Commun. Korean Math. Soc. **24** (2009), No. 1, pp. 67–83

COMMON FIXED POINT THEOREMS WITH APPLICATIONS TO THE SOLUTIONS OF FUNCTIONAL EQUATIONS ARISING IN DYNAMIC PROGRAMMING

Zeqing Liu, Min Liu, Hyeong Kug Kim, and Shin Min Kang

Abstract. Several common fixed point theorems for a few contractive type mappings in complete metric spaces are established. As applications, the existence and uniqueness of common solutions for certain systems of functional equations arising in dynamic programming are discussed.

1. Introduction and preliminaries

Contractive type mappings and corresponding fixed or common fixed point theorems and their applications have been studied and discussed during the past decades, see [1-16] and the references therein. In particular, Ray [16] established two fixed point theorems for the following contractive type mappings

$$d(fx,gy) \le d(hx,hy) - W(d(hx,hy)), \quad \forall x,y \in X. \tag{1.1}$$

Liu [7] produced a few common fixed point theorems for three self mappings f, g and h in a complete metric space (X,d) which satisfy the following condition

$$\begin{aligned} d(fx,gy) \le &\max\{d(hx,hy), d(hx,fx), d(hy,gy)\} \\ &- W(\max\{d(hx,hy), d(hx,fx), d(hy,gy)\}), \quad \forall x,y \in X. \end{aligned} \tag{1.2}$$

As proposed in Bellman and Lee [1], the essential form of the functional equation of dynamic programming is

$$f(x) = \underset{y}{\operatorname{opt}}\{H(x,y,f(T(x,y)))\}, \tag{1.3}$$

where x and y denote the state and decision vectors, respectively. T denotes the transformation of the process, $f(x)$ denotes the optimal return function with the initial state x, and the opt represents sup or inf.

Received January 2, 2008.

2000 *Mathematics Subject Classification.* 54H25, 49L20, 49L99.

Key words and phrases. common fixed point, continuous mapping, complete metric space, system of functional equations, common solution, dynamic programming.

This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (2008352) and the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2007-313-C00040).

Bhakta and Choudhury [2], Bhakta and Mitra [3], Liu [4-6], Liu, Agarwal and Kang [8], Liu and Kang [9-10], Liu and Kim [11], Liu and Ume [12], Liu, Ume and Kang [13], Liu, Xu, Ume and Kang [14], Pathak and Fisher [15] and others established the existence and uniqueness of solutions or common solutions for several classes of functional equations or system of functional equations arising in dynamic programming.

Aroused and motivated by the above achievements in [1-16], we introduce the following contractive type mappings and system of functional equations arising in dynamic programming, respectively,

$$\begin{aligned} & d^t(fx,gy) \\ \le\ & \max\Big\{d^t(hx,hy), d^t(hx,fx), d^t(hy,gy), \\ & \frac{d^t(hx,gy)d^t(hy,fx)}{1+d^t(hx,hy)}, \frac{d^t(hx,gy)d^t(hy,fx)}{1+d^t(fx,gy)}, \\ & \frac{d^t(hx,fx)d^t(hy,gy)}{1+d^t(hx,hy)}, \frac{d^t(hx,fx)d^t(hy,gy)}{1+d^t(fx,gy)}, \\ & \frac{d^t(hx,gy)d^t(hy,fx)d^t(hx,hy)}{1+d^t(hx,hy)d^t(fx,gy)}, \frac{d^t(hx,gy)d^t(hy,fx)d^t(fx,gy)}{1+d^t(hx,hy)d^t(fx,gy)}, \\ & \frac{d^t(hx,fx)d^t(hy,gy)d^t(hx,hy)}{1+d^t(hx,hy)d^t(fx,gy)}, \frac{d^t(hx,fx)d^t(hy,gy)d^t(fx,gy)}{1+d^t(hx,hy)d^t(fx,gy)}\Big\} \\ & - W\Big(\max\Big\{d^t(hx,hy), d^t(hx,fx), d^t(hy,gy), \\ & \quad \frac{d^t(hx,gy)d^t(hy,fx)}{1+d^t(hx,hy)}, \frac{d^t(hx,gy)d^t(hy,fx)}{1+d^t(fx,gy)}, \\ & \quad \frac{d^t(hx,fx)d^t(hy,gy)}{1+d^t(hx,hy)}, \frac{d^t(hx,fx)d^t(hy,gy)}{1+d^t(fx,gy)}, \\ & \quad \frac{d^t(hx,gy)d^t(hy,fx)d^t(hx,hy)}{1+d^t(hx,hy)d^t(fx,gy)}, \\ & \quad \frac{d^t(hx,gy)d^t(hy,fx)d^t(fx,gy)}{1+d^t(hx,hy)d^t(fx,gy)}, \\ & \quad \frac{d^t(hx,fx)d^t(hy,gy)d^t(hx,hy)}{1+d^t(hx,hy)d^t(fx,gy)}, \\ & \quad \frac{d^t(hx,fx)d^t(hy,gy)d^t(fx,gy)}{1+d^t(hx,hy)d^t(fx,gy)}\Big\}\Big), \quad \forall x,y \in X, \end{aligned} \tag{1.4}$$

where t is a positive number, and

$$f_i(x) = \operatorname*{opt}_{y\in D}\{u(x,y) + H_i(x,y,f_i(T(x,y)))\}, \quad \forall x \in S,\ i \in \{1,2,3\}. \tag{1.5}$$

The main aim in this paper is to study the existence and uniqueness of common fixed point for the contractive type mappings (1.4) in complete metric spaces. Under certain conditions, we prove a few common fixed point theorems for the contractive type mappings (1.4) and its some transmutations. As applications, the common fixed point theorems presented are maken full use of to discuss the existence and uniqueness problems of common solutions for the system of functional equations (1.5) and its some transformations.

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$ and $W : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying $0 < W(r) < r$ for all $r \in \mathbb{R}^+ \setminus \{0\}$.

2. Common fixed point theorems

In this section, we establish four common fixed point theorems for the contractive type mappings (1.4) and its transfigurations in a complete metric space.

Theorem 2.1. *Let f, g and h be three mappings from a complete metric space (X, d) into itself, h be continuous, $fh = hf$, $gh = hg$ and $f(X) \cup g(X) \subseteq h(X)$. If there exists $t \in \mathbb{R}^+ \setminus \{0\}$ satisfying (1.4), then f, g and h have a unique common fixed point in X.*

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \cup g(X) \subseteq h(X)$, there exists a sequence $\{x_n\}_{n \geq 0} \subseteq X$ such that $fx_{2n} = hx_{2n+1}$, $gx_{2n+1} = hx_{2n+2}$ for $n \geq 0$. Define $d_n = d(hx_n, hx_{n+1})$ for $n \geq 0$. First of all, we show that

$$d_n^t \leq d_{n-1}^t - W(d_{n-1}^t), \quad \forall n \geq 1. \tag{2.1}$$

In view of (1.4) we acquire that

$$\begin{aligned}
&d^t(fx_{2n}, gx_{2n+1}) \\
&\leq \max\Big\{d^t(hx_{2n}, hx_{2n+1}), d^t(hx_{2n}, fx_{2n}), d^t(hx_{2n+1}, gx_{2n+1}), \\
&\qquad \frac{d^t(hx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fx_{2n})}{1 + d^t(hx_{2n}, hx_{2n+1})}, \frac{d^t(hx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fx_{2n})}{1 + d^t(fx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hx_{2n}, fx_{2n})d^t(hx_{2n+1}, gx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})}, \frac{d^t(hx_{2n}, fx_{2n})d^t(hx_{2n+1}, gx_{2n+1})}{1 + d^t(fx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fx_{2n})d^t(hx_{2n}, hx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})d^t(fx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fx_{2n})d^t(fx_{2n}, gx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})d^t(fx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hx_{2n}, fx_{2n})d^t(hx_{2n+1}, gx_{2n+1})d^t(hx_{2n}, hx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})d^t(fx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hx_{2n}, fx_{2n})d^t(hx_{2n+1}, gx_{2n+1})d^t(fx_{2n}, gx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})d^t(fx_{2n}, gx_{2n+1})}\Big\} \\
&\quad - W\Big(\max\Big\{d^t(hx_{2n}, hx_{2n+1}), d^t(hx_{2n}, fx_{2n}), d^t(hx_{2n+1}, gx_{2n+1}), \\
&\qquad\quad \frac{d^t(hx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fx_{2n})}{1 + d^t(hx_{2n}, hx_{2n+1})},
\end{aligned}$$

$$\frac{d^t(hx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fx_{2n})}{1 + d^t(fx_{2n}, gx_{2n+1})},$$
$$\frac{d^t(hx_{2n}, fx_{2n})d^t(hx_{2n+1}, gx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})},$$
$$\frac{d^t(hx_{2n}, fx_{2n})d^t(hx_{2n+1}, gx_{2n+1})}{1 + d^t(fx_{2n}, gx_{2n+1})},$$
$$\frac{d^t(hx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fx_{2n})d^t(hx_{2n}, hx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})d^t(fx_{2n}, gx_{2n+1})},$$
$$\frac{d^t(hx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fx_{2n})d^t(fx_{2n}, gx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})d^t(fx_{2n}, gx_{2n+1})},$$
$$\frac{d^t(hx_{2n}, fx_{2n})d^t(hx_{2n+1}, gx_{2n+1})d^t(hx_{2n}, hx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})d^t(fx_{2n}, gx_{2n+1})},$$
$$\left.\frac{d^t(hx_{2n}, fx_{2n})d^t(hx_{2n+1}, gx_{2n+1})d^t(fx_{2n}, gx_{2n+1})}{1 + d^t(hx_{2n}, hx_{2n+1})d^t(fx_{2n}, gx_{2n+1})}\right\}, \quad \forall n \geq 0,$$

which deduces that

$$\begin{aligned} d^t_{2n+1} \leq \max\Big\{ & d^t_{2n}, d^t_{2n}, d^t_{2n+1}, 0, 0, \frac{d^t_{2n} d^t_{2n+1}}{1 + d^t_{2n}}, \frac{d^t_{2n} d^t_{2n+1}}{1 + d^t_{2n+1}}, 0, 0, \\ & \frac{d^{2t}_{2n} d^t_{2n+1}}{1 + d^t_{2n} d^t_{2n+1}}, \frac{d^t_{2n} d^{2t}_{2n+1}}{1 + d^t_{2n} d^t_{2n+1}} \Big\} \\ & - W\Big(\max\Big\{ d^t_{2n}, d^t_{2n}, d^t_{2n+1}, 0, 0, \frac{d^t_{2n} d^t_{2n+1}}{1 + d^t_{2n}}, \frac{d^t_{2n} d^t_{2n+1}}{1 + d^t_{2n+1}}, 0, 0, \\ & \frac{d^{2t}_{2n} d^t_{2n+1}}{1 + d^t_{2n} d^t_{2n+1}}, \frac{d^t_{2n} d^{2t}_{2n+1}}{1 + d^t_{2n} d^t_{2n+1}} \Big\} \Big), \quad \forall n \geq 0. \end{aligned} \tag{2.2}$$

Suppose that $d_{2n+1} > d_{2n}$ for some $n \geq 0$. It follows form (2.2) that

$$d^t_{2n+1} \leq d^t_{2n+1} - W(d^t_{2n+1}) < d^t_{2n+1},$$

which is a contradiction. Hence $d_{2n+1} \leq d_{2n}$ for any $n \geq 0$. Thus (2.3) means that $d^t_{2n+1} \leq d^t_{2n} - W(d^t_{2n})$ for any $n \geq 0$. Similarly we conclude that $d^t_{2n} \leq d^t_{2n-1} - W(d^t_{2n-1})$ for each $n \geq 1$. It follows that (2.1) holds.

At present we demonstrate that

$$\lim_{n \to \infty} d_n = 0. \tag{2.3}$$

In view of (2.2) we deduce that

$$\sum_{i=0}^{n} W(d^t_i) \leq d^t_0 - d^t_{n+1} \leq d^t_0, \quad \forall n \geq 0,$$

which yields that the series of nonnegative terms $\sum_{n=0}^{\infty} W(d^t_n)$ is convergent. Therefore

$$\lim_{n \to \infty} W(d^t_n) = 0. \tag{2.4}$$

Due to (2.1) we get that

$$0 \le d_n^t \le d_{n-1}^t \le \cdots \le d_0^t, \quad \forall n \ge 0,$$

which implies that $\lim_{n\to\infty} d_n^t = a$ for some $a \in \mathbb{R}^+$. By (2.4) and the continuity of W we have

$$W(a) = \lim_{n\to\infty} W(d_n^t) = 0,$$

thereupon $a = 0$. It follows that (2.3) holds.

For the sake of showing that $\{hx_n\}_{n\ge 0}$ is a Cauchy sequence, in terms of (2.3), it is sufficient to show that $\{hx_{2n}\}_{n\ge 0}$ is a Cauchy sequence. Suppose that $\{hx_{2n}\}_{n\ge 0}$ is not a Cauchy sequence. Thus there exists a positive number ε such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ such that $2m(k) > 2n(k) > 2k$, and

$$d(hx_{2m(k)}, hx_{2n(k)}) > \varepsilon.$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying the above inequality, so that

$$d(hx_{2m(k)-2}, hx_{2n(k)}) \le \varepsilon \quad \text{and} \quad d(hx_{2m(k)}, hx_{2n(k)}) > \varepsilon. \tag{2.5}$$

It follows that for each even integer $2k$,

$$d(hx_{2m(k)}, hx_{2n(k)}) \le d(hx_{2n(k)}, hx_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

According to (2.3), (2.5) and the above inequality we infer that

$$\lim_{k\to\infty} d(hx_{2n(k)}, hx_{2m(k)}) = \varepsilon. \tag{2.6}$$

It is clear that for any $k \ge 1$,

$$|d(hx_{2m(k)}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)})| \le d_{2n(k)},$$
$$|d(hx_{2m(k)+1}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)+1})| \le d_{2m(k)}$$

and

$$|d(hx_{2m(k)+1}, hx_{2n(k)+2}) - d(hx_{2m(k)+1}, hx_{2n(k)+1})| \le d_{2n(k)+1}.$$

In light of (2.3), (2.6) and the above inequalities, we gain that

$$\begin{aligned}\varepsilon &= \lim_{k\to\infty} d(hx_{2m(k)}, hx_{2n(k)+1}) = \lim_{k\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+1}) \\ &= \lim_{k\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+2}).\end{aligned}$$

By (2.1), we arrive at

$$
\begin{aligned}
& d^t(fx_{2m(k)}, gx_{2n(k)+1}) \\
\le\ & \max\Big\{d^t(hx_{2m(k)}, hx_{2n(k)+1}), d^t(hx_{2m(k)}, fx_{2m(k)}), \\
& d^t(hx_{2n(k)+1}, gx_{2n(k)+1}), \\
& \frac{d^t(hx_{2m(k)}, gx_{2n(k)+1})d^t(hx_{2n(k)+1}, fx_{2m(k)})}{1+d^t(hx_{2m(k)}, hx_{2n(k)+1})}, \\
& \frac{d^t(hx_{2m(k)}, gx_{2n(k)+1})d^t(hx_{2n(k)+1}, fx_{2m(k)})}{1+d^t(fx_{2m(k)}, gx_{2n(k)+1})}, \\
& \frac{d^t(hx_{2m(k)}, fx_{2m(k)})d^t(hx_{2n(k)+1}, gx_{2n(k)+1})}{1+d^t(hx_{2m(k)}, hx_{2n(k)+1})}, \\
& \frac{d^t(hx_{2m(k)}, fx_{2m(k)})d^t(hx_{2n(k)+1}, gx_{2n(k)+1})}{1+d^t(fx_{2m(k)}, gx_{2n(k)+1})}, \frac{I}{M}, \frac{J}{M}, \frac{K}{M}, \frac{L}{M}\Big\}\Big) \\
& - W\Big(\max\Big\{d^t(hx_{2m(k)}, hx_{2n(k)+1}), d^t(hx_{2m(k)}, fx_{2m(k)}), \\
& d^t(hx_{2n(k)+1}, gx_{2n(k)+1}), \\
& \frac{d^t(hx_{2m(k)}, gx_{2n(k)+1})d^t(hx_{2n(k)+1}, fx_{2m(k)})}{1+d^t(hx_{2m(k)}, hx_{2n(k)+1})}, \\
& \frac{d^t(hx_{2m(k)}, gx_{2n(k)+1})d^t(hx_{2n(k)+1}, fx_{2m(k)})}{1+d^t(fx_{2m(k)}, gx_{2n(k)+1})}, \\
& \frac{d^t(hx_{2m(k)}, fx_{2m(k)})d^t(hx_{2n(k)+1}, gx_{2n(k)+1})}{1+d^t(hx_{2m(k)}, hx_{2n(k)+1})}, \\
& \frac{d^t(hx_{2m(k)}, fx_{2m(k)})d^t(hx_{2n(k)+1}, gx_{2n(k)+1})}{1+d^t(fx_{2m(k)}, gx_{2n(k)+1})}, \\
& \frac{I}{M}, \frac{J}{M}, \frac{K}{M}, \frac{L}{M}\Big\}\Big), \quad \forall k \ge 1,
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
I &= d^t(hx_{2m(k)}, gx_{2n(k)+1})d^t(hx_{2n(k)+1}, fx_{2m(k)})d^t(hx_{2m(k)}, hx_{2n(k)+1}), \\
J &= d^t(hx_{2m(k)}, gx_{2n(k)+1})d^t(hx_{2n(k)+1}, fx_{2m(k)})d^t(fx_{2m(k)}, gx_{2n(k)+1}), \\
K &= d^t(hx_{2m(k)}, fx_{2m(k)})d^t(hx_{2n(k)+1}, gx_{2n(k)+1})d^t(hx_{2m(k)}, hx_{2n(k)+1})), \\
L &= d^t(hx_{2m(k)}, fx_{2m(k)})d^t(hx_{2n(k)+1}, gx_{2n(k)+1})d^t(fx_{2m(k)}, gx_{2n(k)+1}), \\
M &= 1 + d^t(hx_{2m(k)}, hx_{2n(k)+1})d^t(fx_{2m(k)}, gx_{2n(k)+1}).
\end{aligned}
$$

As $k \to \infty$ in (2.7), we get that

$$
\varepsilon^t \le \max\Big\{\varepsilon^t, 0, 0, \frac{\varepsilon^{2t}}{1+\varepsilon^t}, \frac{\varepsilon^{2t}}{1+\varepsilon^t}, 0, 0, \frac{\varepsilon^{3t}}{1+\varepsilon^{2t}}, \frac{\varepsilon^{3t}}{1+\varepsilon^{2t}}, 0, 0\Big\}
$$

$$
\begin{aligned}
&- W\Big(\max\Big\{\varepsilon^t, 0, 0, \frac{\varepsilon^{2t}}{1+\varepsilon^t}, \frac{\varepsilon^{2t}}{1+\varepsilon^t}, 0, 0, \frac{\varepsilon^{3t}}{1+\varepsilon^{2t}}, \frac{\varepsilon^{3t}}{1+\varepsilon^{2t}}, 0, 0\Big\}\Big) \\
&= \varepsilon^t - W(\varepsilon^t),
\end{aligned}
$$

which leads to $W(\varepsilon^t) \leq 0$. Hence $\varepsilon = 0$, which is a contradiction. Thus $\{hx_n\}_{n\geq 0}$ is a Cauchy sequence and so it converges to a point $u \in X$ by completeness of X. The continuity of h brings about

$$
\lim_{n\to\infty} hfx_{2n} = \lim_{n\to\infty} hhx_{2n+1} = hu = \lim_{n\to\infty} hhx_{2n+2} = \lim_{n\to\infty} hgx_{2n+1}.
$$

Next we prove that u is a common fixed point of h, f and g. It follows from (1.4) that

$$
\begin{aligned}
&d^t(fhx_{2n}, gu) \\
&\leq \max\Big\{d^t(hhx_{2n}, hu), d^t(hhx_{2n}, fhx_{2n}), d^t(hu, gu), \\
&\qquad \frac{d^t(hhx_{2n}, gu)d^t(hu, fhx_{2n})}{1 + d^t(hhx_{2n}, hu)}, \frac{d^t(hhx_{2n}, gu)d^t(hu, fhx_{2n})}{1 + d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hu, gu)}{1 + d^t(hhx_{2n}, hu)}, \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hu, gu)}{1 + d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, gu)d^t(hu, fhx_{2n})d^t(hhx_{2n}, hu)}{1 + d^t(hhx_{2n}, hu)d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, gu)d^t(hu, fhx_{2n})d^t(fhx_{2n}, gu)}{1 + d^t(hhx_{2n}, hu)d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hu, gu)d^t(hhx_{2n}, hu)}{1 + d^t(hhx_{2n}, hu)d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hu, gu)d^t(fhx_{2n}, gu)}{1 + d^t(hhx_{2n}, hu)d^t(fhx_{2n}, gu)}\Big\} \\
&\quad - W\Big(\max\Big\{d^t(hhx_{2n}, hu), d^t(hhx_{2n}, fhx_{2n}), d^t(hu, gu), \\
&\qquad \frac{d^t(hhx_{2n}, gu)d^t(hu, fhx_{2n})}{1 + d^t(hhx_{2n}, hu)}, \frac{d^t(hhx_{2n}, gu)d^t(hu, fhx_{2n})}{1 + d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hu, gu)}{1 + d^t(hhx_{2n}, hu)}, \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hu, gu)}{1 + d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, gu)d^t(hu, fhx_{2n})d^t(hhx_{2n}, hu)}{1 + d^t(hhx_{2n}, hu)d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, gu)d^t(hu, fhx_{2n})d^t(fhx_{2n}, gu)}{1 + d^t(hhx_{2n}, hu)d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hu, gu)d^t(hhx_{2n}, hu)}{1 + d^t(hhx_{2n}, hu)d^t(fhx_{2n}, gu)}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hu, gu)d^t(fhx_{2n}, gu)}{1 + d^t(hhx_{2n}, hu)d^t(fhx_{2n}, gu)}\Big\}\Big), \quad \forall n \geq 0.
\end{aligned}
$$

As $n \to \infty$ in the above inequality, we conclude that

$$d^t(hu, gu) \le d^t(hu, gu) - W(d^t(hu, gu)),$$

which implies that $hu = gu$. In a similar manner we can show that $hu = fu$.

Using (1.4) we obtain that $\forall n \ge 0$,

$$\begin{aligned}
&d^t(fhx_{2n}, gx_{2n+1}) \\
&\le \max\Big\{d^t(hhx_{2n}, hx_{2n+1}), d^t(hhx_{2n}, fhx_{2n}), d^t(hx_{2n+1}, gx_{2n+1}), \\
&\frac{d^t(hhx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fhx_{2n})}{1 + d^t(hhx_{2n}, hx_{2n+1})}, \frac{d^t(hhx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fhx_{2n})}{1 + d^t(fhx_{2n}, gx_{2n+1})}, \\
&\frac{d^t(hhx_{2n}, fhx_{2n})d^t(hx_{2n+1}, gx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})}, \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hx_{2n+1}, gx_{2n+1})}{1 + d^t(fhx_{2n}, gx_{2n+1})}, \\
&\frac{d^t(hhx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fhx_{2n})d^t(hhx_{2n}, hx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}, \\
&\frac{d^t(hhx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fhx_{2n})d^t(fhx_{2n}, gx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}, \\
&\frac{d^t(hhx_{2n}, fhx_{2n})d^t(hx_{2n+1}, gx_{2n+1})d^t(hhx_{2n}, hx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}, \\
&\frac{d^t(hhx_{2n}, fhx_{2n})d^t(hx_{2n+1}, gx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}\Big\} \\
&- W\Big(\max\Big\{d^t(hhx_{2n}, hx_{2n+1}), d^t(hhx_{2n}, fhx_{2n}), d^t(hx_{2n+1}, gx_{2n+1}), \\
&\qquad \frac{d^t(hhx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fhx_{2n})}{1 + d^t(hhx_{2n}, hx_{2n+1})}, \\
&\qquad \frac{d^t(hhx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fhx_{2n})}{1 + d^t(fhx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hx_{2n+1}, gx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hx_{2n+1}, gx_{2n+1})}{1 + d^t(fhx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hhx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fhx_{2n})d^t(hhx_{2n}, hx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hhx_{2n}, gx_{2n+1})d^t(hx_{2n+1}, fhx_{2n})d^t(fhx_{2n}, gx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hx_{2n+1}, gx_{2n+1})d^t(hhx_{2n}, hx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}, \\
&\qquad \frac{d^t(hhx_{2n}, fhx_{2n})d^t(hx_{2n+1}, gx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}{1 + d^t(hhx_{2n}, hx_{2n+1})d^t(fhx_{2n}, gx_{2n+1})}\Big\}\Big).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we deduce that

$$\begin{aligned} d^t(hu,u) \le & \max\Big\{d^t(hu,u),0,0,\frac{d^{2t}(hu,u)}{1+d^t(hu,u)},\frac{d^{2t}(hu,u)}{1+d^t(hu,u)},0,0, \\ & \frac{d^{3t}(hu,u)}{1+d^{2t}(hu,u)},\frac{d^{3t}(hu,u)}{1+d^{2t}(hu,u)},0,0\Big\} \\ & - W\Big(\max\Big\{d^t(hu,u),0,0,\frac{d^{2t}(hu,u)}{1+d^t(hu,u)},\frac{d^{2t}(hu,u)}{1+d^t(hu,u)},0,0, \\ & \frac{d^{3t}(hu,u)}{1+d^{2t}(hu,u)},\frac{d^{3t}(hu,u)}{1+d^{2t}(hu,u)},0,0\Big\}\Big) \\ = & \, d^t(hu,u) - W(d^t(hu,u)), \end{aligned}$$

which signifies that $u = hu$. That is, $u = hu = fu = gu$.

We finally show that u is a unique common fixed point of h, f and g. If $v \in X$ is any common fixed point of h, f and g, it is easy to see that from (1.4)

$$d^t(u,v) \le d^t(u,v) - W(d^t(u,v)),$$

which means that $0 \le W(d^t(u,v)) \le 0$, that is, $u = v$. This completes the proof. □

Using similar method as in the proof of Theorem 2.1, we have the following three results and omit their proofs.

Theorem 2.2. *Let f, g and h be three mappings from a complete metric space (X,d) into itself, h be continuous, $fh = hf$, $gh = hg$ and $f(X) \cup g(X) \subseteq h(X)$. If there exists $t \in \mathbb{R}^+ \setminus \{0\}$ satisfying*

$$\begin{aligned} & d^t(fx,gy) \\ \le & \max\Big\{d^t(hx,hy),d^t(hx,fx),d^t(hy,gy),\frac{d^t(hx,gy)d^t(hy,fx)}{1+d^t(hx,hy)}, \\ & \frac{d^t(hx,gy)d^t(hy,fx)}{1+d^t(fx,gy)},\frac{d^t(hx,fx)d^t(hy,gy)}{1+d^t(hx,hy)},\frac{d^t(hx,fx)d^t(hy,gy)}{1+d^t(fx,gy)}\Big\} \\ & - W\Big(\max\Big\{d^t(hx,hy),d^t(hx,fx),d^t(hy,gy), \\ & \frac{d^t(hx,gy)d^t(hy,fx)}{1+d^t(hx,hy)},\frac{d^t(hx,gy)d^t(hy,fx)}{1+d^t(fx,gy)}, \\ & \frac{d^t(hx,fx)d^t(hy,gy)}{1+d^t(hx,hy)},\frac{d^t(hx,fx)d^t(hy,gy)}{1+d^t(fx,gy)}\Big\}\Big), \quad \forall x,y \in X, \end{aligned}$$

then f, g and h have a unique common fixed point in X.

Theorem 2.3. *Let f, g and h be three mappings from a complete metric space (X,d) into itself, h be continuous, $fh = hf$, $gh = hg$, $f(X) \cup g(X) \subseteq h(X)$. If* *there exists* $t \in \mathbb{R}^+ \setminus \{0\}$ *satisfying*

$$d^t(fx, gy) \le \max\{d^t(hx, hy), d^t(hx, fx), d^t(hy, gy), \\ - W(\max\{d^t(hx, hy), d^t(hx, fx), d^t(hy, gy)\}), \quad \forall x, y \in X,$$

then f, g *and* h *have a unique common fixed point in* X.

Remark 2.4. Theorem 2.3 extends the Theorem and Corollaries 1 and 2 of Liu [7].

Theorem 2.5. *Let* f *and* g *be two self mappings from a complete metric space* (X, d) *into itself. If there exists* $t \in \mathbb{R}^+ \setminus \{0\}$ *satisfying* $\forall x, y \in X$,

$$\begin{aligned} & d^t(fx, gy) \\ \le & \max\Big\{d^t(x, y), d^t(x, fx), d^t(y, gy), \frac{d^t(x, gy)d^t(y, fx)}{1 + d^t(x, y)}, \\ & \frac{d^t(x, gy)d^t(y, fx)}{1 + d^t(fx, gy)}, \frac{d^t(x, fx)d^t(y, gy)}{1 + d^t(x, y)}, \frac{d^t(x, fx)d^t(y, gy)}{1 + d^t(fx, gy)}, \\ & \frac{d^t(x, gy)d^t(y, fx)d^t(x, y)}{1 + d^t(x, y)d^t(fx, gy)}, \frac{d^t(x, gy)d^t(y, fx)d^t(fx, gy)}{1 + d^t(x, y)d^t(fx, gy)}, \\ & \frac{d^t(x, fx)d^t(y, gy)d^t(x, y)}{1 + d^t(x, y)d^t(fx, gy)}, \frac{d^t(x, fx)d^t(y, gy)d^t(fx, gy)}{1 + d^t(x, y)d^t(fx, gy)}\Big\} \\ & - W\Big(\max\Big\{d^t(x, y), d^t(x, fx), d^t(y, gy), \frac{d^t(x, gy)d^t(y, fx)}{1 + d^t(x, y)}, \\ & \quad \frac{d^t(x, gy)d^t(y, fx)}{1 + d^t(fx, gy)}, \frac{d^t(x, fx)d^t(y, gy)}{1 + d^t(x, y)}, \frac{d^t(x, fx)d^t(y, gy)}{1 + d^t(fx, gy)}, \\ & \quad \frac{d^t(x, gy)d^t(y, fx)d^t(x, y)}{1 + d^t(x, y)d^t(fx, gy)}, \frac{d^t(x, gy)d^t(y, fx)d^t(fx, gy)}{1 + d^t(x, y)d^t(fx, gy)}, \\ & \quad \frac{d^t(x, fx)d^t(y, gy)d^t(x, y)}{1 + d^t(x, y)d^t(fx, gy)}, \frac{d^t(x, fx)d^t(y, gy)d^t(fx, gy)}{1 + d^t(x, y)d^t(fx, gy)}\Big\}\Big), \end{aligned}$$

then f *and* g *have a unique common fixed point in* X.

3. Applications

Throughout this section, we would like to assume that X, Y are both Banach spaces, $S \subseteq X$ is the state space, and $D \subseteq Y$ is the decision space. $B(S)$ denotes the set of all bounded real-valued functions on S. Let

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in S\} \quad \text{for } f, g \in B(S).$$

It is clear that $(B(S), d)$ is a complete metric space.

Now we study the existence and uniqueness of common solutions for the systems of functional equations (1.5) and its distortions arising in dynamic programming in the complete metric space $(B(S), d)$.

Theorem 3.1. *Let* $u : S\times D \to \mathbb{R}, T : S\times D \to S$, H_1, H_2 *and* $H_3 : S\times D\times\mathbb{R} \to \mathbb{R}$ *satisfy the following conditions:*

$(C1)$ u *and* H_i *are bounded for* $i \in \{1,2,3\}$;

$(C2)$ *For all* $(x,q,y) \in S^2 \times D$, $g,h \in B(S)$,

$$
\begin{aligned}
&|H_1(x,y,g(q)) - H_2(x,y,h(q))| \\
&\le \max\Big\{d(A_3g,A_3h), d(A_3g,A_1g), d(A_3h,A_2h), \\
&\quad \frac{d(A_3g,A_2h)d(A_3h,A_1g)}{1+d(A_3g,A_3h)}, \frac{d(A_3g,A_2h)d(A_3h,A_1g)}{1+d(A_1g,A_2h)}, \\
&\quad \frac{d(A_3g,A_1g)d(A_3h,A_2h)}{1+d(A_3g,A_3h)}, \frac{d(A_3g,A_1g)d(A_3h,A_2h)}{1+d(A_1g,A_2h)}, \\
&\quad \frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}, \\
&\quad \frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}, \\
&\quad \frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}, \\
&\quad \frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}\Big\} \\
&\quad - W\Big(\max\Big\{d(A_3g,A_3h), d(A_3g,A_1g), d(A_3h,A_2h), \\
&\qquad \frac{d(A_3g,A_2h)d(A_3h,A_1g)}{1+d(A_3g,A_3h)}, \frac{d(A_3g,A_2h)d(A_3h,A_1g)}{1+d(A_1g,A_2h)}, \\
&\qquad \frac{d(A_3g,A_1g)d(A_3h,A_2h)}{1+d(A_3g,A_3h)}, \frac{d(A_3g,A_1g)d(A_3h,A_2h)}{1+d(A_1g,A_2h)}, \\
&\qquad \frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}, \\
&\qquad \frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}, \\
&\qquad \frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}, \\
&\qquad \frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}\Big\}\Big),
\end{aligned}
$$

where the mappings A_1, A_2 *and* A_3 *are defined as follows:*

$$A_ig_i(x) = \operatorname*{opt}_{y\in D}\{u(x,y) + H_i(x,y,g_i(T(x,y)))\} \tag{3.1}$$

for all $x \in S$, $g_i \in B(S)$, $i \in \{1,2,3\}$;

$(C3)$ $A_1(B(S)) \cup A_2(B(S)) \subseteq A_3(B(S))$, $A_1A_3 = A_3A_1$, $A_2A_3 = A_3A_2$;

($C4$) *For any sequence* $\{g_n\}_{n\geq 1} \subset B(S)$ *and* $g \in B(S)$,

$$\lim_{n\to\infty} d(g_n, g) = 0 \;\Rightarrow\; \lim_{n\to\infty} d(A_3(g_n), A_3(g)) = 0.$$

Then the system of functional equations (1.5) *possesses a unique common solution in* $B(S)$.

Proof. It follows from ($C1$) and ($C2$) that A_1, A_2 and A_3 are self mappings in $B(S)$. Presume that $\mathrm{opt}_{y\in D} = \sup_{y\in D}$. For any $h, g \in B(S)$, $x \in S$ and $\varepsilon > 0$, there exist $y, z \in D$ such that

$$A_1g(x) < u(x, y) + H_1(x, y, g(T(x, y))) + \varepsilon, \tag{3.2}$$

$$A_2h(x) < u(x, z) + H_2(x, z, h(T(x, z))) + \varepsilon. \tag{3.3}$$

Meanwhile we get that

$$A_1g(x) \geq u(x, z) + H_1(x, z, g(T(x, z))), \tag{3.4}$$

$$A_2h(x) \geq u(x, y) + H_2(x, y, h(T(x, y))). \tag{3.5}$$

From (3.2) and (3.5) we infer that

$$\begin{aligned} & A_1g(x) - A_2h(x) \\ & < H_1(x, y, g(T(x, y))) - H_2(x, y, h(T(x, y))) + \varepsilon \\ & \leq |H_1(x, y, g(T(x, y))) - H_2(x, y, h(T(x, y)))| + \varepsilon. \end{aligned} \tag{3.6}$$

Similarly, we know that from (3.3) and (3.4)

$$\begin{aligned} & A_1g(x) - A_2h(x) \\ & > H_1(x, z, g(T(x, z))) - H_2(x, z, h(T(x, z))) - \varepsilon \\ & \geq -|H_1(x, z, g(T(x, z))) - H_2(x, z, h(T(x, z)))| - \varepsilon. \end{aligned} \tag{3.7}$$

Combining (3.6), (3.7) and ($C2$), we gain that

$$\begin{aligned} & d(A_1g, A_2h) \\ = & \sup_{x\in S} |A_1g(x) - A_2h(x)| \\ \leq & \sup_{x\in S} \max\{|H_1(x, y, g(T(x, y))) - H_2(x, y, h(T(x, y)))|, \\ & \qquad |H_1(x, z, g(T(x, z))) - H_2(x, z, h(T(x, z)))|\} + \varepsilon \\ \leq & \max\Big\{d(A_3g, A_3h), d(A_3g, A_1g), d(A_3h, A_2h), \\ & \frac{d(A_3g, A_2h)d(A_3h, A_1g)}{1 + d(A_3g, A_3h)}, \frac{d(A_3g, A_2h)d(A_3h, A_1g)}{1 + d(A_1g, A_2h)}, \\ & \frac{d(A_3g, A_1g)d(A_3h, A_2h)}{1 + d(A_3g, A_3h)}, \frac{d(A_3g, A_1g)d(A_3h, A_2h)}{1 + d(A_1g, A_2h)}, \\ & \frac{d(A_3g, A_2h)d(A_3h, A_1g)d(A_3g, A_3h)}{1 + d(A_3g, A_3h)d(A_1g, A_2h)}, \end{aligned}$$

$$\frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)},$$
$$\frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)},$$
$$\frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}\Big\}$$
$$-W\Big(\max\Big\{d(A_3g,A_3h),d(A_3g,A_1g),d(A_3h,A_2h),$$
$$\frac{d(A_3g,A_2h)d(A_3h,A_1g)}{1+d(A_3g,A_3h)},\frac{d(A_3g,A_2h)d(A_3h,A_1g)}{1+d(A_1g,A_2h)},$$
$$\frac{d(A_3g,A_1g)d(A_3h,A_2h)}{1+d(A_3g,A_3h)},\frac{d(A_3g,A_1g)d(A_3h,A_2h)}{1+d(A_1g,A_2h)},$$
$$\frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)},$$
$$\frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)},$$
$$\frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)},$$
$$\frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}\Big\}\Big)+\varepsilon.$$

Letting ε tend to zero, we have

$$\begin{aligned}
&d(A_1g,A_2h)\\
\le\ &\max\Big\{d(A_3g,A_3h),d(A_3g,A_1g),d(A_3h,A_2h),\\
&\frac{d(A_3g,A_2h)d(A_3h,A_1g)}{1+d(A_3g,A_3h)},\frac{d(A_3g,A_2h)d(A_3h,A_1g)}{1+d(A_1g,A_2h)},\\
&\frac{d(A_3g,A_1g)d(A_3h,A_2h)}{1+d(A_3g,A_3h)},\frac{d(A_3g,A_1g)d(A_3h,A_2h)}{1+d(A_1g,A_2h)},\\
&\frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)},\\
&\frac{d(A_3g,A_2h)d(A_3h,A_1g)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)},\\
&\frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_3g,A_3h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)},\\
&\frac{d(A_3g,A_1g)d(A_3h,A_2h)d(A_1g,A_2h)}{1+d(A_3g,A_3h)d(A_1g,A_2h)}\Big\}
\end{aligned}\tag{3.8}$$

$$
\begin{aligned}
&- W\Big(\max \Big\{ d(A_3g, A_3h), d(A_3g, A_1g), d(A_3h, A_2h),\\
&\qquad \frac{d(A_3g, A_2h)d(A_3h, A_1g)}{1 + d(A_3g, A_3h)}, \frac{d(A_3g, A_2h)d(A_3h, A_1g)}{1 + d(A_1g, A_2h)},\\
&\qquad \frac{d(A_3g, A_1g)d(A_3h, A_2h)}{1 + d(A_3g, A_3h)}, \frac{d(A_3g, A_1g)d(A_3h, A_2h)}{1 + d(A_1g, A_2h)},\\
&\qquad \frac{d(A_3g, A_2h)d(A_3h, A_1g)d(A_3g, A_3h)}{1 + d(A_3g, A_3h)d(A_1g, A_2h)},\\
&\qquad \frac{d(A_3g, A_2h)d(A_3h, A_1g)d(A_1g, A_2h)}{1 + d(A_3g, A_3h)d(A_1g, A_2h)},\\
&\qquad \frac{d(A_3g, A_1g)d(A_3h, A_2h)d(A_3g, A_3h)}{1 + d(A_3g, A_3h)d(A_1g, A_2h)},\\
&\qquad \frac{d(A_3g, A_1g)d(A_3h, A_2h)d(A_1g, A_2h)}{1 + d(A_3g, A_3h)d(A_1g, A_2h)} \Big\} \Big).
\end{aligned}
$$

In a similar way we conclude that (3.8) holds for $\text{opt}_{y\in D} = \inf_{y\in D}$. It follows from Theorem 2.1 with $t = 1$ that there exists a unique function $w \in B(S)$ fitting $A_1w = A_2w = A_3w = w$. Consequently the system of functional equations (1.5) possesses a unique common solution $w \in B(S)$. This completes the proof. □

As in the proof of Theorem 3.1, we get the following three results and omit their proofs.

Theorem 3.2. *Let $u : S\times D \to \mathbb{R}, T : S\times D \to S$, $H_1, H_2, H_3 : S\times D\times\mathbb{R} \to \mathbb{R}$ and A_1, A_2 and A_3 satisfy $(C1)$, $(C3)$, $(C4)$, (3.1) and*

$(C5)$ *For all $(x, q, y) \in S^2 \times D$, $g, h \in B(S)$,*

$$
\begin{aligned}
&|H_1(x, y, g(q)) - H_2(x, y, h(q))|\\
\le\ & \max \Big\{ d(A_3g, A_3h), d(A_3g, A_1g), d(A_3h, A_2h),\\
&\qquad \frac{d(A_3g, A_2h)d(A_3h, A_1g)}{1 + d(A_3g, A_3h)}, \frac{d(A_3g, A_2h)d(A_3h, A_1g)}{1 + d(A_1g, A_2h)},\\
&\qquad \frac{d(A_3g, A_1g)d(A_3h, A_2h)}{1 + d(A_3g, A_3h)}, \frac{d(A_3g, A_1g)d(A_3h, A_2h)}{1 + d(A_1g, A_2h)} \Big\}\\
&- W\Big(\max \Big\{ d(A_3g, A_3h), d(A_3g, A_1g), d(A_3h, A_2h),\\
&\qquad \frac{d(A_3g, A_2h)d(A_3h, A_1g)}{1 + d(A_3g, A_3h)}, \frac{d(A_3g, A_2h)d(A_3h, A_1g)}{1 + d(A_1g, A_2h)},\\
&\qquad \frac{d(A_3g, A_1g)d(A_3h, A_2h)}{1 + d(A_3g, A_3h)}, \frac{d(A_3g, A_1g)d(A_3h, A_2h)}{1 + d(A_1g, A_2h)} \Big\} \Big).
\end{aligned}
$$

Then the system of functional equations (1.5) *possesses a unique common solution in $B(S)$.*

Theorem 3.3. *Let* $u : S\times D \to \mathbb{R}$, $T : S\times D \to S$, $H_1, H_2, H_3 : S\times D\times\mathbb{R} \to \mathbb{R}$ *and* A_1, A_2 *and* A_3 *satisfy* $(C1)$, $(C3)$, $(C4)$, (3.1) *and*

$(C6)$ *For all* $(x, q, y) \in S^2 \times D$, $g, h \in B(S)$,

$$\begin{aligned}&|H_1(x,y,g(q)) - H_2(x,y,h(q))|\\ &\le \max\{d(A_3g, A_3h), d(A_3g, A_1g), d(A_3h, A_2h)\}\\ &\quad - W(\max\{d(A_3g, A_3h), d(A_3g, A_1g), d(A_3h, A_2h)\}).\end{aligned}$$

Then the system of functional equations (1.5) *possesses a unique common solution in* $B(S)$.

Theorem 3.4. *Let* $u : S\times D \to \mathbb{R}$, $T : S\times D \to S$, H_1 *and* $H_2 : S\times D\times\mathbb{R} \to \mathbb{R}$ *satisfy the following conditions:*

$(C7)$ u *and* H_i *are bounded for* $i \in \{1, 2\}$;

$(C8)$ *For all* $(x, q, y) \in S^2 \times D$, $g, h \in B(S)$,

$$\begin{aligned}&|H_1(x,y,g(q)) - H_2(x,y,h(q))|\\ &\le \max\Big\{d(g,h), d(g,A_1g), d(h,A_2h), \frac{d(g,A_2h)d(h,A_1g)}{1+d(g,h)},\\ &\qquad \frac{d(g,A_2h)d(h,A_1g)}{1+d(A_1g,A_2h)}, \frac{d(g,A_1g)d(h,A_2h)}{1+d(g,h)}, \frac{d(g,A_1g)d(h,A_2h)}{1+d(A_1g,A_2h)},\\ &\qquad \frac{d(g,A_2h)d(h,A_1g)d(g,h)}{1+d(g,h)d(A_1g,A_2h)}, \frac{d(g,A_2h)d(h,A_1g)d(A_1g,A_2h)}{1+d(g,h)d(A_1g,A_2h)},\\ &\qquad \frac{d(g,A_1g)d(h,A_2h)d(g,h)}{1+d(g,h)d(A_1g,A_2h)}, \frac{d(g,A_1g)d(h,A_2h)d(A_1g,A_2h)}{1+d(g,h)d(A_1g,A_2h)}\Big\}\\ &\quad - W\Big(\max\Big\{d(g,h), d(g,A_1g), d(h,A_2h), \frac{d(g,A_2h)d(h,A_1g)}{1+d(g,h)},\\ &\qquad \frac{d(g,A_2h)d(h,A_1g)}{1+d(A_1g,A_2h)}, \frac{d(g,A_1g)d(h,A_2h)}{1+d(g,h)}, \frac{d(g,A_1g)d(h,A_2h)}{1+d(A_1g,A_2h)},\\ &\qquad \frac{d(g,A_2h)d(h,A_1g)d(g,h)}{1+d(g,h)d(A_1g,A_2h)}, \frac{d(g,A_2h)d(h,A_1g)d(A_1g,A_2h)}{1+d(g,h)d(A_1g,A_2h)},\\ &\qquad \frac{d(g,A_1g)d(h,A_2h)d(g,h)}{1+d(g,h)d(A_1g,A_2h)}, \frac{d(g,A_1g)d(h,A_2h)d(A_1g,A_2h)}{1+d(g,h)d(A_1g,A_2h)}\Big\}\Big),\end{aligned}$$

where the mappings A_1 *and* A_2 *are defined as follows:*

$$A_ig_i(x) = \operatorname*{opt}_{y\in D}\{u(x,y) + H_i(x,y,g_i(T(x,y)))\}, \quad \forall x \in S,\ g_i \in B(S),\ i \in \{1,2\}.$$

Then the following system of functional equations

$$f_i(x) = \operatorname*{opt}_{y\in D}\{u(x,y) + H_i(x,y,f_i(T(x,y)))\}, \quad \forall x \in S,\ i \in \{1,2\}$$

possesses a unique common solution in $B(S)$.

References

[1] R. Bellman and E. S. Lee, *Functional equations arising in dynamic programming*, Aequationes Math. **17** (1978), 1–18.

[2] P. C. Bhakta and S. R. Choudhury, *Some existence theorems for functional equations arising in dynamic programming II*, J. Math. Anal. Appl. **131** (1988), 217–231.

[3] P. C. Bhakta and S. Mitra, *Some existence theorems for functional equations arising in dynamic programming*, J. Math. Anal. Appl. **98** (1984), 348–362.

[4] Z. Liu, *Existence theorems of solutions for certain classes of functional equations arising in dynamic programming*, J. Math. Anal. Appl. **262** (2001), 529–553.

[5] ———, *Coincidence theorems for expansive mappings with applications to the solutions of functional equations arising in dynamic programming*, Acta Sci. Math. (Szeged) **65** (1999), 359–369.

[6] ———, *Compatible mappings and fixed points*, Acta Sci. Math. (Szeged) **65** (1999), 371–383.

[7] ———, *A note on unique common fixed point*, Bull. Cal. Math. Soc. **85** (1993), 469–472.

[8] Z. Liu, R. P. Agarwal, and S. M. Kang, *On solvability of functional equations and system of functional equations arising in dynamic programming*, J. Math. Anal. Appl. **297** (2004), 111–130.

[9] Z. Liu and S. M. Kang, *Properties of solutions for certain functional equations arising in dynamic programming*, J. Global Optim. **34** (2006), 273–292.

[10] ———, *Existence and uniqueness of solutions for two classes of functional equations arising in dynamic programming*, Acta Math. Appl. Sin. **23** (2007), 195–208.

[11] Z. Liu and J. K. Kim, *A common fixed point theorem with applications in dynamic programming*, Nonlinear Funct. Anal. & Appl. **1** (2006), 11–19.

[12] Z. Liu and J. S. Ume, *On properties of solutions for a class of functional equations arising in dynamic programming*, J. Optim. Theory Appl. **117** (2003), 533–551.

[13] Z. Liu, J. S. Ume, and S. M. Kang, *Some existence theorems for functional equations arising in dynamic programming*, J. Korean Math. Soc. **43** (2006), no. 1, 11–28.

[14] Z. Liu, Y. Xu, J. S. Ume, and S. M. Kang, *Solutions to two functional equations arising in dynamic programming*, J. Comput. Appl. Math. **192** (2006), 251–269.

[15] H. K. Pathak and B. Fisher, *Common fixed point theorems with applications in dynamic programming*, Glasnik Mate. **31** (1996), 321–328.

[16] B. N. Ray, *On common fixed points in metric spaces*, Indian J. pure appl. Math. **19** (1988), 960–962.

Zeqing Liu
Department of Mathematics
Liaoning Normal University
Liaoning 116029, P. R. China
E-mail address: zeqingliu@sina.com.cn

Min Liu
Department of Mathematics
Liaoning Normal University
Liaoning 116029, P. R. China
E-mail address: min_liu@yeah.net

Hyeong Kug Kim
Department of Mathematics
Gyeongsang National University
Jinju 660-701, Korea
E-mail address: math@gnu.ac.kr

Shin Min Kang
Department of Mathematics and
The Research Institute of Natural Science
Gyeongsang National University
Jinju 660-701, Korea
E-mail address: smkang@gnu.ac.kr