C^1-STABLE INVERSE SHADOWING CHAIN COMPONENTS FOR GENERIC DIFFEOMORPHISMS

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ABSTRACT. Let f be a diffeomorphism of a compact C^∞ manifold, and let p be a hyperbolic periodic point of f. In this paper we introduce the notion of C^1-stable inverse shadowing for a closed f-invariant set, and prove that (i) the chain recurrent set $\mathcal{R}(f)$ of f has C^1-stable inverse shadowing property if and only if f satisfies both Axiom A and no-cycle condition, (ii) C^1-generically, the chain component $C_f(p)$ of f associated to p is hyperbolic if and only if $C_f(p)$ has the C^1-stable inverse shadowing property.

1. Introduction

It has been a problem in differentiable dynamical systems during last decades to understand the influence of a robust dynamic property (i.e., a property that holds for a system and all nearby ones) on the behavior of the tangent map of the system. For instance, K. Lee, K. Moriyasu, and K. Sakai [9] proved that for the chain component $C_f(p)$ of f containing a hyperbolic periodic point p, $C_f(p)$ has the C^1-stable shadowing property if and only if $C_f(p)$ is the hyperbolic homoclinic class p. Moreover K. Sakai [16] proved that if $C_f(p)$ is C^1-stably shadowable and the $C_f(p)$-germ of f is expansive then $C_f(p)$ is hyperbolic, and K. Lee and M. Lee [8] showed that the homoclinic class $H_f(p)$ of f containing a hyperbolic periodic point p, $H_f(p)$ is C^1-stably expansive if and only if $H_f(p)$ is hyperbolic.

In this paper, we introduce the notion of C^1-stable inverse shadowing property and, study the case when the chain component $C_f(p)$ of f associated to a hyperbolic periodic point p has the C^1-stable inverse shadowing property.

Let us be more precise. Let X be a compact metric space with metric d, and let $Z(X)$ denote the space of homeomorphisms on X with the C^0-metric d_0. Let $f \in Z(X)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ in X is called a δ-*pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b - 1$. For given $x, y \in M$, we

write $x \looparrowright y$ if for any $\delta > 0$, there is a δ-pseudo-orbit $\{x_i\}_{i=a}^b (a < b)$ of f such that $x_a = x$ and $x_b = y$. The set of points $\{x \in M : x \looparrowright x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. It is easy to see that the set is closed and $f(\mathcal{R}(f)) = \mathcal{R}(f)$. If we denote the set of periodic points of f by $P(f)$, then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is the non-wondering set of f. The relation $\looparrowright$ induces on $\mathcal{R}(f)$ an equivalence relation in $\mathcal{R}(f)$, whose classes are called *the chain components* of f. Let $\Lambda \subset X$ be a closed f-invariant set. We say that $f|_\Lambda$ has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^b \subset \Lambda$ of $f(-\infty \le a < b \le \infty)$, there is $y \in M$, f-ϵ-shadowing the pseudo-orbit; i.e., $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b - 1$. This property does not depend on the metric used and is preserved under topological conjugacy. Note that $f|_\Lambda$ has the shadowing property if and only if $f^n|_\Lambda$ has the shadowing property for any $n \in \mathbb{Z} \setminus \{0\}$.

Let $X^{\mathbb{Z}}$ be the compact metric space of all two sided sequences $\xi = \{x_k : k \in \mathbb{Z}\}$ in X, endowed with the product topology. For a constant $\delta > 0$ and $f \in Z(X)$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f.

A mapping $\varphi : X \to \Phi_f(\delta) \subset X^{\mathbb{Z}}$ satisfying $\varphi(x)_0 = x$, $x \in X$, is said to be a δ-*method* for f. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k \in \mathbb{Z}}$. Say that φ is *continuous δ-method* for f if φ is continuous. The set of all δ-methods [respectively, continuous δ-methods] for f will be denoted by $\mathcal{T}_0(f, \delta)$ [respectively, $\mathcal{T}_c(f, \delta)$]. Every $g \in Z(X)$ with $d_0(f, g) < \delta$ induces a continuous δ-method $\varphi_g : X \to X^{\mathbb{Z}}$ for f by defining $\varphi_g(x) = \{g^k(x) : k \in \mathbb{Z}\}$. Let $\mathcal{T}_h(f, \delta)$ denote the set of all continuous δ-methods for f which are induced by homeomorphisms g on X with $d_0(f, g) < \delta$. Define $\mathcal{P}_\alpha(f, \delta)$ by

$$\mathcal{P}_\alpha(f, \delta) = \bigcup_{\varphi \in \mathcal{T}_\alpha(f, \delta)} \varphi(X)$$

for $\alpha = 0, c, h, d$. Clearly we have

$$\mathcal{P}_h(f, \delta) \subset \mathcal{P}_c(f, \delta) \subset \mathcal{P}_0(f, \delta) = \Phi_f(\delta).$$

Let $\Lambda \subset X$ be a closed f-invariant set. We say that $f|_\Lambda$ has the *inverse shadowing property* with respect to a class $\mathcal{T}_\alpha, (\mathcal{T}_\alpha$–I.S) if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-method $\varphi \in \mathcal{T}_\alpha(f, \delta)$, there is a map $s : \Lambda \to M$ satisfying $d(f^n(x), \varphi(s(x))_n) < \epsilon$ for all $x \in \Lambda$ and all $n \in \mathbb{Z}$. Clearly we have the following relations among the various notions of inverse shadowing.

$$\mathcal{T}_0 - \text{I.S} \Rightarrow \mathcal{T}_c - \text{I.S} \Rightarrow \mathcal{T}_h - \text{I.S} \Rightarrow \mathcal{T}_d - \text{I.S}.$$

where I.S denotes the inverse shadowing property.

Let M be a closed C^∞ manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \text{Diff}(M)$, and let $\Lambda \subset M$ be a closed f-invariant set. We say that Λ is

locally maximal if there is a compact neighborhood U of Λ such that

$$\bigcap_{n \in \mathbb{Z}} f^n(U) = \Lambda.$$

Definition 1. We say that an f-invariant set Λ has the C^1-*stable inverse shadowing property* if Λ is locally maximal in U and there is a C^1-neighborhood of $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g}$ has the inverse shadowing property with respect to the class $\mathcal{T}_c$. Here $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$ and which is called the *continuation* of $\Lambda_f = \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$.

We say simply that f has the C^1-*stable inverse shadowing property* if $\Lambda = M$ in the above definition. S. Pilyugin [12] proved that a structurally stable diffeomorphism has the inverse shadowing property and any diffeomorphism belonging to the C^1-interior of the set of diffeomorphisms with inverse shadowing property is structurally stable.

Thus we can restate the above facts as follows.

Theorem 1. *A diffeomorphism f has the C^1-stable inverse shadowing property if and only if f satisfies both Axiom A and the strong transversality condition.*

Recently, two C^1-open generic sets were introduced in [1] which are *tame* diffeomorphisms and *wild* diffeomorphisms, that is, first tame diffeomorphisms have a finite number of homoclinic classes and whose non-wondering sets admit partitions into a finite number of disjoint transitive sets and wild diffeomorphisms have an infinite number of (disjoint different) homoclinic classes. In fact in [1], they proved that there is a residual set $\mathcal{R} \subset \text{Diff}(M)$ such that if $f \in \mathcal{R}$ is tame, then the following two conditions are equivalent: (a) f satisfies both Axiom A and the no-cycle condition, (b) f has the shadowing property. In [9], the authors showed that if the chain component $C_f(p)$ of f containing a hyperbolic periodic point p has the C^1-stable shadowing property then it is hyperbolic homoclinic class of p.

Next we pass to the chain recurrent set $\mathcal{R}(f)$ of $f \in \text{Diff}(M)$. In [6], Hurley proved that the map $f \mapsto \mathcal{R}(f)$ is upper semi continuous. More precisely, for any neighborhood U of $\mathcal{R}(f)$, there is $\delta > 0$ such that if $d_0(f, g) < \delta$ ($g \in \text{Diff}(M)$), then $\mathcal{R}(g) \subset U$. Here d_0 is the usual C^0-metric on $\text{Diff}(M)$. Thus we can prove the following result of this paper based on the facts in [5].

Theorem 2. *The chain recurrent set $\mathcal{R}(f)$ of f has the C^1-stable inverse shadowing property if and only if f satisfies both Axiom A and the no-cycle condition.*

Let f satisfy Axiom A. Then it well-known that $\Omega(f) = \mathcal{R}(f)$ if and only if f satisfies the no-cycle condition. Hence the C^1-stable inverse shadowing property on $\mathcal{R}(f)$ is characterized as the Ω-stability of the system by Theorem 2.

For dynamical systems satisfying Axiom A, the hyperbolic basic set is a really basic subsystem possessing lots of important dynamical properties and investigated very well in view of stability theory and ergodic theory. As stated

before, in the shadowing theory of dynamical systems, chain components are the natural candidates to replace hyperbolic basic sets. In fact, in the C^1-generic context, every chain component with a periodic point is a homoclinic class of the periodic point (see [2]).

The main purpose of this paper is to characterize chain components containing a hyperbolic periodic point by making use of the inverse shadowing property under the C^1-open condition. To be more precise, let $p \in P(f)$ be a hyperbolic saddle with the prime period $\pi(p) > 0$; that is, there is no eigenvalues of $D_p f^{\pi(p)}$ with modulus equal to 1, at least one of them greater than 1, at least one of them is smaller than 1. Remark that there are a neighborhood U of p and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exists a unique hyperbolic periodic point $p_g \in U$ of g with same period as p and index(p_g)=index(p). Here index(p) is the *index* of p, namely, the dimension of the stable eigenspace E_p^s of p. Such the point p_g is called the *continuation* of p.

Denoted by $C_f(p)$ the chain component of f containing p. If p is a sink or a source periodic point, then $C_f(p)$ is the periodic orbit itself.

It is well known that if p is a hyperbolic periodic point of f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \quad \text{and}$$

$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. Then $W^s(p, f)$ is called *stable manifolds* and $W^u(p, f)$ is called *unstable manifold* of p with respect to f.

A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of f associated to p, and it is said to be a *transversal homoclinic point* of f if the above intersection is transversal at x; i.e., $x \in W^s(p) \pitchfork W^u(p)$. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. The closure of the transversal homoclinic points of f associated to p is called the *transversal homoclinic class* of f associated to p, and it is denoted by $H_f^T(p)$. It is clear that both $H_f(p)$ and $H_f^T(p)$ are compact f-invariant sets. Homoclinic classes are the natural candidates to replace hyperbolic basic sets in nonhyperbolic theory. Several recent papers ([8, 9, 16]) explore their "hyperbolic-like" properties, many of which hold only for generic diffeomorphisms.

Let q be a hyperbolic periodic point of f. We say that p and q are *homoclinic related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \phi \text{ and } W^u(p) \pitchfork W^s(q) \neq \phi.$$

It is clear that if $p \sim q$ then index(p) = index(q); i.e., $\dim W^s(p) = \dim W^s(q)$. By the Smale's transverse homoclinic point theorem, $H_f^T(p)$ coincides with the closure of the set of hyperbolic periodic points q of f such that $p \sim q$. When a homoclinic class is not hyperbolic, it may contain periodic points having different indices. Actually, there are examples of diffeomorphisms with homoclinic classes containing hyperbolic periodic points with different indices in a robust

way, for instance [8, 9, 16]. These systems do not have the inverse shadowing property in general. Such dynamical phenomena easily give pseudo orbit which cannot be inverse shadowable. In this paper we prove the hyperbolicity of chain components containing a hyperbolic saddle periodic point under the C^1-open condition on the inverse shadowing property by ruling out heterodimensional cycles.

Denote by $\mathcal{O}_f(p)$ the periodic f-orbit of p, and set $H_f(\mathcal{O}_f(p)) = H_f(p) \cup \cdots \cup H_f(f^{\pi(p)-1})(p))$. Then, $H_f(\mathcal{O}_f(p)) \subset C_f(p)$ but these sets do not coincide in general (see [16]). Obviously, $H_f(\mathcal{O}_f(p))$ is a closed f-invariant set, and it is known that $f|_{H_f(\mathcal{O}_f(p))}$ is transitive. Observe that if $q \in H_f(\mathcal{O}_f(p)) \cap P(f)$ is hyperbolic, then q is neither sink or source.

Let $\Lambda \subset M$ be a f-invariant closed set. Recall that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and constant $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E_x^s}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E_x^u}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

Given an open subset $\mathcal{U}$ of Diff(M), a subset $\mathcal{R}$ of $\mathcal{U}$ is said to be *residual* in $\mathcal{U}$ if $\mathcal{R}$ contains a countable intersection of open and dense subsets of $\mathcal{U}$. If P is a property of $f \in \mathcal{U}$, we say that this property is *generic* in $\mathcal{U}$ if $\{f \in \mathcal{U} : f$ satisfies $P\}$ is residual in $\mathcal{U}$. From [2], we know that generically $H_f^T(p) = C_f(p)$ for any hyperbolic saddle point $p \in M$. Let $p \in P(f)$ be as before, and let $C_f(p)$ be the chain component of f containing p. Then we get the following result which is the main theorem of this paper.

Theorem 3. *Generically, the chain component $C_f(p)$ has the C^1-stable inverse shadowing property if and only if $C_f(p)$ is the hyperbolic homoclinic class of p.*

2. Proof of the Theorem 2

Let (X, d) be as before, and for $\epsilon > 0$ we denote by $B_\epsilon(A)$ the closed ϵ-ball $\{x \in X : d(x, A) \leq \epsilon\}$ of a subset A of X. Denote by $Z(X)$ the set of homeomorphisms of X with the usual C^0-metric d_0. For the proof of the following lemma, refer [6, 9].

Lemma 2.1. *Let $f \in Z(X)$, and let $\mathcal{R}(f)$ be the chain recurrent set of f. For any $\epsilon > 0$, there is $\delta > 0$ such that if $d_0(f, g) < \delta$ $(g \in Z(X))$, then $\mathcal{R}(g) \subset B_\epsilon(\mathcal{R}(f))$.*

Let M be as before, and let $f \in \text{Diff}(M)$. The following so-called Franks' lemma will play essential roles in our proofs.

Lemma 2.2. *Let $\mathcal{U}(f)$ be any given C^1-neighborhood of f. Then there exists $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_N\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \leq \epsilon$ for all $1 \leq i \leq N$, there*

exists $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \ldots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \leq i \leq N$.

Proof. See [4, Lemma 1.1]. □

Remark 1. Note that the identity map *id* of the unit interval I and a rotation map ρ of the unit circle S^1 do not have the inverse shadowing property.

End of the proof of Theorem 2. First we suppose that f satisfies both Axiom A and no-cycle condition. Then $\mathcal{R}(f) = \Omega(f) = \overline{P(f)}$ is hyperbolic. And so $\mathcal{R}(f)$ is locally maximal; i.e., there exists a compact neighborhood U of $\mathcal{R}(f)$ such that

$$\Omega(f) = \mathcal{R}(f) = \bigcap_{n \in \mathbb{Z}} g^n(U).$$

By the stability of locally maximal hyperbolic sets, we can take a compact neighborhood U of $\mathcal{R}(f)$ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $f|_{\mathcal{R}(f)}$ and $g|_{\Lambda_g}$ are conjugate, where $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$. Since $\mathcal{R}(f)$ is hyperbolic for f, $f|_{\mathcal{R}(f)}$ has the C^1-stable inverse shadowing property. Thus $g|_{\Lambda_g}$ has the inverse shadowing property.

Next we suppose that $\mathcal{R}(f)$ has the C^1-stable inverse shadowing property. Then there are a compact neighborhood U of $\mathcal{R}(f)$ and a C^1-neighborhood $\mathcal{U}(f)$ of f and such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g}$ has the inverse shadowing property, where $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$. Choose $\epsilon > 0$ satisfying $B_\epsilon(\mathcal{R}(f)) \subset U$. By Lemma 2.1, there is $\delta > 0$ such that if $d_1(f, g) < \delta$ for $g \in \mathcal{U}(f)$, then

$$\mathcal{R}(g) \subset B_\epsilon(\mathcal{R}(f)) \subset U,$$

where d_1 is the usual C^1-metric on Diff(M). Put $\mathcal{U}_0(f) = \{g \in \mathcal{U}(f) : d_1(f, g) < \delta\}$. Then for each $g \in \mathcal{U}_0(f)$, $\mathcal{R}(g) \subset U$ and so $\mathcal{R}(g) \subset g^n(U)$ for all $n \in \mathbb{Z}$. This means that $\mathcal{R}(g) \subset \bigcap_{n \in \mathbb{Z}} g^n(U) = \Lambda_g$ for $g \in \mathcal{U}_0(f)$. And so $g|_{\mathcal{R}(g)}$ has the inverse shadowing property.

Denote by $\mathcal{F}(M)$ the set of $f \in \text{Diff}(M)$ such that there is a C^1-neighborhood $\mathcal{U}(f)$ of f with property that every $p \in P(g)(g \in \mathcal{U}(f))$ is hyperbolic. It is proved by Hayashi [5] that $f \in \mathcal{F}(M)$ if and only if f satisfies both Axiom A and no-cycle condition. Therefore, to complete the proof of the theorem it is enough to show that if $\mathcal{R}(f)$ has the C^1-stable inverse shadowing property then $f \in \mathcal{F}(M)$.

Let $\epsilon > 0$ and $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ be the corresponding number and C^1-neighborhood given by Lemma 2.2 with respect to $\mathcal{U}(f)$. Suppose that there exists a non-hyperbolic periodic point $q \in P(g)$ for some $g \in \mathcal{U}_0(f)$. To simplify notation in this proof, we assume that $g(q) = q$ (other case is similar). Then by making use of Lemma 2.2 we linearize g at q with respect to the exponential coordinates $\exp_q$; that is, by choosing $\alpha > 0$ small enough we construct g_1 C^1-nearby g such that

$$g_1(x) = \begin{cases} \exp_q \circ D_q g \circ \exp_q^{-1}(x) & \text{if } x \in B_{\alpha/4}(q), \\ g(x) & \text{if } x \notin B_\alpha(q). \end{cases}$$

Clearly, $g_1(q) = q$. Moreover, if necessary, by slightly modifying $D_q g$ we may assume further that there is an eigenvalue λ with multiplicity one such that $|\lambda| = 1$ (of course, such a modification is done by a linear map whose distance from $D_q g$ is within ϵ). Thus there is a g_1-invariant normally hyperbolic small arc $\mathcal{I}_q$ center at q (resp. a g_1-invariant normally hyperbolic small circle $\mathcal{C}_q$ with a small diameter center at q) such that $g_1{}^k|_{\mathcal{I}_q} = id$ for some $k > 0$ (resp. $g_{1_{\mathcal{C}_q}}$ is conjugated to an irrational rotation map) if λ is real (resp. complex). It is easy to see that both $\mathcal{I}_q$ and $\mathcal{C}_q$ are contained in $\mathcal{R}(g_1)$. Since $g_1|_{\mathcal{R}(g_1)}$ has the inverse shadowing property for $g_1 \in \mathcal{U}_0(f)$, both $g_1^k|_{\mathcal{I}_q}$ and $g_1|_{\mathcal{C}_q}$ have the inverse shadowing property. But by the Remark 1, this is a contradiction. This completes the proof of Theorem 2.

3. Proof Theorem 3

Above all, we prove that "if" part; i.e., if $C_f(p)$ is hyperbolic then it has the C^1-stable inverse shadowing property. From Proposition 9.1 in [16], if $C_f(p)$ is hyperbolic then it is locally maximal(in U). Thus, by the local stability of a hyperbolic set, there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, and $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is hyperbolic. And so, $g|_{\Lambda_g}$ has the inverse shadowing property. This means $C_f(p)$ has the C^1-stable inverse shadowing property for any $g \in \mathcal{U}(f)$ so that "if" part is proved.

Next, we will prove "only if" part by making use of the techniques developed by Mañé in [10]. Hereafter in this section, let $f \in \text{Diff}(M)$ and $p \in P(f)$ be hyperbolic. For the sake of simplicity, in the following results and the proofs we restrict our attentions to a hyperbolic fixed point $f(p) = p$. Of course, we can prove the same results for any hyperbolic periodic point.

Lemma 3.1 ([2]). *There exists a residual set $\mathcal{R} \subset \text{Diff}(M)$ such that for any $f \in \mathcal{R}$, $C_f(p) = H_f^T(p)$, where p is a hyperbolic periodic point of f.*

Suppose that a closed f-invariant set Λ has the C^1-stable inverse shadowing property. Then we can choose a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g}$ has the inverse shadowing property.

Lemma 3.2. *Suppose that Λ has the C^1-stable inverse shadowing property for f. Then there exists a C^1-neighborhood $\mathcal{V}(f) \subset \mathcal{U}(f)$ of f such that for each $g \in \mathcal{V}(f)$, every $q \in \Lambda_g \cap P(g)$ is hyperbolic, where $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$ for a small neighborhood U of Λ.*

Proof. Choose $\epsilon > 0$ and $\mathcal{U}_0(f)$ (by Lemma 2.2) corresponding to $\mathcal{U}(f)$. Suppose there is $g \in \mathcal{U}_0(f)$ and p such that p is non-hyperbolic periodic point of g. To simplify the notation in the proof, assume $g(q) = q$. Then there is at least one eigenvalue λ of $D_q g$ such that $|\lambda| = 1$, where $\lambda \in \mathbb{R}$ or $\mathbb{C}$. We can take a linear isomorphism $L : T_p M \to T_{g(q)} M(= T_q M)$ such that

  (i) $\|L - D_q g\| < \epsilon$

(ii) $L = \begin{pmatrix} A & O & O \\ O & B & O \\ O & O & C \end{pmatrix}$ with respect to some splitting

$$T_q M = E_q^c \oplus E_q^s \oplus E_q^u,$$

where $A : E_q^c \to E_q^c$ has eigenvalue $|\lambda| = 1$ such that $\dim E_q^c = 1$ if $\lambda \in \mathbb{R}$ or $\dim E_q^c = 2$ if $\lambda \in \mathbb{C}$, $B : E_q^s \to E_q^s$ is contraction, and $C : E_q^u \to E_q^u$ is expanding.

(iii) There exists $m > 0$ such that $L^m|_{E_q^c} = id$ on E_q^c. By Lemma 2.2, there exists $\alpha > 0$ with $B_{4\alpha}(q) \subset U$ and there exists $g \in \mathcal{U}_0(f)$ such that

$$g_1(x) = \begin{cases} \exp_q \circ L \circ \exp_q^{-1}(x) & \text{if } x \in B_\alpha(q), \\ g(x) & \text{if } x \notin B_{4\alpha}(q). \end{cases}$$

Then $g_1(q) = g(q) = q$. By the definition of exponential map, there is $0 < \delta \leq \alpha$ such that

$$\exp_q : T_q M(\delta) \to B_\delta(q)$$

is a diffeomorphism. Choose $0 < \delta_1 < \delta$ such that

$$g_1^m|_{\exp_q(T_q M(\delta_1))} = \exp_q \circ L^m \circ \exp_q^{-1}|_{\exp_q(T_q M(\delta_1))}.$$

Put $E_q^c(\delta_1) = E_q^c \cap T_q M(\delta_1)$. Then $L^m|_{E_q^c} = id$ on $E_q^c(\delta_1)$. Note that $\exp_q(E_q(\delta_1))$ is a small arc $\mathcal{I}_q$ or a small disk $\mathcal{C}_q$, and $\mathcal{I}_q \subset U$ and $\mathcal{C}_q \subset U$. We know that $g^m|_{\mathcal{I}_q} = id$ and $g|_{\mathcal{C}_q}$ is conjugate to irrational map. Since $g_1|_{\Lambda_g}$ has the inverse shadowing property, $g_1^m|_{\mathcal{I}_p}$ and $g_1|_{\mathcal{C}_q}$ must have the inverse shadowing property. But by Remark 1, this is a contradiction.   □

Let $f \in \text{Diff}(M)$. We say that a compact f-invariant set $\Lambda \subset M$ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E \oplus F$ and there exist constants $C > 0, 0 < \lambda < 1$ such that

$$\|D_x f^n|_{E_x}\| \cdot \|D_{f^n(x)} f^{-n}|_{F_{f^n(x)}}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ such that $\dim E_x \ (x \in \Lambda)$ is constant, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood V of Λ such that for each $g \in \mathcal{U}(f), \bigcap_{n \in \mathbb{Z}} g^n(V)$ admits a dominated splitting

$$T_{\bigcap_{n \in \mathbb{Z}} g^n(V)} M = \widetilde{E}(g) \oplus \widetilde{F}(g)$$

with $\dim \widetilde{E}(g) = \dim E$.

Suppose that $C_f(p)$ has the inverse shadowing property; i.e., there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of $C_f(p)$ such that for any $g \in \mathcal{U}(f), g|_{\Lambda_g}$ has the inverse shadowing property. By Lemma 3.1 and the Smale's transverse homoclinic point theorem, the set of hyperbolic periodic point $q \in P(f)$ which is homoclinically related to p is dense in $H_f^T(p) = C_f(p)$. Moreover, by Lemma 3.2, the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by $Dg(g \in \mathcal{U}_0(f))$ along the hyperbolic periodic points $q \in \Lambda_g \cap P(g)$ is uniformly hyperbolic (see [10]). That is, there exists $\epsilon > 0$ such that for any $g \in \mathcal{U}_0(f), q \in \Lambda_g \cap P(g)$, and any

sequence of linear maps $L_i : T_{g^i(q)}M \to T_{g^{i+1}(q)}M$ with $\|L_i - D_{g^i(q)}g\| < \epsilon (i = 1, 2, \ldots, \pi(q)-1)$, $\prod_{i=0}^{\pi(q)-1} L_i$ is hyperbolic. Here $\mathcal{U}_0(f)$ is the C^1-neighborhood of f given by Lemma 3.2. with respect to $\mathcal{U}(f)$. Thus we have the following result.

Proposition 3.3 ([10]). *Suppose that the chain component $C_f(p)$ has the C^1-stable inverse shadowing property, and let $\mathcal{V}(f)$ as the Lemma* 2.1. *Then there are constants $C > 0$, $0 < \lambda < 1$ and $m > 0$ such that*

(a) *for any $g \in \mathcal{V}(f)$, if $q \in \Lambda_g \cap P(g)$ has minimum period $\pi(q) \geq m$, then*

$$\prod_{i=0}^{k-1} \|D_{g^{im}(q)}g^m|_{E^s_{g^{im}(q)}}\| < C\lambda^k \ and \ \prod_{i=0}^{k-1} \|D_{g^{-im}(q)}g^m|_{E^u_{g^{-im}(q)}}\| < C\lambda^k$$

where $k = [\pi(q)/m]$.

(b) *$C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E \oplus F$ with $\dim E = \mathrm{index}(p)$.*

(c) *if $q \in C_f(p) \cap P(f)$, then*

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{f^{im}(q)}f^m|_{E^s_{f^{im}(q)}}\| < 0, \ and$$

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{f^{-im}(q)}f^{-m}|_{E^u_{f^{-im}(q)}}\| < 0.$$

By Lemma 3.1, C^1-generically $H^T_f(p) = C_f(p)$, and so, we will use the result of [10] to show the hyperbolicity of $C_f(p)$. We denote the index(p) by $j(0 < j < \dim M)$ and let $P_i(f|_{H^T_f(p)}) = \tilde{\Lambda}_i$ be the set of periodic points $q \in H^T_f(p) \cap P(f) = C_f(p) \cap P(f)$ such that index$(q)= i$ for all $0 < i < \dim M$. Finally, set

$$\Lambda_i(f) = \overline{P_i(f|_{H^T_f(p)})} = \overline{P(f|_{C_f(p)})} = \overline{\tilde{\Lambda}}_i.$$

Recall that, in general, a non-hyperbolic homoclinic class $H^T_f(p)$ contains saddle periodic points with different indices. Thus the chain component $C_f(p)$ may contain saddle periodic points with different indices in general. However, if $C_f(p)$ has C^1-stable inverse shadowing property, then such cases cannot happen. And so, we have the following result.

Proposition 3.4. *Suppose that C^1-generically, $C_f(p)$ has the C^1-stable inverse shadowing property. Then for any $q \in C_f(p) \cap P(f)$, index$(q)=$index(p).*

Proof. Suppose that $C_f(p)$ has the C^1-stable inverse shadowing property. Let $\mathcal{U}(f)$ be a C^1-neighborhood of f, and let U be a compact neighborhood of $C_f(p)$ as in the property. Then by Lemma 3.1, generically, $C_f(p) = H^T_f(p)$, and so $C_f(p) = \Lambda_j(f) = P_j(f|_{H^T_f(p)})$, where $j = \mathrm{index}(p)$. Suppose that there exists $q \in C_f(p) \cap P(f)$ such that index$(q) \neq$ index(p). Since $q \in C_f(p) \cap P(f)$,

we know that $q \rightsquigarrow p$ and $p \rightsquigarrow q$, i.e., $q \leftrightsquigarrow p$. Since both p and q are hyperbolic by Lemma 3.2, it is not hard to show that there is a heterodimensional cycle between p and q in $C_f(p)$ by making use of the inverse shadowing property of $f|_{C_f(p)}$. More precisely, there are $x \in W^u(q) \cap W^s(p)$ and $y \in W^s(q) \cap W^u(p)$ such that

$$\dim W^u(q) + \dim W^s(p) < \dim M \quad \text{or} \quad \dim W^s(q) + \dim W^u(p) < \dim M.$$

Since $x \in W^u(q) \cap W^s(p)$ and $y \in W^s(q) \cap W^u(p)$, we know that $x \in C_f(p)$ and $y \in C_f(p)$.
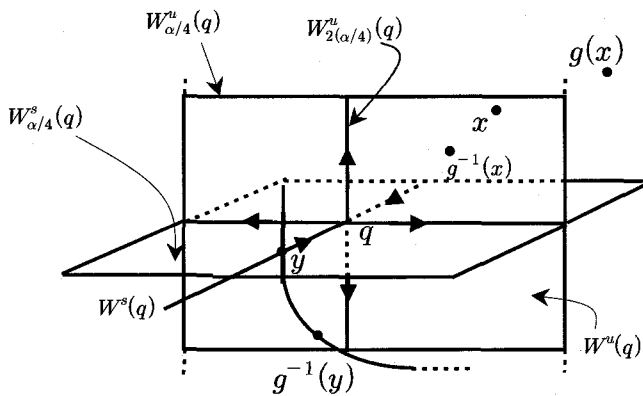

FIGURE 1. non-transverse intersection point

We consider the $\dim W^s(p) + \dim W^u(p) < \dim M$ (other case is similar); that is,

$$\text{index}(q) < \text{index}(p).$$

By the Proposition 3.3(b), there exists a dominated splitting $T_{C_f(p)}M = E \oplus F$ with dim $E =$index(p). Thus, shrinking $\mathcal{U}(f)$ if necessary, for any $g \in \mathcal{U}(f)$, we may assume that $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$ admits a dominated splitting $T_{\bigcap_{n \in \mathbb{Z}} g^n(U)}M = \widetilde{E}(g) \oplus \widetilde{F}(g)$ with dim $\widetilde{E}(g) =$ index(p).

To get a contradiction, we are going show that there exists $\epsilon > 0$ such that for any $\delta > 0$ there are a δ-method $\varphi \in \Phi_f(\delta)$ and $x \in C_f(p)$ such that if for any $y \in M$ then $d(f^k(x), \varphi(y)_k) \geq \epsilon$ for some $k \in \mathbb{Z}$. For the sake of simplicity, we suppose $f(q) = q$ (general case is similar). Note that y is a non-transverse intersection point between $W^s(q)$ and $W^u(p)$. A method $\varphi \in \Phi_f(\delta)$ is complete if for any $p \in M$ there exists a δ-pseudo orbit $\xi = \{x_i\}_{i \in \mathbb{Z}} \in \Phi_g(\delta)$ such that $p = x_0$ (see [3, 12]). We may assume that $x \notin W^u_{2(\alpha/4)}(q)$. Take $\epsilon = d(x, W^u_{2(\alpha/4)}(q))/2 > 0$. Let $0 < \delta \leq \epsilon$ be a constant which can be selected by the inverse shadowing property of $g|_{\Lambda_g}$.

FIGURE 2. construction of δ-pseudo orbit

Choose k_1, k_2, k_3, and $k_4 > 0$ such that $d(g^{k_4}(y), g^{-k_3}(x)) < \delta$ and $d(g^{k_2}(x), g^{-k_1}(y)) < \delta$. Then we can construct δ-pseudo orbit $\xi = \varphi_g(y) \in \mathcal{T}_c(g, \delta)$. That is,

$$\varphi_g(y) = \{\ldots, g^{-1}(y), y, \ldots, g^{k_4-1}(y), g^{-k_3}(x), \ldots, x, g(x), \ldots, g^{k_2-1}(x), g^{-k_1}(y), \ldots\}$$

is a δ-pseudo orbit of g. Since $g|_{\Lambda_g}$ has the inverse shadowing property, $\varphi_g(y)$ ϵ-shadows $O_f(y)$. Since $y \in W^u(q)$, we have $d(g^n(y), q) \to 0$ as $n \to \infty$. And so, for $z_0 \in B_\epsilon(y)$, we can define a map h by

$$h(z_0) = \begin{cases} g^i(y) & \text{for } -k_1 - 1 \le i \le k_4 - 1, \\ g^i(x) & \text{for } -k_2 \le i \le k_3. \end{cases}$$

Then $d(g, h) < \delta$. Thus $\varphi_g(y)$ is a δ-pseudo orbit for g. If $z_0 = y$, then it is complete, and there exists $l > 0$ such that $d(g^l(y), h^l(y)) \ge \epsilon$. This mean that any point near by y can not be shadowable.

And we can show that if $z_0(= z) \in I_y$ then it is not inverse shadowable. By making use of Lemma 2.2 we linearize f at q with respect to the exponential coordinates $\exp_q$; that is, by choosing $\alpha > 0$ small enough we construct g C^1-nearby f such that

$$g(x) = \begin{cases} \exp_q \circ D_q f \circ \exp_q^{-1}(x) & \text{if } x \in B_{\alpha/4}(q), \\ g(x) & \text{if } x \notin B_\alpha(q). \end{cases}$$

Clearly, $g(q) = q$, and identity the map with its derivative $D_q g$ in a small neighborhood, say, $B_{\alpha/4}(q)$ by assuming that $B_{\alpha/4}(q) \subset \mathbb{R}^{\dim M}$ and q is the origin O. We may suppose that both x and y are in $\text{int} B_{\alpha/4}(q)$ by iterating them by g and g^{-1}, respectively, if necessary. By perturbing the map again at $g^{-1}(y)$ we construct a small arc $I_y \subset B_{\alpha/4}(q)$, which is a neighborhood of y in the unstable manifold $W^u(p_g)$ of p_g. Since $g|_{B_{\alpha/4}(q)}$ is identified with
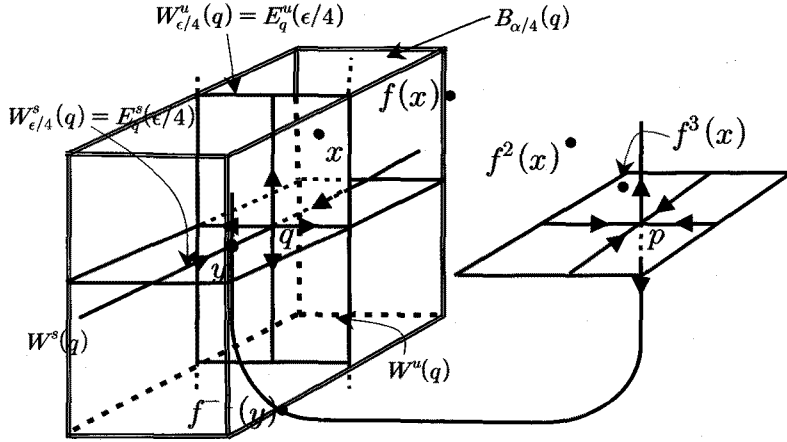
FIGURE 3. small neighborhood q

its linear part $D_q g$ we may suppose that the neighborhood $B_{\alpha/4}(q)$ is of the form $W^s_{\alpha/4}(q) \times W^u_{\alpha/4}(q)$, where $W^s_{\alpha/4}(q)$ and $W^u_{\alpha/4}(q)$ are the local stable and local unstable manifolds of q, respectively. Actually, we can setting $W^s_{\alpha/4}(q) = E^s_q(\alpha/4)$ and $W^u_{\alpha/4}(q) = E^u_q(\alpha/4)$, where $E^s_q(\alpha/4)$ and $E^u_q(\alpha/4)$ are the local linear stable and the local unstable manifolds of q. Put $E^u_1 = \widetilde{E}_q(g) \cap E^u_q$ and $E^u_2 = \widetilde{F}_q(g)$. Then $E^u_q = E^u_1 \oplus E^u_2$. Recall that $\bigcap_{n \in \mathbb{Z}} g^n(U) = \Lambda_g$ admits a dominated splitting $\widetilde{E}(g) \oplus \widetilde{F}(g)$. By the uniqueness of the dominated splitting, we have $T_y I_y = \widetilde{F}_y(g)$, $\widetilde{F}_q(g) \subset E^u_q$ and $E^s_q \subset \widetilde{E}_q(g)$. Let $\epsilon > 0$ be given, and let $0 < \delta \leq \epsilon$ be a number corresponding to the definition of the inverse shadowing property of Λ_g.

By the λ-lemma, we know that I_y accumulated by $E^u_2(q)$; i.e., $I_y \subset g(I_y)$ and $g^n(I_y) \to E^u_2(\alpha/4)$ as $n \to \infty$. Thus we can construct an δ-pseudo orbit $\varphi_\delta \in \Phi_g(\delta)$. Let $z \in B_\epsilon(y) \subset int I_y$ and $z \in W^u(p) \cap W^u_{\alpha/4}(q)$. Then for $z \in B_\epsilon(y)$, we have $g(z) \notin B_\epsilon(y)$. Set $z_1 = \varphi_\delta(z)_1$ and take $z_1 \in B_\epsilon(g(y)) \cap B_\delta(g(z))$ such that $d(g(z), z_1) < \delta$. Since $z \in W^u(p) \cap W^u_{\alpha/4}(q)$ and by definition, $g^n(z) \to p$ as $n \to -\infty$, we can choose a sufficiently large constant $k > 0$ such that $B_\epsilon(g^k(y)) \cap B_\delta(g(z_{k-1})) = \phi$. This contradiction completes the proof. $\square$

Proposition 3.5. *There exists a residual set $\mathcal{R} \subset \mathrm{Diff}(M)$ such that for the chain component $C_f(p)$ of $f \in \mathcal{R}$ containing a hyperbolic periodic saddle p, $C_f(p)$ has C^1-stable inverse shadowing property if and only if $C_f(p)$ is the hyperbolic homoclinic class of p.*
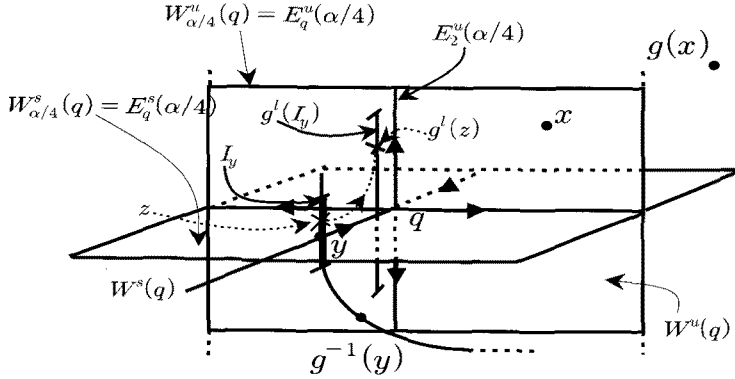
FIGURE 4. small neighborhood of q

Let us recall the Mañé's ergodic closing lemma in (for more detail, see [10]). Denote by $B_\epsilon(f, x)$ an ϵ-tubular neighborhood of f-orbit of x; i.e.,

$$B_\epsilon(f, x) = \{y \in M : d(f^n(x), y) < \epsilon \text{ for some } n \in \mathbb{Z}\}.$$

Let Σ_f be the set of points $x \in M$ such that for any C^1-neighborhood $\mathcal{U}(f)$ and $\epsilon > 0$, there are $g \in \mathcal{U}(f)$ and $y \in P(g)$ satisfying $g = f$ on $M \setminus B_\epsilon(f, x)$ and $d(f^k(x), g^k(y)) \leq \epsilon$ for $0 \leq k \leq \pi(y)$. Then for any f-invariant probability measure μ, we have $\mu(\Sigma_f) = 1$.

Remark 2. Let $f \in \text{Diff}(M)$ and Λ be a closed f-invariant set. Then $f|_\Lambda$ has the inverse shadowing property with respect to $\mathcal{T}_\alpha(\alpha = 0, c, h)$ if and only if $f^n|_\Lambda$ has the inverse shadowing property with respect to $\mathcal{T}_\alpha$ for $n \in \mathbb{Z} \setminus \{0\}$ (see [7]).

End of the proof of Proposition 3.4. Suppose that the chain component $C_f(p)$ has the C^1-stable inverse shadowing property, and let U be a compact neighborhood of $C_f(p)$ with $\bigcap_{n \in \mathbb{Z}} f^n(U) = C_f(p)$. By Lemma 3.1, C^1-generically, $C_f(p) = H_f^T(p)$ where p is a hyperbolic periodic point of f. And so, we know that $C_f(p) = \Lambda_j(f)$, where $0 < j = \text{index}(p) < \dim M$. Let $\mathcal{V}(f)$ be the C^1-neighborhood of f given by Proposition 3.3. To get the conclusion, it is sufficient to show that $\Lambda_j(f)$ is hyperbolic.

Fix any neighborhood $U_j (\subset U)$ of $\Lambda_j(f)$ (note that by Proposition 3.3, $\Lambda_i(f) = \overline{P_i(f_{|C_f(p)})} = \emptyset$ if $i \neq j$). That is $C_f(p) = \Lambda_j(f) \subset U_j \subset U$, (index$(p) = j$). First of all, we clam the following. *Let $\widetilde{\mathcal{U}}(f) \subset \mathcal{V}(f)$ be a small connected C^1-neighborhood of f. If $g \in \widetilde{\mathcal{U}}(f)$ satisfy $g = f$ on $M \setminus U_j$, then* index$(q) = $ index(p) *for any $q \in \Lambda_g \cap P(g)$.*

Proof. If this property is false, then there are $\widetilde{g} \in \widetilde{\mathcal{U}}(f)$ and $q \in \Lambda_{\widetilde{g}} \cap P(\widetilde{g})$ such that $\widetilde{g} = f$ on $M \setminus U_j$ and index$(q) \neq$ index(p). Suppose that $\widetilde{g}^n(q) = q$, $i_0 = $ index(q), and define $\nu : \widetilde{\mathcal{U}}(f) \to \mathbb{Z}$ by

$$\nu(g) = \sharp\{y \in \Lambda_g \cap P(g) : g^n(y) = y \text{ and index}(y) = i_0\}.$$

By Lemma 3.2, the function ν is continuous, and since $\widetilde{\mathcal{U}}(f)$ is connected, it is constant. But the property of $\widetilde{g}$ implies $\nu(\widetilde{g}) > \nu(f)$. This is a contradiction. $\square$

Proof of Proposition 3.4. We know that C^1-generically, the chain component $C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E^s \oplus E^u$ such that dim E^s =index(p) by Proposition 3.3(b). Let $\epsilon > 0$ and $\widetilde{\mathcal{U}}_0(f)$ be given by Lemma 2.3 with respect to $\widetilde{\mathcal{U}}(f)$. Then, by adapting the techniques in the proof of [10, Theorem B], we will show that for all $x \in C_f(p)$,

$$\liminf_{n \to \infty} \|D_x f^n_{|E^s}\| = 0 \quad \text{and} \quad \liminf_{n \to \infty} \|D_x f^{-n}_{|E^u}\| = 0.$$

Alternatively, we will prove the case of $\liminf_{n \to \infty} \|D_x f^n_{|E^s}\| = 0$, and $\limsup_{n \to \infty} \|D_x f^n_{|E^s}\| = 0$. Here we only prove the formal case. However, it is enough to show that for any $x \in C_f(p)$, there exist $n = n_x > 0$ such that

$$\prod_{j=0}^{n-1} \|Df^m|_{E^s_{f^{mj}(x)}}\| < 1.$$

Suppose not. Then there is $x \in C_f(p)$ such that

$$\prod_{j=0}^{n-1} \|Df^m|_{E^s_{f^{mj}(x)}}\| \geq 1$$

for all $n \geq 0$. Thus

$$\frac{1}{n} \sum_{j=0}^{n-1} \log\|Df^m|_{E^s_{f^{mj}(x)}}\| \geq 0$$

for all $n \geq 0$. Define a probability measure

$$\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^{mj}(x)}.$$

Then there exists $\mu_{n_k} (k \geq 0)$ such that $\mu_{n_k} \to \mu_0 \in \mathcal{M}_{f^m}(M)$, as $k \to \infty$, where M is compact metric space. Thus

$$\int \log\|Df^m|_{E^s_x}\| d\mu_0 = \lim_{k \to \infty} \int \log\|Df^m|_{E^s_x}\| d\mu_{n_k}$$

$$= \lim_{k \to \infty} \int \frac{1}{n} \sum_{j=0}^{n-1} \log\|Df^m|_{E^s_{f^{mj}(x)}}\| \geq 0.$$

By Mañè ([10], (3) page 521),

$$\int_{C_f(p)} \log\|Df^m|_{E_x^s}\|d\mu_0 = \int_{C_f(p)} \frac{1}{n}\sum_{j=0}^{n-1} \log\|D_{f^{mj}(z)}f^m|_{E_{f^{mj}(z)}}\|d\mu_0 \geq 0,$$

where μ_0 is a f^m-invariant measure. Let

$$B_\epsilon(f,x) = \{y \in M : d(f^n(x),y) < \epsilon \text{ for some } n \in \mathbb{Z}\},$$

and $\Sigma_f = \{x \in M : d(f^n(x),y) < \epsilon, \text{ there exist } g \in \mathcal{U}(f) \text{ and } y \in P(g) \text{ such that } g = f \text{ on } M \setminus B_\epsilon(f,x) \text{ and } d(f^i(x),f^i(y)) \leq \epsilon \text{ for } 0 \leq i \leq \pi(y)\}$.

Note that if $x \notin P(f)$, $0 \leq \pi(y) = N$ such that $d(f^N(x), f^N(y)) = d(f^N(x), y) \to 0$ as $N \to \infty$, then $d(x,y) \to 0$. So, this is a contradiction.

For any $\mu \in \mathcal{M}_f(M)$, $\mu(\Sigma_f) = 1$. Then, for any $\mu \in \mathcal{M}_f(C_f(p))$,

$$\mu(C_f(p) \cap \Sigma_f) = 1,$$

since $\mu(C_f(p)) = 1$ and $\mu(\Sigma_f) = 1$. Hence it defines an f-invariant probability measure ν on $C_f(p)$ by

$$\nu = \frac{1}{m}\sum_{l=0}^{m-1} f_*^l(\mu_0).$$

We obtain

$$\begin{aligned}
0 &= \nu(C_f(p) - C_f(p) \cap \Sigma(f)) \\
&= \frac{1}{m}\sum_{l=0}^{m-1} \mu_0(f^l(C_f(p) - C_f(p) \cap \Sigma(f)) \\
&= \frac{1}{m}\{\mu_0(C_f(p) - C_f(p) \cap \Sigma(f)) + \mu_0(f(C_f(p) - C_f(p) \cap \Sigma(f))) \\
&\quad + \mu_0(f^2(C_f(p) - C_f(p) \cap \Sigma(f))) + \cdots \\
&\quad + \mu_0(f^{m-1}(C_f(p) - C_f(p) \cap \Sigma(f)))\} \\
&= \frac{1}{m}\{\mu_0(C_f(p) - C_f(p) \cap \Sigma(f)) + \mu_0(C_f(p) - C_f(p) \cap \Sigma(f)) \\
&\quad + \mu_0(C_f(p) - C_f(p) \cap \Sigma(f)) + \cdots + \mu_0(C_f(p) - C_f(p) \cap \Sigma(f))\} \\
&= \mu_0(C_f(p) - C_f(p) \cap \Sigma(f)).
\end{aligned}$$

Thus, $C_f(p) = C_f(p) \cap \Sigma(f)$ almost everywhere. Therefore

$$\int_{C_f(p) \cap \Sigma(f)} \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\|Df^m|_{E_{f^{mj}(z)}^s}\|d\mu_0 \geq 0.$$

There exist $z_0 \in C_f(p) \cap \Sigma(f)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\|Df^m|_{E_{f^{mj}(z_0)}}\| \geq 0.$$

This is a contradiction.

By Proposition 3.3, we can take $\lambda < \lambda_0 < 1$ and $n_0 > 0$ such that

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^m|_{E_{f^{mj}(z_0)}}\| \geq \log\lambda_0, \quad \text{if } n \geq n_0.$$

Since $z_0 \notin P(f)$, $z_0 \in C_f(p) \cap \Sigma(f)$, $\Lambda_j(f) = C_f(p)$ and $\Lambda_i(f) = \phi$ if $i \neq j$. Therefore, there is g C^1-nearby f and $\bar{z}_0 \in P(g)$ such that $g = f$ on $M \setminus U_j$ and $d(f^l(z_0), g^l(\bar{z}_0)) = $ small, for $0 \leq l \leq n = \pi(\bar{z}_0, g)$.

Choose $n = \pi(\bar{z}_0, g) \geq m$ such that $k = [n/m] \geq n_0$, $K\lambda^k < \lambda_0^k$ and $(\lambda/\lambda_0)^k C^m \leq 1/2$, where $C = \sup_{x \in M} \|D_x f^{-1}\| > 0$.

Since

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^m|_{E_{f^{mj}(z_0)}}\| \geq \log\lambda_0, \quad \text{if } n \geq n_0.$$

By Proposition 3.3,

$$\|D_{f^n(z_0)} f^{-n}|_{E^u_{f^n(z_0)}}\| \leq 1/2.$$

By dominated splitting of $C_f(p)$,

$$\|D_{f^{n-m(i+1)}(z_0)} f^m|_{E^s_{f^{n-m(i+1)}(z_0)}}\| \cdot \|D_{f^{n-mi}(z_0)} f^{-m}|_{E^u_{f^{n-mi}(z_0)}}\| \leq \lambda.$$

Therefore, $\|D_{f^{n-mi}(z_0)} f^{-m}|_{E^u_{f^{n-mi}(z_0)}}\| \leq \lambda \|D_{f^{n-m(i+1)}(z_0)} f^m|_{E^s_{f^{n-m(i+1)}(z_0)}}\|^{-1}.$

From

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^m|_{E_{f^{mj}(z_0)}}\| \geq \log\lambda_0, \quad \text{if } n \geq n_0.$$

We have

$$\log \prod_{i=0}^{n-1} \|Df^m|_{E^s_{f^{mi}(z_0)}}\| \geq \log\lambda_0^n = \prod_{i=0}^{n-1} \|Df^m|_{E^s_{f^{mi}(z_0)}}\| \geq \lambda_0^n.$$

Thus

$$\lambda_0^{-n} \geq \prod_{i=0}^{n-1} \|Df^m|_{E^s_{f^{mi}(z_0)}}\|^{-1}.$$

And so,

$$\|D_{f^n(p)} f^{-n}|_{E^u_{f^n(p)}}\| \leq \prod_{i=0}^{n-1} \|Df^m|_{E^s_{f^{mi}(z_0)}}\|^{-1}$$

$$\leq \lambda^k \prod_{i=0}^{k-1} \|Df^m|_{E^s_{f^{mi}(z_0)}}\|^{-1} C^m \leq \lambda^k \cdot \lambda_0^{-k} \cdot C^m \leq 1/2.$$

By Mañé's Ergodic Closing Lemma we can find $\bar{g} \in \widetilde{\mathcal{U}}_0(f)$ ($\bar{g} = f$ on $M \setminus U_j$) and $\bar{z} \in \Lambda_{\bar{g}} \cap P(\bar{g})$ nearby z.

Moreover, we know that index($\bar{z}$) = index(p) since $\bar{g} = f$ on $M \setminus U_j$. By applying Lemma 2.3, we can construct $\hat{g} \in \widetilde{\mathcal{U}}(f) \subset \mathcal{V}(f)$ C^1-nearby $\bar{g}$ such that

$$\lambda_0^k \leq \prod_{i=0}^{k-1} \|D_{\hat{g}^{im}(\bar{z})}\hat{g}^m_{|E^s_{\hat{g}^{im}(\bar{z})}}\|$$

(see [10, pp. 523–524]). On the other hand, by Proposition 3.3(a) we see that

$$\prod_{i=0}^{k-1} \|D_{\hat{g}^{im}(\bar{z})}\hat{g}^m_{|E^s_{\hat{g}^{im}(\bar{z})}}\| < K\lambda^k.$$

We can choose the period $\pi(\bar{z})$ ($> n_0$) of $\bar{z}$ as large as $\lambda_0^k \geq K\lambda^k$. Here $k = [\pi(\bar{z})/m]$. This is a contradiction. Thus,

$$\liminf_{n \to \infty} \|D_x f^n_{|E_x}\| = 0$$

for all $x \in C_f(p)$.

We can show the assertion

$$\liminf_{n \to \infty} \|D_x f^{-n}_{|F_x}\| = 0$$

in a similar way for all $x \in C_f(p)$, and so $C_f(p)$ is hyperbolic. This completes the proof of Theorem 3. $\qquad\square$

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