quadratic, that is, they are given by

$$p_n^e = H(\mathbf{P}_{n-1}) = p_{n-1}, \quad D(p_n) = 1 - p_n, \qquad (3.1)$$
$$S(p_n^e) = S(p_{n-1}) = 4p_{n-1}(1 - p_{n-1}).$$

respectively. Note that the price p_n in (3.1) is a scaled price such that $0 \leq p_n \leq 1$, and the quadratic supply function $S(x) = 4x(1-x)$ is a so called *logistic map*, which is known to exhibit chaotic dynamics on the whole interval $[0,1]$ (e.g., [15]). This kind of non-monotonic supply curve can be justified in an actual market, e.g., by an *income effect* in an agricultural market (e.g., [18], pp 339). This income effect, of course, may be applied to our fish market as well. In other words, as prices of fish are getting higher, the income of fishermen is getting higher, and so the production of fish might be getting less due to their taking more leisure time.

Now, with these choices of H, D, and S, the local market equilibrium price dynamics, $D(p_n) = S(p_n^e)$, is given by

$$1 - p_n = 4p_{n-1}(1 - p_{n-1}), \quad \text{or}, \qquad (3.2)$$
$$p_n = 1 - 4p_{n-1}(1 - p_{n-1}).$$

Hence, our local market dynamics f for the global market model (2.7) is given by

$$f(x) = 1 - 4x(1 - x) = (1 - 2x)^2, \qquad (3.3)$$

where x may be assumed to be restricted to the interval $0 \leq x \leq 1$, since for $x \notin [0,1]$, the dynamics of f is very simple, i.e., $f^n(x) \to +\infty$ as $n \to +\infty$, and so only for $x \in [0,1]$, $f^n(x) \in [0,1]$ for all $n \in \mathbf{Z}^+$ and exhibits interesting chaotic dynamics.

The map f has two fixed points, one at $p^* = \frac{1}{4}$ where $f'(p^*) = -2 < -1$, and the other at $q^* = 1$ where $f'(q^*) = 4 > 1$, and so both fixed points are repellors.

Note that at $p^* = \frac{1}{4}$, $D(\frac{1}{4}) = S(\frac{1}{4}) = \frac{3}{4}$, and prices near p^* diverges in an oscillatory way from p^*, while at $q^* = 1$, $D(1) = S(1) = 0$, and prices near and less than q^* decreases in a monotone way from q^* and so prices fluctuates between these two repellors p^* and q^* in a chaotic way.

Now, let us consider the stationary solutions of the global market dynamics (2.7), where the local market dynamics f is given by (3.3), i.e.,

$$(3.4)\quad p_j(n+1) = (1-\alpha)p_j(n) + \alpha\{1-2p_j(n)\}^2 + \varepsilon\{p_{j-1}(n) - 2p_j(n) + p_{j+1}(n)\}.$$

Hereafter, we will slightly loosen our restriction $0 \le p_j(n) \le 1$ so that $p_j(n) \ge 0$ because interesting dynamics can occur for $p_j(n) > 1$ in the global market dynamics.

To obtain the stationary solutions, we set $p_j(n) = \phi_j$ in (3.4), then we have

$$\phi_{j+1} = -\beta - \phi_{j-1} + (2+5\beta)\phi_j - 4\beta\phi_j^2, \tag{3.5}$$

where $\beta = \alpha/\varepsilon > 0$. Equation (3.5) is a 2nd order nonlinear difference equation which has two fixed points $\phi_1^* = \frac{1}{4}$ and $\phi_2^* = 1$ for $0 < \alpha \le 1$. For $\alpha = 0$, (3.5) is reduced to a linear equation $\phi_{j+1} = 2\phi_j - \phi_{j-1}$ with general solution of the form $\phi_j = c_1 + c_2 j$ (c_1, c_2 are arbitrary constants) and so any constant solution $\phi_j = \phi_0 \ge 0$ is a fixed point. Note that these fixed points are the spatially homogeneous static solutions of (3.5). Hereafter, we assume that $0 < \alpha \le 1$, i.e., $\beta > 0$. Letting $x_j = \phi_{j-1}$, $y_j = \phi_j$, then (3.5) can be reduced to a 2D discrete dynamical system defined by

$$\begin{aligned} x_{j+1} &= y_j \\ y_{j+1} &= -\beta - x_j + (2+5\beta)y_j - 4\beta y_j^2 \end{aligned} \tag{3.6}$$

Note that an orbit $\{\cdots, (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), \cdots\}$ of (3.6) corresponds in a one-to-one fashion to a solution $\{\cdots, y_{-1} = \phi_{-1}, y_0 = \phi_0, y_1 = \phi_1, \cdots\}$ of (3.5). In fact, the system (3.6) is a translational dynamical system and is generated by a 2D map $S_\beta : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$S_\beta(x,y) = (y, -\beta - x + (2+5\beta)y - 4\beta y^2). \tag{3.7}$$

The map S_β clearly has two fixed points $Q_1 = (x_1^*, y_1^*) = (\frac{1}{4}, \frac{1}{4})$ and $Q_2 = (x_2^*, y_2^*) = (1,1)$. Note that the derivative of $S_\beta(x,y)$ is given by

$$DS_\beta(x,y) = \begin{bmatrix} 0 & 1 \\ -1 & 2+5\beta-8\beta y \end{bmatrix} \tag{3.8}$$

and so $\det DS_\beta(x,y) = 1$ for all $(x,y) \in \mathbf{R}^2$, $\beta > 0$. Hence, $S_\beta(x,y)$ is an *orientation* and *area-preserving map*. From (3.8), it follows that the matrix $DS_\beta(\frac{1}{4}, \frac{1}{4})$ has eigenvalues $\lambda_{1,2} = \frac{1}{2}\{(2+3\beta) \pm \sqrt{(2+3\beta)^2-4}\}$ and so $\lambda_1 > 1 > \lambda_2 > 0$ for all $\beta > 0$, and the matrix $DS_\beta(1,1)$ has eigenvalues $\mu_{1,2} = \frac{1}{2}\{(2-3\beta) \pm \sqrt{(2-3\beta)^2-4}\}$ and so $0 > \mu_1 > -1 > \mu_2$ for $\beta > \frac{4}{3}$ and $\mu_{1,2} = -1, -1$ for $\beta = \frac{4}{3}$ and $\mu_{1,2}$ are complex conjugate eigenvalues lying on the unit circle for $0 < \beta < \frac{4}{3}$.

In other words, the fixed point $Q_1 = (\frac{1}{4}, \frac{1}{4})$ is a hyperbolic fixed point (saddle point) for all $\beta > 0$, while the fixed point $Q_2 = (1,1)$ is a saddle point for $\beta > \frac{4}{3}$, a parabolic point for $\beta = \frac{4}{3}$, and an elliptic fixed point (center) for $0 < \beta < \frac{4}{3}$. Hence, even with simple looking at the movement of the eigenvalues $\mu_{1,2}$ on the unit circle as the parameter β increases from 0 and crosses the value $\frac{4}{3}$, we can notice that the local dynamics of $S_\beta(x,y)$ near the fixed point $Q_2 = (1,1)$ would be very complicated.

In fact, our map S_β given in (3.7) turns out to be dynamically same as the celebrated Hénon map $H_{a,b} : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$H_{a,b}(x,y) = (a + by - x^2, x), \tag{3.9}$$

where a, b are real parameters and $b = -1$ corresponding to our map S_β.

Remark 3.1. The Hénon map was originally defined by Hénon himself in the form

$$T_{a,b}(x,y) = (1 + y - ax^2, bx), \tag{3.10}$$

which represents one of the canonical forms for general quadratic maps with constant Jacobian determinant. But, this map $T_{a,b}$ can be transformed into the form $H_{a,b}$ given in (3.9) with the parameter values a and b unchanged, by the simple scaling $x \to x/a, y \to by/a$ for $a \neq 0$ and $b \neq 0$. Note that if $a = 0$, then $T_{a,b}$ becomes a linear map, while in the map $H_{a,b}$ given in (3.9), $a = 0$ has no special significance.

Now, we first show the dynamical equivalence between our map and the Hénon map.

Lemma 3.1. *Our map S_β given in (3.7) is topologically conjugate to the Hénon map $H_{a,b}$ defined by (3.9) with the parameters $a = (\frac{3}{2}\beta)^2 - 1$ and $b = -1$, via the affine transformation given by $h(x,y) = (c_1 y + c_2, c_1 x + c_2)$ with $c_1 = 4\beta$ and $c_2 = -\frac{1}{2}(5\beta + 2)$ where $\beta = \alpha/\varepsilon > 0$.*

Hence, by reflecting with respect to the line $y = x$, multiplying by 4β, and translating downward along the line $y = x$, every orbit of S_β is transformed to the orbit of the Hénon map (and vice versa). Note that $\det DH_{a,b}(x,y) = -b$ and so the Hénon map with $b = -1$, i.e., $H_{a,-1}(x,y) = (a - y - x^2, x)$ is also an orientation and area preserving map. We will write $H_a = H_{a,-1}$ hereafter. Now that we know

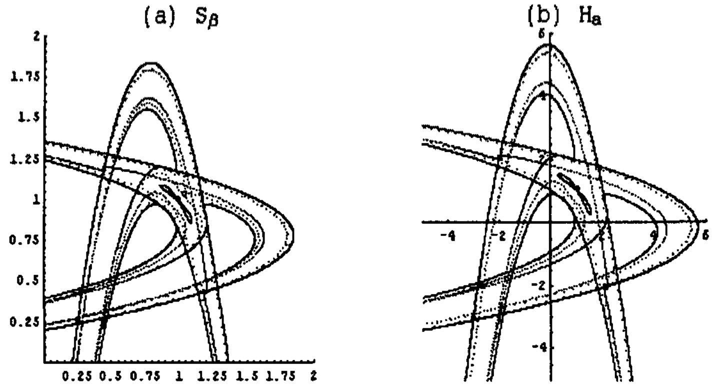

Figure 1. When $\beta = 4.1/3$; (a) the stable (horizontal one) and the unstable manifold (vertical one) of the fixed point $(\frac{1}{4}, \frac{1}{4})$ and those of the fixed point $(1, 1)$ for S_β. (b) the stable (vertical one) and the unstable manifold (horizontal one) of the corresponding fixed point $(-3.05, -3.05)$ and those of $(1.05, 1.05)$ for the Hénon map H_a with corresponding value $a = 3.2025$. The conjugacy between them is obvious.

from the Lemma 3.1 that the dynamics of our map (3.7) is the same as that of the area-preserving Hénon map H_a, all we have to do is just examine the behavior of the Hénon map H_a. See the stable and unstable manifolds of our map S_β and H_a in Figure 1. Both pictures clearly show the existence of a *transversal homoclinic point* to the hyperbolic fixed point and the tangential connection (looking like the figure ∞) of the stable and the unstable manifold at the other hyperbolic fixed point.

Over the last thirty years, so many researchers have been devoting themselves to investigating the dynamics of the Hénon map, however, unfortunately, until this time, it is still far away from being complete to understand its extremely complicated dynamical nature, even in the subcase of the area-preserving map H_a. Drawing the bifurcation diagram for the Hénon map $H_{a,b}$ in the parameter (a, b)-space is still left as an open problem even in the area-preserving case $b = \pm 1$.

Let us first consider the local bifurcation problem about the map S_β. Recall that the fixed point $Q_1 = (\frac{1}{4}, \frac{1}{4})$ is always hyperbolic and so no local bifurcations occur there. But, the fixed point $Q_2 = (1, 1)$ becomes nonhyperbolic (parabolic) when $\beta = \frac{4}{3}$ and is elliptic when $\beta < \frac{4}{3}$ and is hyperbolic when $\beta > \frac{4}{3}$, and hence we expect very complicated local bifurcation to occur when β passes through the value $\frac{4}{3}$.

In fact, Sterling et al ([24]) have reported various kinds of local bifurcations of low periodic orbits up to period 6 for the area-preserving Hénon map H_a and also have proved that there are no local bifurcations for the Hénon map $H_{a,b}$ when a and b are in the range given by $\frac{1+|b|}{\sqrt{a}} < 2\sqrt{1-2/\sqrt{5}}$. This parameter bound for no local bifurcations are coincident with the bound given by Devaney and Nitecki ([16]) for the existence of *hyperbolic horseshoe*. Now, using Lemma 3.1, we can convert their results in terms of the parameter β for our map S_β and show the bifurcation table and the parameter bound in the following lemma.

Lemma 3.2. *Our map S_β undergoes a cascade of bifurcations of attracting periodic orbits (part of which are shown up to period six in the Table* 1*) near the fixed point $Q_2 = (1,1)$ as the parameter β increases from* 0 *and passes through the value* 4/3. *This local bifurcation ends when $\beta = \frac{2}{3}(1+\sqrt{5}) \approx 2.15738$ and for $\beta > 2.15738$, S_β has a hyperbolic horseshoe and hence has an invariant Cantor set on which the behavior of S_β is chaotic.*

β values	parent	type	children
1/3=0.333333	Fixed point	1/6	Two period-6 orbits
$(5-\sqrt{5})/6 = 0.460655$	Fixed point	1/5	Two period-5 orbits
2/3=0.666667	Fixed point	1/4	Two period-4 orbits
$2\sqrt{2}/3 = 0.942809$		sn	Two period-3 orbits
1.0	Fixed point	1/3	One period-3 orbit
1.0	period-3 orbit	pd	One period-6 orbit
$(5+\sqrt{5})/6 = 1.20601$	Fixed point	2/5	Two period-5 orbits
4/3=1.33333	Fixed point	pd	One period-2 orbit
4/3=1.33333	period-6 orbit	pf	Two period-6 orbits
1.44555		sn	Two period-6 orbits
$\sqrt{19}/3 = 1.45297$	period-2 orbit	1/3	One period-6 orbit
$2\sqrt{5}/3 = 1.49071$	period-2 orbit	pd	One period-4 orbit
1.70642		sn	Two period-5 orbits
1.72297		sn	Two period-6 orbits

Table 1: The fixed point means the fixed point $Q_2 = (1,1)$ and *parent* refers to the orbit that is undergoing the bifurcation and *children* represents the created orbits through the bifurcation. *type* is one of sn, pf, pd, or m/n, corresponding to a saddle-node, pitchfork, period-doubling, or rotational bifurcation, respectively. A rotational bifurcation denoted by m/n occurs when the winding number of an elliptic parent orbit becomes m/n or when the eigenvalues of the fixed point are $e^{2\pi i m/n}$.

Note that the local bifurcations given in the Table 1 is simply a very small part of the whole scenario and this extremely complicated cascade of bifurcations

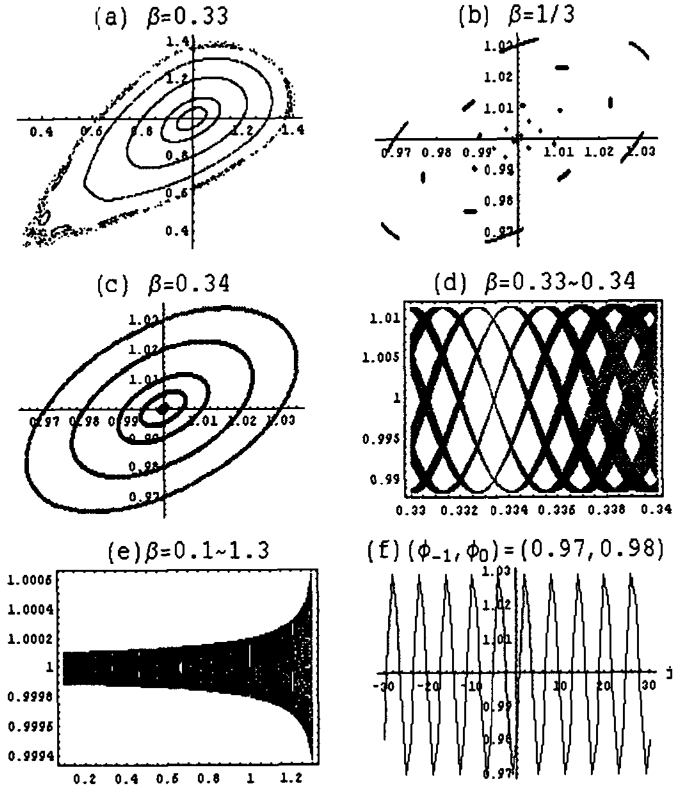

Figure 2. (a),(b) and (c) show the orbits of S_β with different initial conditions near the fixed point $(1,1)$: (a) and (c) show the orbits when $\beta = 0.33, 0.34$ respectively. In both of (a) and (c), the rotation numbers are irrational, and so each orbit is dense on the invariant ellipse. (b) When $\beta = 1/3$, since the rotation number is $1/6$, each orbit is attracted to a period-6 orbit. The other repelling period-6 orbit cannot be observed on the computer. (d) Bifurcation diagram as β varies from 0.33 to 0.34 for a fixed initial condition $(0.99, 0.99)$. The interval $(0.33, 0.34)$ has 400 step-points and at each step the y-values of the last 60 points among the 400 iterations are ploted. (e) Bifurcation diagram for $0.1 < \beta < 1.3$. All the bifurcating periodic orbits are contained in a bounded region. (f) The static solution with initial conditions $(\phi_{-1}, \phi_0) = (0.97, 0.98)$ when $\beta = 1/3$. This solution is attracted to the period-6 static solution.

successively occurs until β reaches the value $\beta \approx 2.15738$ and hence when our map S_β has a hyperbolic horseshoe, all the created periodic orbits with any period still exist and are included in an invariant Cantor set on which the dynamics of S_β becomes chaotic. As an illustration, the first case in the Table 1 is shown in the Figure 2.

The Lemma 3.2 simply says that just like the 1D map Q_μ, right after the end of the successive local bifurcation of periodic orbits near the fixed point $(1,1)$, S_β

becomes chaotic. Devaney and Nitecki have shown in the below (Lemma 3.3) that this chaotic motion occurs in a bounded region.

Now let us consider global dynamics of the map S_β. Devaney and Nitecki have already examined the *nonwandering set* and the global dynamics of the Hénon map $H_{a,b}$ and defined the following three crucial a-values and the radius R of a square;

$$(3.11) \qquad a_0 = -(1+|b|)^2/4, \quad a_1 = 2(1+|b|)^2, \quad a_2 = (5+2\sqrt{5})(1+|b|)^2/4$$
$$R = \frac{1}{2}\{1+|b|+\sqrt{(1+|b|)^2+4a}\}$$

and have shown that for any fixed $b \neq 0$ (and hence also includes our case $b = -1$);
(i) if a is small enough, i.e., $a < a_0$, then the nonwandering set $\Omega(H_{a,b})$ of $H_{a,b}$ is empty, i.e., all points in the plane eventually escape to infinity.
(ii) if $a \geq a_0$, then $\Omega(H_{a,b})$ is contained in a square $U = \{(x,y) | \, |x| \leq R, |y| \leq R\}$.
(iii) if $a \geq a_1$, then $\Lambda = \cap_{n\in \mathbf{Z}} H^n_{a,b}(U)$ is a topological horseshoe, i.e., there exists a continuous semi-conjugacy of $\Omega(H_{a,b}) \subset \Lambda$ onto the 2-shift.
(iv) if $a > a_2$, then $\Lambda = \Omega(H_{a,b})$ has a hyperbolic structure and is topologically conjugate to the 2-shift. That is, $\Omega(H_{a,b})$ is the compact invariant Cantor set obtained from the Smale horseshoe construction and so the points in the $\Omega(H_{a,b})$ remain there forever and moves in a chaotic fashion. Note that the parameter a_2 is the same as the one given by Sterling et al ([24]).

Now our map S_β corresponds in a one to one fashion to $H_{a,b}$ with $b = -1$ through the parameter relation $\beta = \frac{2}{3}\sqrt{1+a}$ (Lemma 3.1), and so by converting the above results by Devaney and Nitecki into our case, we have the following lemma.

Lemma 3.3. *The map S_β, depending on the values of $\beta > 0$, shows the following global dynamics:*
(i) *For $\beta > 0$, the nonwandering set $\Omega(S_\beta)$ is contained in a square*

$$V = \left\{(x,y) | \, \frac{1}{4} \leq x \leq 1+\frac{1}{2\beta}, \ \frac{1}{4} \leq y \leq 1+\frac{1}{2\beta}\right\}.$$

(ii) *For $\beta \geq 2$, $\Lambda = \cap_{n\in \mathbf{Z}} S^n_\beta(V)$ is a topological horseshoe, i.e., there exists a continuous semi-conjugacy of $\Omega(S_\beta) \subset \Lambda$ onto the 2-shift.*
(iii) *For $\beta > \frac{2}{3}(1+\sqrt{5}) \approx 2.15738$, $\Lambda = \Omega(S_\beta)$ is uniformly hyperbolic and $S_\beta|\Lambda$ is topologically conjugate to the 2-shift.*

Note that the left vortex of V in Lemma 3.3 is just the fixed point $Q_1 = (\frac{1}{4}, \frac{1}{4})$ for S_β and the right vortex of V is $(1+\frac{1}{2\beta}, 1+\frac{1}{2\beta})$ and so as $\beta > 0$ increases, the right vortex gets closer to the fixed point $Q_2 = (1,1)$, say, $(1.25, 1.25)$ for $\beta = 2$ and $(1.23, 1.23)$ for $\beta \approx 2.15738$.

From the Lemma 3.2 and 3.3, we can see that if $\beta > 0$, then the nonwandering set, say, fixed points, all the bifurcating periodic orbits near $(1,1)$, all the bounded chaotic motion of S_β are contained in the square V. Specifically, for $0 < \beta < 2$, all the created periodic orbits are contained in V, and for $2 \le \beta \le 2.15738$, the bounded chaotic orbit of S_β which is semi-conjugate to the 2-shift, is also confined in V, and for $\beta > 2.15738$, there is also in V a compact invariant Cantor set of S_β in which the chaotic motion of points is conjugate to the 2-shift.

Speaking in terms of static solutions of our global market dynamics (by taking y-components of an orbit of a point under S_β), as β increases from 0 until $\beta \approx 2.15738$, in the infinite strip between $\phi_j = 1/4$ and $\phi_j = 1 + 1/2\beta$, our map S_β undergoes a successive bifurcation of periodic orbits of any period and for $2 \le \beta \le 2.15738$, there also exist a compact invariant set of values of static solutions whose motion along the spatial coordinates j is semi-conjugate to the 2-shift. For $\beta > 2.15738$, there is no local bifurcation for S_β and the motion of the static solutions along the spatial coordinates j is topologically conjugate to the 2-shift.

Chaotic orbits can occur even near the fixed point $(1/4, 1/4)$. Kirchgraber and Stoffer ([21]) have proved that:
(i) for the orientation and area preserving Hénon map H_a (i.e., $b = -1$), a transversal homoclinic orbit exists for $a \ge \frac{17}{64} = 0.265625$ (i.e., $\beta \ge \frac{3}{4} = 0.75$ for our map S_β). And with computer assistance they also have shown that:
(ii) this result holds even for $a \ge -0.866360$ (i.e., $\beta \ge 0.243712$ for our map S_β).
Kirchgraber and Stoffer's result supports an old conjecture proposed by Devaney and Nitecki that the Hénon map H_a admits a transversal homoclinic point for $a > -1$ (i.e., $\beta > 0$ for our case). Now let us rewrite the Kirchgraber and Stoffer's results in terms of our map S_β in the following lemma.

Lemma 3.4. *For $\beta \ge 0.243712$, our map S_β admits a transversal homoclinic orbit to the hyperbolic fixed point $Q_1 = (\frac{1}{4}, \frac{1}{4})$. Consequently, near the hyperbolic fixed point $Q_1 = (\frac{1}{4}, \frac{1}{4})$, there exists an invariant Cantor set on which the dynamics of S_β is topologically conjugate to a 2-shift and so is chaotic.*

Now, combining the results of the above four Lemmas, we can state the following Theorem.

Theorem 3.5. *The static solutions $p_j(n) = \phi_j$ satisfy the 2nd order nonlinear difference equation given by (3.5):*

$$\phi_{j+1} = -\beta - \phi_{j-1} + (2 + 5\beta)\phi_j - 4\beta\phi_j^2,$$

where $\beta = \alpha/\varepsilon > 0$. Equation (3.5) *has two fixed points (i.e., the spatially homogeneous static solutions) $\phi_j = \frac{1}{4}$ and $\phi_j = 1$ for $0 < \alpha \leq 1$, and $\phi_j = \phi_0 \geq 0$ for $\alpha = 0$, for all $\varepsilon > 0$. The static solutions of* (3.5) *exhibit the following local and global dynamics:*

(i) *If $0 < \beta \leq 2.15738$, then the static solution $\phi_j = 1$ undergoes a successive bifurcation of periodic orbits of any period in the infinite strip $I = \{\{\phi_j\} | \frac{1}{4} \leq \phi_j \leq 1 + \frac{1}{2\beta}, j \in \mathbf{Z}\}$.*

(ii) *If $\beta > 2.15738$, then there exist compact invariant Cantor sets of real numbers on which the motions of static solutions are chaotic and are conjugate to the 2-shift.*

(iii) *If $\beta \geq 0.243712$, then in the strip I, there also exist "homoclinic" static solutions which converge to the spatially homogeneous solution $\phi_j = \frac{1}{4}$ as $j \to \pm\infty$ and compact invariant Cantor sets of real numbers on which the motions of static solutions along the spacial coordinates j are topologically conjugate to the 2-shift.*

Therefore, according to Theorem 3.5, the bounded static solutions are as follows:
(i)the static spatially homogeneous solutions, i.e., $\{\phi_j\} = \{\frac{1}{4}\}$ and $\{\phi_j\} = \{1\}$ for $0 < \alpha \leq 1$, and $\{\phi_j\} = \{\phi_0\}$ for $\alpha = 0$.

(ii) the infinitely many spatially-periodic static solutions with any period created through the successive bifurcation of periodic orbits near the fixed point $(1, 1)$ for the map S_β.

(iii) the bounded spatially-chaotic static solutions created right after the successive bifurcation of periodic orbits.

(iv) the homoclinic static solutions which converge to the spatially homogeneous solution $\phi_j = \frac{1}{4}$ as $j \to \pm\infty$.

(v) the bounded spatially-chaotic static solutions created due to the transversal homoclinic orbits to the hyperbolic fixed point $(1/4, 1/4)$ for the map S_β.

APPENDIX

Proof of Lemma 3.1. By using the relationship $a = \frac{1}{4}(9\beta^2 - 4)$ and $b = -1$, we can immediately check that the commutativity relation $h \circ S_\beta(x, y) = H_{a,b} \circ h(x, y)$ holds. □

Proof of Lemma 3.3. By Lemma 3.1, our map $S_\beta(x, y) = (y, -\beta - x + (2+5\beta)y - 4\beta y^2)$ is transformed to the Hénon map $H_{a,b}(X, Y) = (a + bY - X^2, X)$ with $a = (\frac{3}{2}\beta)^2 - 1$ and $b = -1$ by the change of coordinates $(X, Y) = h(x, y) = (c_1 y + c_2, c_1 x + c_2)$ with $c_1 = 4\beta$ and $c_2 = -\frac{1}{2}(5\beta + 2)$. In the case $b = -1$, the three crucial a-values a_0, a_1,

a_2 and the radius $R(a)$ become $a_0 = -1, a_1 = 8, a_2 = 5+2\sqrt{5}$ and $R = 1+\sqrt{1+a}$ respectively. Also using the parameter relation $\beta = \frac{2}{3}\sqrt{1+a}$, these values correspond to $\beta_0 = 0, \beta_1 = 2, \beta_2 = \frac{2}{3}(1+\sqrt{5})$ and $R(\beta) = 1 + \frac{3}{2}\beta$ respectively. Moreover, using the change of coordinates $(X, Y) = h(x, y)$, the X-side of the rectangle $|X| \leq R(a)$ is transformed to $-1 - \frac{3}{2}\beta \leq 4\beta y - \frac{1}{2}(5\beta + 2) \leq 1 + \frac{3}{2}\beta$. This gives $\frac{1}{4} \leq y \leq 1 + \frac{1}{2\beta}$. Similarly for the $|Y| \leq R(a)$. □

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