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UNIFYING A MULTITUDE OF COMMON FIXED POINT THEOREMS EMPLOYING AN IMPLICIT RELATION

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ABSTRACT. A general common fixed point theorem for two pairs of weakly compatible mappings using an implicit function is proved without any continuity requirement which generalizes the result due to Popa [20, Theorem 3]. In process, several previously known results due to Fisher, Kannan, Jeong and Rhoades, Imdad and Ali, Imdad and Khan, Khan, Shahzad and others are derived as special cases. Some related results and illustrative examples are also discussed. As an application of our main result, we prove an existence theorem for the solution of simultaneous Hammerstein type integral equations.

1. Introduction

As established in Jungck [12], a common fixed point theorem in metric spaces generally involves conditions on contraction, commutativity and continuity of the involved mappings besides a suitable containment of range of one mapping into the range of other. To prove a new metrical common fixed point theorem one is always required to improve one or more of these conditions. Of all these four conditions, the means of improving contraction condition to prove new results on fixed and common fixed point is much discussed which continue to attract the attention of the researchers of this domain. But in this paper we use an implicit function due to Popa [20].

The tradition of improving commutativity condition in common fixed point theorems was initiated by Sessa [21]. Inspired by the definition of weak commutativity of Sessa [21], researchers of this domain, introduced several definitions of weak commutativity such as: Compatible mappings, Compatible mappings of type (A), Compatible mappings of type (B), Compatible mappings of type (P), Compatible mappings of type (C), Biased maps, R-weakly commuting mappings and some others whose lucid comparison and illustration

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can be found in Murthy [17]. In our subsequent work, we use the most natural of these weak conditions namely 'weak compatibility' due to Jungck [14] defined as follows:

Definition 1.1. Let f and g be self-mappings of a nonempty set X. Then the pair (f, g) is said to be weak compatible if for any $x \in X$ with $fx = gx$ implies $fgx = gfx$.

The first ever endeavor to improve the continuity requirement in fixed point theorem can be traced back to Kannan [15] wherein he noticed that there do exist maps with fixed point possessing discontinuities in their domain. However, in common fixed point theorems such an effort is due to Singh and Mishra [23], who were able to relax continuity requirement completely when they replaced the completeness of the space with the completeness of range of any one map of the involved four maps. At the same time Pant [19] introduced the notion of reciprocal continuity to improve continuity requirement in common fixed point theorem. Here, we opt to use the way of Singh and Mishra [23].

In an attempt to improve a common fixed point theorem of Imdad and Khan [8], Popa [20] defined an implicit function (to be described in Section 2) and constructed some examples of it. In what follows, we strengthen the implicit function of Popa [20] by adding some more examples to this effect and demonstrate that how an implicit function can imply several rational inequalities which also include many unknown natural contraction conditions as well. In this paper by combining the ideas of Popa [20] and Singh and Mishra [23], we prove a general common fixed point theorem without continuity requirement wherein the commutativity is restricted to the points of coincidence while the completeness requirement of the space is weakened to a set of four alternative natural conditions. In process results of Ahmad and Imdad [1], Fisher [5], Kannan [15], Imdad and Ali [9], Jungck [13], Khan [16], Chugh and Kumar [2] and others are deduced as special cases.

2. Implicit relation

Let $\Re_+$ be the set of nonnegative real numbers, P be the subset of $\Re_+^6$ formed from the 6-tuples of the type $(t_1, t_2, 0, 0, t_5, t_6)$, where $t_1, t_2, t_5, t_6 \in \Re_+$, and let Ψ be a family of lower semi-continuous functions $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ satisfying the following conditions:

F_1 : F is non-increasing in variables t_5 and t_6,

F_2 : there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with

$$F(u, v, v, u, u + v, 0) \leq 0 \text{ or } F(u, v, u, v, 0, u + v) \leq 0$$

we have $u \leq h.v$.

Example 2.1. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - a_1 \left[\frac{t_3^2 + t_4^2}{t_3 + t_4}\right] - a_2 t_2 - a_3[t_5 + t_6],$$

where $a_i \geq 0$ $(i = 1, 2, 3)$ with at least one a_i non zero and $a_1 + a_2 + 2a_3 < 1$.

Example 2.2. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 - \frac{ct_3t_4 + bt_5t_6}{t_3 + t_4},$$

where $a, b, c \geq 0$ and $1 < 2a + c < 2$.

Example 2.3. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^3 + t_2^2 + t_1 - a\left[\frac{t_3^3 + t_4^3 + t_2t_5t_6}{t_3^2 + t_4^2}\right], \text{ where } a \in (0, 1).$$

Details and verifications of all above examples can be found in Popa [20]. To this effect, we also construct some examples of implicit functions which deduce several known as well as unknown rational inequalities.

Example 2.4. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - a.\max\left\{t_2, t_3, t_4, \frac{t_5}{2}, \frac{t_6}{2}\right\}, \text{ where } a \in (0, 1).$$

F_1 : Obvious.

F_2 : Let $u > 0$ and $F(u, v, v, u, u + v, 0) = u - a.\max\{v, v, u, \frac{1}{2}(u + v)\} \leq 0$. If $u \geq v$, then $u \leq au < u$, a contradiction. Thus $u < v$ and $u \leq av$, where $a \in (0, 1)$.

If $F(u, v, u, v, 0, u + v) \leq 0$, then similar argument established F_2.

Example 2.5. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - a_1\left[\frac{t_3^2 + t_4^2}{t_3 + t_4}\right] - a_2[t_5 + t_6] - a_3t_2,$$

where $a_i \geq 0$ $(i = 1, 2, 3)$ with at least one a_i non zero and $2a_1 + 2a_2 + a_3 < 1$.

Example 2.6. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - (a_1t_2 + a_2t_3 + a_3t_4 + a_4t_5 + a_5t_6), \text{ where } \sum_{i=1}^{5} a_i < 1.$$

Example 2.7. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - a.\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}, \text{ where } a \in (0, 1).$$

Example 2.8. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 - b\left[\frac{t_3t_5 + t_4t_6}{t_3 + t_4}\right],$$

where $a, b \geq 0$ and $a + b < 1$.

Example 2.9. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 - bt_2 \left[\frac{t_5 + t_6}{t_3 + t_4}\right],$$

where $a, b \geq 0$ and $a + b < 1$.

Example 2.10. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - a.\max\left\{t_3, t_4, t_2 \left[\frac{t_5 + t_6}{t_3 + t_4}\right]\right\}, \text{ where } a \in (0, 1).$$

Example 2.11. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 - \frac{t_5 t_6}{t_3 + t_4}, \text{ where } a \in (0, 1).$$

Example 2.12. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 - bt_2^2 \left[\frac{\sqrt{t_5 t_6 + 1}}{t_3 + t_4}\right], \text{ where } a, b \geq 0 \text{ and } 2a + b < 2.$$

Example 2.13. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 - P \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - at_2 \left[\frac{t_5 + t_6}{t_3 + t_4}\right], \text{ where } a \in (0, 1).$$

Since verification of requirements (F_1 and F_2) for Examples 2.5–2.13 is straightforward, hence details are omitted.

3. Results

We prove our main result as follows:

Theorem 3.1. *Let A, B, S and T be self-mappings of a metric space (X, d) with $A(X) \subset T(X)$ and $B(X) \subset S(X)$ satisfying*
(3.1.1)

$$F(d(Ax, By), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)) \leq 0$$

for all distinct $x, y \in X$ with $d(Sx, Ax) + d(Ty, By) \neq 0$ where $F \in \Psi$ or,

(3.1.2) $$d(Ax, By) = 0 \text{ whenever } d(Sx, Ax) + d(Ty, By) = 0.$$

If one of $A(X), B(X), S(X)$ or $T(X)$ is a complete subspace of X, then
(a) *(A, S) has a point of coincidence,*
(b) *(B, T) has a point of coincidence.*

Moreover, if the pairs (A, S) and (B, T) are weak compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $A(X) \subset T(X)$, we find a point x_1 in X such that $Ax_0 = Tx_1$. Also, since $B(X) \subset S(X)$, we choose a point x_2 with $Bx_1 = Sx_2$. Thus in general for the point x_{2n} one find a point x_{2n+1} such that $Ax_{2n} = Tx_{2n+1}$ and then a point x_{2n+2} with $Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$. Repeating such arguments one can construct a sequence

$\{z_n\}$ such that $z_{2n} = Ax_{2n} = Tx_{2n+1}$ and $z_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$.

We now consider the following two cases:

Case I: Suppose $d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n+1}, Bx_{2n+1}) \neq 0$ for $n = 0, 1, 2, \ldots$. Then using inequality (3.1.1), we have

$$F(d(Ax_{2n}, Bx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}), d(Sx_{2n}, Bx_{2n+1}), d(Tx_{2n+1}, Ax_{2n})) \leq 0$$

$$\text{or } F(d(Ax_{2n}, Bx_{2n+1}), d(Bx_{2n-1}, Ax_{2n}), d(Bx_{2n-1}, Ax_{2n}), d(Ax_{2n}, Bx_{2n+1}), d(Bx_{2n-1}, Ax_{2n}) + d(Ax_{2n}, Bx_{2n+1}), 0) \leq 0$$

In view of (F_2), we have

$$d(Ax_{2n}, Bx_{2n+1}) \leq h.d(Ax_{2n}, Bx_{2n-1}).$$

Again, if $d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n-1}, Bx_{2n-1}) \neq 0$ for $n = 0, 1, 2, \ldots$, then as earlier using (F_2), we have

$$d(Ax_{2n}, Bx_{2n-1}) \leq h.d(Ax_{2n-2}, Bx_{2n-1})$$

and so,

$$d(Ax_{2n}, Bx_{2n+1}) \leq h^{2n}.d(Ax_0, Bx_1) \text{ for } n = 0, 1, 2, \ldots.$$

Thus, in all $\{z_n\}$ is a Cauchy sequence in X. Now suppose that $S(X)$ is a complete subspace of X, then the subsequence $z_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ converges to some z in $S(X)$. As $\{z_n\}$ is a Cauchy sequence which contains a convergent subsequence $\{z_{2n+1}\}$, therefore the sequence $\{z_n\}$ also converges implying thereby the convergence of the subsequence $\{z_{2n}\}$ being a subsequence of convergent sequence $\{z_n\}$. Let $u \in S^{-1}(z)$, then $Su = z$. To prove $Au = z$, set $x = u$ and $y = x_{2n+1}$ in inequality (3.1.1), then

$$F(d(Au, Bx_{2n+1}), d(Su, Tx_{2n+1}), d(Su, Au), d(Tx_{2n+1}, Bx_{2n+1}), d(Su, Bx_{2n+1}), d(Tx_{2n+1}, Au)) \leq 0$$

which on letting $n \to \infty$ reduces to

$$F(d(Au, Su), 0, d(Su, Au), 0, 0, d(Su, Au)) \leq 0$$

implying thereby $d(Au, Su) \leq 0$. Hence, $Au = Su$ which shows that the pair (A, S) has a point of coincidence. This proves (a).

Since $A(X) \subset T(X)$, $Au = z$ implies that $z \in T(X)$. Let $v \in T^{-1}(z)$, then $Tv = z$. Again using the earlier arguments, it can be easily shown that $Bv = z$, implying thereby $Tv = Bv = z$ which establishes (b). If one assumes $T(X)$ to be a complete subspace of X, then analogues arguments can be produced to establish (a) and (b). The remaining two cases pertain essentially to the earlier ones. Indeed, if $B(X)$ is complete, then $z \in B(X) \subset S(X)$. Similarly, if $A(X)$ is complete, then $z \in A(X) \subset T(X)$. Thus in every case (a) and (b) are completely established.

Moreover, if the pairs (A, S) and (B, T) are weak compatible at u and v respectively, then

$$z = Au = Su = Bv = Tv,$$

$$Az = A(Su) = S(Au) = Sz \text{ and } Bz = B(Tv) = T(Bv) = Tz.$$

Now in order to prove $Az = z$, we note that

$$d(Az, Sz) + d(Bv, Tv) = 0$$

which due to (3.1.2), amounts to say that

$$d(Az, Bv) = d(Az, z) = 0$$

yielding thereby $Az = z$, hence $Az = Sz = z$. Similarly, one can show that $z = Bz = Tz$. Thus z is a common fixed of A, B, S and T. The uniqueness of common fixed point follows easily.

Case II: Now suppose that $d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n+1}, Bx_{2n+1}) = 0$ for some n. Then proof follows on the lines of Popa [20], hence it is omitted. This completes the proof. □

Remark 3.1. Theorem 3.1 presents a generalized and improved form of Theorem 3 due to Popa [20] without any continuity requirement besides limiting commutativity requirement to the points of coincidence along with replacement of the completeness of the space with four alternative natural conditions.

Corollary 3.1. *The conclusions of Theorem 3.1 remain true if implicit relation* (3.1.1) *is replaced by any one of the following retaining condition* (3.1.2) *whenever it is relevant.*

$$(\mathbf{a_1}) \qquad \begin{aligned} d(Ax, By) \leq\ & a_1 \frac{d^2(Sx, Ax) + d^2(Ty, By)}{d(Sx, Ax) + d(Ty, By)} + a_2 d(Sx, Ty) \\ & + a_3[d(Sx, By) + d(Ty, Ax)] \end{aligned}$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a_i \geq 0$ $(i = 1, 2, 3)$ *with at least one* a_i *non-zero and* $a_1 + a_2 + 2a_3 < 1$.

$(\mathbf{a_2})$

$$d(Ax, By) \leq \frac{c.d(Sx, Ax).d(Ty, By) + b.d(Sx, By).d(Ty, Ax)}{d(Sx, Ax) + d(Ty, By)} + a.d(Sx, Ty)$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a, b, c \geq 0$ *and* $1 < 2a + c < 2$.

$$(\mathbf{a_3}) \qquad \begin{aligned} & d^3(Ax, By) + d^2(Sx, Ty) + d(Ax, By) \\ & \leq a \frac{d^3(Sx, Ax) + d^3(Ty, By) + d(Sx, Ty).d(Sx, By).d(Ty, Ax)}{d^2(Sx, Ax) + d^2(Ty, By)} \end{aligned}$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a \in (0, 1)$.

$(\mathbf{a_4})$

$$d(Ax, By) \leq a. \max\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{d(Sx, By)}{2}, \frac{d(Ty, Ax)}{2}\right\}$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a \in (0, 1)$.

$$(\mathbf{a_5}) \quad d(Ax, By) \leq a_1 \frac{d^2(Sx, Ax) + d^2(Ty, By)}{d(Sx, Ax) + d(Ty, By)} + a_2[d(Sx, By) + d(Ty, Ax)] + a_3 d(Sx, Ty)$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a_i \geq 0$ $(i = 1, 2, 3)$ *with at least one* a_i *non-zero and* $2a_1 + 2a_2 + a_3 < 1$.

$$(\mathbf{a_6}) \quad d(Ax, By) \leq a_1 d(Sx, Ty) + a_2 d(Sx, Ax) + a_3 d(Ty, By) + a_4 d(Sx, By) + a_5 d(Ty, Ax), \text{ where } \sum_{i=1}^{5} a_i < 1.$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$

$(\mathbf{a_7})$

$$d(Ax, By) \leq a. \max \left\{ d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{d(Sx, By) + d(Ty, Ax)}{2} \right\}$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a \in (0, 1)$.

$(\mathbf{a_8})$

$$d(Ax, By) \leq a \frac{d(Sx, Ax).d(Sx, By) + d(Ty, By).d(Ty, Ax)}{d(Sx, Ax) + d(Ty, By)} + b.d(Sx, Ty)$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a + b < 1$.

$$(\mathbf{a_9}) \quad d(Ax, By) \leq a.d(Sx, Ty) + b.d(Sx, Ty) \left[\frac{d(Sx, By) + d(Ty, Ax)}{d(Sx, Ax) + d(Ty, By)} \right]$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a + b < 1$.

$(\mathbf{a_{10}})$

$$d(Ax, By) \leq a \max \left\{ d(Sx, Ax), d(Ty, By), d(Sx, Ty), \left[\frac{d(Sx, By) + d(Ty, Ax)}{d(Sx, Ax) + d(Ty, By)} \right] \right\}$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a \in (0, 1)$.

$$(\mathbf{a_{11}}) \quad d(Ax, By) \leq a.d(Sx, Ty) + \frac{d(Sx, By).d(Ty, Ax)}{d(Sx, Ax) + d(Ty, By)}$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a \in (0, 1)$.

$$(\mathbf{a_{12}}) \quad d(Ax, By) \leq a.d(Sx, Ty) + b.d(Sx, Ty)^2 \frac{\sqrt{d(Sx, By).d(Ty, Ax) + 1}}{d(Sx, Ax) + d(Ty, By)}$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a, b \geq 0$ *and* $2a + b < 2$.

$$(\mathbf{a_{13}}) \quad d(Ax, By) \leq a.d(Sx, Ty) \left[\frac{d(Sx, By) + d(Ty, Ax)}{d(Sx, Ax) + d(Ty, By)} \right]$$

for all distinct $x, y \in X$ *with* $d(Sx, Ax) + d(Ty, By) \neq 0$, *where* $a \in (0, 1)$.

Proof. The proof of Corollary 3.1 follows from Theorem 3.1 and Examples 2.1-2.13. □

Remark 3.2. Corollaries corresponding to contraction conditions $(a_1), (a_4)$, $(a_5), (a_6)$ and (a_7) are known results due to Imdad and Ali [9], Chugh and Kumar [2], Khan [16], Hardy and Rogers [7] and Jungck [13] (also Shahzad [22]), respectively. We also point out that some of above corollaries are new to the literature (e.g. Corollaries corresponding to a_3 and $a_8 - a_{13}$).

As an application of Theorem 3.1, we prove a result for four finite families of self-mappings which runs as follows:

Theorem 3.2. *Let $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_p\}$, $\{S_1, S_2, \ldots, S_n\}$ and $\{T_1, T_2, \ldots, T_q\}$ be four finite families of self-mappings of a metric space (X, d) with*

$$A = A_1A_2\cdots A_m,\ B = B_1B_2\cdots B_p,\ S = S_1S_2\cdots S_n \text{ and } T = T_1T_2\cdots T_q$$

satisfying conditions (3.1.1) and (3.1.2) with $A(X) \subset T(X), B(X) \subset S(X)$. If one of $A(X), B(X), S(X)$ or $T(X)$ is a complete subspace of X, then

(c) *(A, S) has a point of coincidence.*

(d) *(B, T) has a point of coincidence.*

Further, if $A_iA_j = A_jA_i$, $B_kB_l = B_lB_k$, $S_rS_s = S_sS_r$, $T_tT_u = T_uT_t$, $A_iB_k = B_kA_i$ and $S_rT_t = T_tS_r$ for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, p\}$, $r, s \in I_3 = \{1, 2, \ldots, n\}$ and $t, u \in I_4 = \{1, 2, \ldots, q\}$, then (for all $i \in I_1, k \in I_2, r \in I_3$ and $t \in I_4$) A_i, B_k, S_r and T_t have a common fixed point.

Proof. The conclusions (c) and (d) are immediate as A, B, S and T satisfy all the conditions of Theorem 3.1. Now appealing to componentwise commutativity of various pairs, one immediately concludes that $AS = SA$ and $BT = TB$ and hence, obviously both the pairs (A, S) and (B, T) are weak compatible. Note that all the conditions of Theorem 3.1 (for mappings A, B, S and T) are satisfied ensuring the existence of unique common fixed point z. Now one needs to show that z remains the fixed point of all the component maps. For this consider

$$\begin{aligned} A(A_iz) &= ((A_1A_2\cdots A_m)A_i)z = (A_1A_2\cdots A_{m-1})((A_mA_i)z) \\ &= (A_1\cdots A_{m-2})(A_{m-1}A_i(A_mz)) \\ &= (A_1\cdots A_{m-2})(A_iA_{m-1}(A_mz)) = \cdots \\ &= A_1A_i(A_2A_3A_4\cdots A_mz) = A_iA_1(A_2A_3\cdots A_mz) = A_i(Az) = A_iz. \end{aligned}$$

Similarly, one can show that,

$$\begin{aligned} & A(B_kz) = B_k(Az) = B_kz,\ B(B_kz) = B_k(Bz) = B_kz, \\ & B(A_iz) = A_i(Bz) = A_iz,\ S(S_rz) = S_r(Sz) = S_rz, \\ & S(T_tz) = T_t(Sz) = T_tz,\ T(T_tz) = T_t(Tz) = T_tz, \\ \text{and}\quad & T(S_rz) = S_r(Tz) = S_rz, \end{aligned}$$

which show that (for all i, r, k and t) A_iz and B_kz are other fixed points of the pair (A, B) whereas S_rz and T_tz are other fixed points of the pair (S, T).

Now appealing to the uniqueness of common fixed points of both the pairs separately, we get

$$z = A_i z = S_r z = B_k z = T_t z,$$

which shows that z is a common fixed point of A_i, S_r, B_k and T_t for all i, r, k and t. □

By setting $A_1 = A_2 = \cdots = A_m = G$, $B_1 = B_2 = \cdots = B_p = H$, $S_1 = S_2 = \cdots = S_n = I$ and $T_1 = T_2 = \cdots = T_q = J$ in Theorem 3.2, we deduce the following:

Corollary 3.2. *Let G, H, I and J be self-mappings of a metric space (X, d) with $G^m(X) \subset J^q(X)$ and $H^p(X) \subset I^n(X)$ satisfying the condition*

$$F(d(G^m x, H^p y), d(I^n x, J^q y), d(G^m x, I^n x), d(H^p y, J^q y), \\ d(I^n x, H^p y), d(J^q y, G^m x)) \leq 0$$

if $d(G^m x, I^n x) + d(H^p y, J^q y) \neq 0$ or $d(G^m x, H^p y) = 0$ whenever $d(G^m x, I^n x) + d(H^p y, J^q y) = 0$, for all $x, y \in X$ where m, n, p and q are fixed positive integers. If one of $G^m(X), H^p(X), I^n(X)$ or $J^q(X)$ is a complete subspace of X, then G, H, I and J have a unique common fixed point provided $GI = IG$ and $HJ = JH$.

Remark 3.3. By restricting four families as $\{A_1, A_2\}, \{B_1, B_2\}, \{S_1\}$ and $\{T_1\}$ in Theorem 3.2, we deduce a substantial but partial generalization of the main result of Imdad and Khan [8] as such a result will deduce stronger commutativity condition besides relaxing continuity requirements and weakening completeness requirement of the space to four alternative natural conditions.

Remark 3.4. Corollary 3.2 is a slight but partial generalization of Theorem 3.1 as the commutativity requirements (i.e., $GI = IG$ and $HJ = JH$) in this corollary are stronger as compared to weak compatibility in Theorem 3.1.

Remark 3.5. A result similar to Corollary 3.1 can be derived from Corollary 3.2 for iterates of maps. For the sake of brevity, we have not included the details.

4. Illustrative examples

Now we furnish examples to demonstrate the validity of the hypotheses and degree of generality of Theorem 3.1 over earlier result due to Popa [20].

Example 4.1. Consider X=[2, 20] with usual metric. Define self-mappings A, B, S and T on X as

$$Ax = 2,\ x \in \{2\} \cup (5, 20],\ Ax = 4,\ 2 < x \leq 5,$$
$$Bx = 2,\ x \in \{2\} \cup (5, 20],\ Bx = 3,\ 2 < x \leq 5,$$
$$S2 = 2,\ Sx = 8,\ 2 < x \leq 5,\ Sx = (x+1)/3,\ x > 5 \text{ and}$$
$$T2 = 2,\ Tx = 12 + x,\ 2 < x \leq 5,\ Tx = x - 3,\ x > 5.$$

One may note that the pairs (A,S) and (B,T) commute at 2 which is their common coincidence point. Also $A(X) = \{2,4\} \subset [2,17] = T(X)$ and $B(X) = \{2,3\} \subset [2,7] \cup \{8\} = S(X)$. All the needed pairwise commutativity at coincidence point 2 are immediate. Define $F(t_1,t_2,\ldots,t_6) : \Re^6_+ - P \rightarrow \Re$ as

$$F(t_1,t_2,\ldots,t_6) = t_1 - a_1\left[\frac{t_3^2+t_4^2}{t_3+t_4}\right] - a_2t_2 - a_3[t_5+t_6]$$

where $a_i \geq 0$ $(i = 1,2,3)$ with at least one a_i non zero and $a_1 + a_2 + 2a_3 < 1$.

By a routine calculation one can verify that contraction conditions (3.1.1) and (3.1.2) are satisfied for $a_1 = \frac{1}{5}$ and $a_2 = a_3 = \frac{1}{4}$. If $x,y \in \{2\} \cup (5,20]$ then $d(Ax,By) = 0$ and verification is trivial. If $x \in (2,5]$ and $y > 5$, then

$$a_1\left[\frac{d^2(Sx,Ax)+d^2(Ty,By)}{d(Sx,Ax)+d(Ty,By)}\right] + a_2d(Sx,Ty) + a_3[d(Sx,By)+d(Ty,Ax)]$$

$$= \frac{1}{5}\left[\frac{4^2+|y-5|^2}{4+|y-5|}\right] + \frac{1}{4}|y-11| + \frac{1}{4}[6+|y-7|]$$

$$\geq \begin{cases} \frac{4}{5}+\frac{1}{4}(24-2y) > 2 = d(Ax,By), & \text{if } y \in (5,7] \\ \frac{4}{5}+\frac{10}{4} = \frac{33}{10} > 2 = d(Ax,By), & \text{if } y \in (7,11] \\ \frac{4}{5}+\frac{1}{4}(2y-12) > 2 = d(Ax,By), & \text{if } y > 11. \end{cases}$$

Similarly, one can verify the other cases. Thus all the conditions of Theorem 3.1 are satisfied and 2 is the unique common fixed point of A, B, S and T.

On the other hand, the pairs (A,S) and (B,T) are neither compatible nor reciprocally continuous. To substantiate this, consider the sequence $\{x_n = 5 + \frac{1}{n}\}$ in X. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2$, $\lim_{n\to\infty} ASx_n = 4$ and $\lim_{n\to\infty} SAx_n = 2$ which shows that the pair (A,S) is noncompatible whereas $\lim_{n\to\infty} ASx_n = 4 \neq A2$ establishes that the pair (A,S) is not reciprocally continuous. Similarly, one can show that the pair (B,T) is neither compatible nor reciprocally continuous as well. One may note that all the mappings in this example are discontinuous at their unique common fixed point 2.

Here it is also worth noting that none of the theorems from [1, 5, 7, 8, 10, 13, 16, 22] can be used in the context of this example as all the involved maps are discontinuous whereas all earlier theorems require the continuity of at least one involved map.

Example 4.2. Let $X = \{0, 1, 1/2, 1/2^2, 1/2^3, \ldots\}$ be a metric space with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Define mappings $A,T : X \rightarrow X$ by $A(0) = 1/2^2$, $A(1/2^n) = 1/2^{n+2}$ and $T(0) = 1/2$, $T(1/2^n) = 1/2^{n+1}$ for $n = 0,1,2,\ldots$ respectively. Also set $B = A$ and $S = T$. Clearly

$$A(X) = \{1/2^2, 1/2^3, \ldots\} \subset \{1/2, 1/2^2, 1/2^3, \ldots\} = T(X).$$

Considering the same implicit function as in Example 4.1, one can verify that the contraction condition (3.1.1) (also (3.1.2)) is satisfied for $a_1 = 1/8$, $a_2 =$

$1/6$ and $a_3 = 1/4$. For example, choose $x = 0$ and $y = \frac{1}{2}$, then

$$a_1 \left[\frac{d^2(Tx,Ax) + d^2(Ty,Ay)}{d(Tx,Ax) + d(Ty,Ay)} \right] + a_2 d(Tx,Ty) + a_3[d(Tx,Ay) + d(Ty,Ax)]$$

$$= \frac{1}{8} \left[\frac{|\frac{1}{2} - \frac{1}{2^2}|^2 + |\frac{1}{2^2} - \frac{1}{2^3}|^2}{|\frac{1}{2} - \frac{1}{2^2}| + |\frac{1}{2^2} - \frac{1}{2^3}|} \right] + \frac{1}{6} \left| \frac{1}{2} - \frac{1}{2^2} \right| + \frac{1}{4} \left[\left| \frac{1}{2} - \frac{1}{2^3} \right| + \left| \frac{1}{2^2} - \frac{1}{2^2} \right| \right]$$

$$= \frac{1}{8}\left(\frac{5}{24}\right) + \frac{1}{6}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{3}{8}\right) = \frac{31}{192} > \frac{1}{8} = d(Ax,Ay).$$

Thus all the conditions of Theorem 3.1 are satisfied except the completeness of the subspaces $A(X)$ and $T(X)$. Note that A and T have no point of coincidence. Here it is fascinating to note that in the set up of Theorem 3.1 even the completeness of the space cannot ensure the existence of coincidence point as the space X is complete in the present example. Also note that mappings A and T are not continuous at 0.

5. An application to Hammerstein type integral equations

It is well known that fixed point theorems have fruitful applications in existence theory of solutions of differential as well as integral equations. Proving existence theorems via fixed and common fixed point theorems continues to be an interesting area of research and there already exists considerable literature on this theme. Here, we especially recall Dhage [3, 4] wherein existence theorems for differential and integral equations are proved as an application of common fixed point theorems. As fixed point theorems are utilized to prove existence theorems for Volterra-Hammerstein integral equations and Hammerstein type integral equations, common fixed point theorems can be utilized to prove results on existence of solutions of simultaneous Volterra-Hammerstein integral equations and simultaneous Hammerstein type integral equations. It is interesting to note that when the control functions are identical, the existence of the solution can very well be established by using the classical Jungck theorem [12]. To substantiate our viewpoint, as an application of one of our common fixed point theorem, we prove an existence theorem for the solution of simultaneous Hammerstein type integral equations in $L_\infty[a,b]$.

Consider following simultaneous Hammerstein type integral equations:

$$x(t) = y(t) + f_i(t, x(t)) + \lambda \int_a^b k(t,s) g_i(s, x(s)) ds \tag{5.1}$$

for all $a \le t \le b$ and $i = 1, 2$, where $y(t) \in L_\infty[a,b]$ is known function. Here it may be explicitly mentioned that such simultaneous equations appear in a problem associated with free surface seepage from nonlinear channels constructed by two different types of materials under the control functions $f_i(., x(.))$ and $g_i(., x(.))$. Apart from this example, there do exist natural instances from other branches of engineering wherein Hammerstein type integral equations occur.

For further concrete instances, one can consult books by Griffel [6] and Joshi and Bose [11].

Now, we proceed to prove an existence theorem as an application of Theorem 3.1. In order to state our result, we are required to assume the following terminology and setting.

Condition (a): there exists a sequence $\{x_n\}$ in $L_\infty[a,b]$ such that

$$x_n(t) - y(t) - \lambda \int_a^b k(t,s) g_1(s, x_n(s))ds,\ f_1(t, x_n(t)) \to \gamma(t) \in L_\infty[a,b]$$

implies
(5.2)

$$\lim_{n\to\infty} \sup_{a\le t\le b} \Big| f_1\Big(t, x_n(t) - y(t) - \lambda \int_a^b k(t,s) g_1(s, x_n(s))ds\Big) - x_n(t) + 2y(t)$$
$$+\lambda \int_a^b k(t,s)\Big[g_1(s, x_n(s)) + g_1\Big(s, x_n(s) - y(s) - \lambda \int_a^b k(s,\tau) g_1(\tau, x_n(\tau))d\tau\Big)\Big]ds\Big| = 0$$

and

$$\lim_{n\to\infty} \sup_{a\le t\le b} \Big| f_1(t, x_n(t) - y(t) - \lambda \int_a^b k(t,s) g_1(s, f_1(s, x_n(s)))ds - f_1(t, f_1(t, x_n(t)))\Big| = 0,$$

Condition (b): there exists a $x(t) \in L_\infty[a,b]$ such that

$$x(t) - y(t) - \lambda \int_a^b k(t,s) g_2(s, x(s))ds = f_2(t, x(t))$$

implies

$$\sup_{a\le t\le b} \Big| x(t) - 2y(t) - \lambda \int_a^b k(t,s)\Big[g_2(s, x(s)) + g_2\Big(s, x(s) - y(s)$$
$$- \lambda \int_a^b k(t,\tau) g_2(\tau, x(\tau))d\tau\Big)\Big]ds - f_2(t, x(t))\Big|$$
$$\le \sup_{a\le t\le b} \Big| f_2\Big(t, x(t) - y(t) - \lambda \int_a^b k(t,s) g_2(s, x(s)ds\Big)$$
$$- x(t) + y(t) + \lambda \int_a^b k(t,s) g_2(s, x(s))ds\Big|,$$

$$\sup_{a \le t \le b} \int_a^b |k(t,s)| ds = M < \infty. \tag{5.3}$$

Condition (c): $g_i(s,x)$ are continuous in s and x and satisfy

$$|g_1(s,x(s)) - g_2(s,y(s))| \le N_1 \|x(s) - y(s)\| \tag{5.4}$$

for all $s \in [a,b]$ and $x, y \in L_\infty[a,b]$ with $N_1 > 0$.

Condition (d): $f_i(s,x)$ are continuous in s and x and satisfy

$$|f_1(s,x(s)) - f_2(s,y(s))| \ge N_2 \|x(s) - y(s)\| \tag{5.5}$$

for all $s \in [a,b]$ and $x, y \in L_\infty[a,b]$ with $N_2 > 1$.

Theorem 5.1. *The simultaneous integral equations described by* (5.1) *under conditions* (5.2)-(5.5) *have a unique solution in* $L_\infty[a,b]$ *provided* $k = \dfrac{1 + |\lambda| N_1 M}{N_2} < 1$.

Proof. Define self mappings A, B, S and T on $L_\infty[a,b]$ as follows:

$$Ax(t) = x(t) - \alpha(t) - \lambda \int_a^b k(t,s) g_1(s, x(s)) ds,$$

$$Bx(t) = x(t) - \alpha(t) - \lambda \int_a^b k(t,s) g_2(s, x(s)) ds,$$

$$Sx(t) = f_1(t, x(t)) \text{ and } Tx(t) = f_2(t, x(t)).$$

Notice that $A(L_\infty[a,b]) \subset T(L_\infty[a,b]) = L_\infty[a,b]$ and $B(L_\infty[a,b]) \subset S(L_\infty[a,b]) = L_\infty[a,b]$. In order to verify the contraction condition (a_1) of Corollary 3.1, for arbitrary $x(t), y(t) \in L_\infty[a,b]$, we have

$$\begin{aligned} \|Ax - By\| &= \|x - y\| + |\lambda| M N_1 \|x - y\| \\ &= (1 + |\lambda| M N_1) \|x - y\| \\ &\le \Big(\frac{1 + |\lambda| N_1 M}{N_2}\Big) \|Sx - Ty\| = k \|Sx - Ty\|. \end{aligned}$$

Hence all the conditions of Corollary 3.1 (corresponding to contraction condition (a_1)) are satisfied, therefore there exists an $x(t) \in L_\infty[a,b]$ such that $x = Ax = Bx = Sx = Tx$, i.e., A, B, S and T have a unique common fixed point in $L_\infty[a,b]$. Hence the simultaneous equations (5.1) have a unique solution in $L_\infty[a,b]$. ☐

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References

[1] A. Ahmad and M. Imdad, *A common fixed point theorem for four mappings satisfying a rational inequality*, Publ. Math. Debrecen **41** (1992), no. 3-4, 181–187.

[2] R. Chugh and S. Kumar, *Common fixed points for weakly compatible maps*, Proc. Indian Acad. Sci.(Math. Sci.) **111** (2001), no. 2, 241–247.

[3] B. C. Dhage, *A fixed point theorem in Banach algebras involving three operators with applications*, Kyungpook Math. J. **44** (2004), 145–155.

[4] ———, *Global attractivity results for nonlinear functional integral equations via a Krasnoselskii type fixed point theorem*, Nonlinear Anal.(TMA), in press.

[5] B. Fisher, *Common fixed point and constant mappings satisfying rational inequality*, Math. Sem. Notes **6** (1978), 29–35.

[6] D. H. Griffel, *Applied Functional Analysis*, Ellis Horwood Limited, Chichester, 1981.

[7] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), 201–206.

[8] M. Imdad and Q. H. Khan, *Six mappings satisfying a rational inequality*, Rad. Mat. **9** (1999), 251–260.

[9] M. Imdad and Javid Ali, *Pairwise coincidentally commuting mappings satisfying a rational inequality*, Italian J. Pure Appl. Math. **20** (2006), 87–96.

[10] G. S. Jeong and B. E. Rhoades, *Some remarks for improving fixed point theorems for more than two maps*, Indian J. Pure Appl. Math. **28** (1997), no. 9, 1177–1196.

[11] M. C. Joshi and R. K. Bose, *Some Topics in Nonlinear Functional Analysis*, Wiley Eastern Limited, New Delhi, 1985.

[12] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83** (1976), 261–263.

[13] ———, *Compatible mappings and common fixed points (2)*, Internat. J. Math. Math. Sci. **11** (1988), 285–288.

[14] ———, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. **4** (1996), no. 2, 199–215.

[15] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–76.

[16] T. I. Khan, *A study of common fixed point theorems under certain weak conditions of commutativity*, Ph. D. thesis, Aligarh Muslim University, Aligarh, India, 2002.

[17] P. P. Murthy, *Important tools and possible applications of metric fixed point theory*, Nonlinear Anal.(TMA) **47** (2001), 3479–3490.

[18] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994), 436–440.

[19] ———, *Common fixed points of four mappings*, Bull. Calcutta Math. Soc. **90** (1998), 281–286.

[20] V. Popa, *Some fixed point theorems for weakly compatible mappings*, Rad. Mat. **10**(2001), 245–252.

[21] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. **32** (1982), no. 46, 149–153.

[22] N. Shahzad, *Invariant approximations, generalized I-contractions and R-subweakly commuting maps*, Fixed Point Theory Appl. **2** (2005), no. 1, 79–86.

[23] S. L. Singh and S. N. Mishra, *Remarks on Jachymski's fixed point theorems for compatible maps*, Indian J. Pure Appl. Math. **28** (1997), no. 5, 611–615.

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