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ON PERIODIC BOUNDARY VALUE PROBLEMS OF HIGHER ORDER NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS WITH p-LAPLACIAN

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Abstract. Motivated by [Linear Algebra and its Appl. **420** (2007), 218–227] and [Linear Algebra and its Appl. **425** (2007), 171–183], we, in this paper, study the solvability of periodic boundary value problems of higher order nonlinear functional difference equations with p-Laplacian. Sufficient conditions for the existence of at least one solution of this problem are established.

1. Introduction

Nonlinear difference equations of order greater than one, that is higher order dynamic systems, are of paramount importance in applications where the $(n+1)^{st}$ generation (or state) of the system depends on the previous k generations (or state). Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. We refer the readers to text book [1, 2, 11] and the references cited there. The difference equations which result from various discrete analogue of differential equations are of the form

$$y_{n+1} - y_n + f(n, y_n, y_{n-1}, \dots, y_{n-k}) = 0, \quad n = 0, 1, \dots.$$

There are large amount of papers discussed the properties, such as permanence, global stability and oscillation properties, of above mentioned equation.

Recently, Ji and Yang [8, 9, 10] studied the eigenvalue problems for boundary value problems of the second order difference equations

$$\Delta(r_{i-1}\Delta y_{i-1}) - b_i y_i + \lambda y_i = 0,\ 1 \le i \le n,\ y_0 - \tau y_1 = y_{n+1} - \delta y_n = 0,$$

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where $\tau, \delta \in [0,1]$ with $\tau+\delta \neq 2$ or $\tau+\delta=2$. The focuses in these papers are the structure of their eigenvalues and comparisons of all eigenvalues as the coefficients a_i, b_i, r_i change.

In this paper, we study the boundary value problems consisting of the higher order nonlinear functional difference equation with p-Laplacian

(1)
$$\Delta[p(n)\phi(\Delta x(n))] = f(n, x(n+1), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n)), \quad n \in [0, T-1],$$

and the following boundary conditions

$$(2) \qquad \begin{cases} x(0) = x(T), \quad \Delta x(0) = \Delta x(T), \\ x(i) = \gamma(i), \quad i \in [-\tau, -1], \\ x(i) = \psi(i), \quad i \in [T+2, T+\delta], \end{cases}$$

where $p(n)$ is a positive sequence with $p(0) = p(T)$, $\phi : R \to R$ a homeomorphism satisfying $\phi(ab) = \phi(a)\phi(b)$ for all $a \geq 0, b \geq 0$ and $\phi(x)x \geq 0$ for all $x \in R$. The inverse function of ϕ is defined by ϕ^{-1}, $\tau_i : [0, T-1] \to N$, $i = 1, \ldots, m$, sequences with $T \geq 1$, γ, ψ sequences, and

$$\tau = \max\left\{0, \max_{n\in[0,T-1]}\{0, \tau_i(n)\} : \ i = 1, \ldots, m\right\},$$

$$\delta = -\min\left\{0, \min_{n\in[0,T-1]}\{0, \tau_i(n)\} : \ i = 1, \ldots, m\right\},$$

$f(n, u)$ continuous about $u = (x_0, \ldots, x_m)$ for each n. $[a, b] = \{a, \ldots, b\}$ if $a \leq b$ and $[a, b] = \emptyset$ if $a > b$.

Boundary conditions (2) are called the periodic boundary conditions, one may see text books [1, 2] and papers [3, 4, 8, 9, 10, 12, 13, 14, 15] and the references therein. The motivation of this paper is due to papers [4, 5, 6, 11].

In [6], Cabada and Otero-Espinar established the existence and comparison results for difference ϕ-Laplacian boundary value problems consisting of the equation

$$(3) \qquad -\Delta[\phi(\Delta x(k))] = f(k, x(k+1)), \quad k \in \{0, 1, \ldots, N-1\},$$

and one of the following boundary conditions

$$(4) \qquad \Delta x(0) = N_0, \ \Delta x(N) = N_1,$$

and

$$(5) \qquad x(0) - x(N) = C_0, \ \ \Delta x(0) - \Delta x(N) = C_1.$$

The methods used in papers [5, 6] and the reference cited there are lower and upper solutions methods and monotone iterative technique and comparison principles. In [6], in order to obtain the solutions of above mentioned problems, the following assumptions are used.

(H_1). $\phi : R \to R$ is a strictly increasing homeomorphism and ϕ^{-1} is a H-Lipschitzian function on R; i.e.,

$$|\phi^{-1}(x) - \phi^{-1}(y)| \leq H|x-y|, \ \ x, y \in R.$$

(H_1^*). $\phi : R \to R$ is a strictly increasing homeomorphism and ϕ^{-1} is a locally H-Lipschitzian function on R; i.e., for every compact interval $[h_1, h_2]$ there exists $H > 0$ such that for all $x, y \in [h_1, h_2]$

$$|\phi^{-1}(x) - \phi^{-1}(y)| \le H|x - y|.$$

(H_3). Suppose that α and β are lower and upper solutions of problem (3) and (4), see the definition in [6], respectively. There exists $M < 0$ for which

$$f(k,x)-f(k,y) \le M(y-x), \text{ for all } \beta(k) \le y \le x \le \alpha(k),\ k \in I=[0,\dots,N-1].$$

In paper [4], the authors investigated the boundary value problem

$$\begin{cases} -\Delta(p(n-1)\Delta y(n-1)) + q(n)y(n) = f(n, y(n)), & n \in [1, N], \\ y(1) = y(N+1),\ p(1)\Delta y(1) = p(N+1)\Delta y(N+1), \end{cases}$$

by using a fixed point theorem in cones in Banach space, the existence results for positive solutions of above problem were established. Above boundary value problem can be rewritten as follows

$$\begin{cases} \Delta(p(n)\Delta y(n)) = q(n+1)y(n+1) - f(n+1, y(n+1)), & n \in [0, N-1], \\ y(0) = y(N),\ p(0)\Delta y(0) = p(N)\Delta y(N). \end{cases}$$

It is easy to see that this kind of problem is a special case of boundary value problem (1) and (2) (BVP (1) and (2) for short) in the case where $\phi(x) = x$ and $f(n, x_0, x_1, \dots, x_m)$ in (1) is replaced by $q(n+1)x_0 - f(n+1, x_0)$.

We find that there is no paper discussed the existence of solutions of BVP (1) and (2).

The purposes of this paper are to establish sufficient conditions for the existence of at least one solution of BVP (1) and (2) by using coincidence degree theory. It is interesting that we allow that f to be sublinear, at most linear or superlinear.

This paper is organized as follows. In Section 2, we give the main results, and in Section 3, an example to illustrate the main results will be presented.

2. Main results

To get existence results for solutions of BVP (1) and (2), we need the following fixed point theorem, which was used to solve multi-point boundary value problems for differential equations in many papers but was not used to solve boundary value problems for difference equations.

Lemma 2.1 ([12]). *Let X and Y be Banach spaces. Suppose $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with* Ker$L = \{0\}$*, $N_1, N_2 : X \to Y$ is L-compact on any open bounded subset of X. If Ω is an open bounded subset, and $Lx = N_2x$ has a solution in $\Omega \subset X$ and*

$$Lx \ne \lambda N_1 x + (1-\lambda)N_2 x \text{ for all } x \in D(L) \cap \partial\Omega \text{ and } \lambda \in [0,1],$$

then there is at least one $x \in \Omega$ so that $Lx = N_1x$.

Choose $X = R^{T+\tau+\delta+1} \times R^{T+1}$ and let X be endowed with the norm

$$||(x,y)||_X = \max\left\{\max_{n\in[-\tau,T+\delta]}|x(n)|,\ \max_{n\in[1,T+1]}|y(n)|\right\}$$

for $(x,y) \in X$. Choose $Y = R^{T+1} \times R^T \times R^2$ and let Y be endowed with the norm

$$||(u,v,w)||_Y = \max\left\{\max_{n\in[0,T]}|u(n)|,\ \max_{n\in[0,T-1]}|v(n)|,\ \max_{n=1,2}|w(n)|\right\}$$

for $(u,v,w) \in Y$. It is easy to see that X and Y are real Banach spaces. Choose

$$(*)\quad \text{Dom}L = \left\{(x,y) \in X:\ \begin{array}{l} x(i) = 0,\ i \in [-\tau,\ldots,-1], \\ x(i) \in R,\ \ i \in [0,T+1], \\ x(i) = 0,\ \ i \in [T+2,\ldots,T+\delta], \end{array}\right\} \times R^{T+1}.$$

Let $(x,y) \in \text{Dom } L \cap X$, define $L : \text{Dom } L \cap X \to Y$, and

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Delta x(n),\ n = 0,\ldots,T \\ \Delta y(n),\ n = 0,\ldots,T-1 \\ x(T) \\ y(T) \end{pmatrix},$$

and let $(x,y) \in \text{Dom } L \cap X$, define $N_1 : X \to Y$ by

$$N_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \phi^{-1}\left(\frac{y(n)}{p(n)}\right), & n = 0,\ldots,T \\ f(n, w(n+1), w(n-\tau_1(n)),\ldots,w(n-\tau_m(n))), & n = 0,\ldots,T-1 \\ x(0) & \\ y(0) & \end{pmatrix}$$

and $N_2 : X \to Y$ by

$$N_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left.\begin{matrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix}\right\} T+1 \\ \left.\begin{matrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix}\right\} T \\ x(0) \\ y(0) \end{pmatrix}$$

for all $(x,y) \in X$, where $w(n) = x(n) + x_0(n)$ and

$$(**) \qquad x_0(n) = \begin{cases} \gamma(n), & n \in [-\tau, -1], \\ 0, & n \in [0, T+1], \\ \psi(n), & n \in [T+2, T+\delta]. \end{cases}$$

It is easy to check the following results.

(i) $\mathrm{Ker}L = \{0 \in X\}$.

(ii) L is a Fredholm operator of index zero.

(iii) Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap \mathrm{Dom}L \neq \emptyset$. Then N_1 and N_2 are L-compact on $\overline{\Omega}$.

(iv) Let Ω be an open bounded nonempty subset of X. Then $Lx = N_2 x$ has a solution in $\Omega \subset X$.

(v) If $(x,y) \in X$ is a solution of $L(x,y) = N_1(x,y)$, then $x + x_0$ is a solution of BVP(1) and (2).

(vi) If $y(n) = p(n)\phi(\Delta x(n))$ for $n = 0, \ldots, T$, since $p(0) = p(T)$, it is easy to see that $x(0) = x(T), x(1) = x(T+1)$ if and only if $x(0) = x(T), y(0) = y(T)$.

Theorem L. *Suppose that there exist numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p_i(n), r(n) (i = 0, \ldots, m)$, functions $g(n, x_0, \ldots, x_m)$, $h(n, x_0, \ldots, x_m)$ such that $f(n, x_0, \ldots, x_m) = g(n, x_0, \ldots, x_m) + h(n, x_0, \ldots, x_m)$ and*

$$g(n, x_0, x_1, \ldots, x_m)x_0 \geq \beta |x_0|^{\theta+1},$$

and

$$|h(n, x_0, \ldots, x_m)| \leq \sum_{s=0}^{m} p_i(n)|x_i|^{\theta} + r(n),$$

for all $n \in \{1, \ldots, T\}$, $(x_0, x_1, \ldots, x_m) \in R^{m+1}$. Then BVP (1) *and* (2) *has at least one solution if*

$$(6) \qquad ||p_0|| + T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| < \beta.$$

Proof. Let $\Omega_1 = \{(x,y) : L(x,y) = \lambda N_1(x,y) + (1-\lambda)N_2(x,y),\ ((x,y),\lambda) \in [\mathrm{Dom}L] \times (0,1)\}$, we prove that Ω_1 is bounded.

For $(x,y) \in \Omega_1$, we have $L(x,y) = \lambda N_1(x,y) + (1-\lambda)N_2(x,y)$, $\lambda \in [0,1]$, so

(7)

$$\begin{cases} \Delta x(n) = \lambda \phi^{-1}\left(\frac{y(n)}{p(n)}\right), \ n = 0, \ldots, T, \\ \Delta y(n) = \lambda f(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))), \ n = 0, \ldots, T-1, \\ x(T) = x(0), \\ y(T) = y(0). \end{cases}$$

It follows, for $n = 0, \ldots, T-1$, that

$$\begin{aligned} &\Delta[p(n)\phi(\Delta w(n))]w(n+1) \\ &= \lambda\phi(\lambda) f(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))w(n+1). \end{aligned}$$

Since $y(0) = y(T)$ and $x(0) = x(T)$ imply that $x(0) = x(T), x(1) = x(T+1)$, it is easy to see from the definition of $w(n) = x(n) + x_0(n)$, and (**) that $w(T) = w(0)$ and $w(T+1) = w(1)$, then

$$\begin{aligned}
&\sum_{n=0}^{T-1} \Delta[p(n)\phi(\Delta w(n))]w(n+1) \\
&= \sum_{n=0}^{T-1}[p(n+1)\phi(\Delta w(n+1)) - p(n)\phi(\Delta w(n))][w(n+2) - \Delta w(n+1)] \\
&= \sum_{n=0}^{T-1}[p(n+1)\phi(\Delta w(n+1))w(n+2) - p(n)\phi(\Delta w(n))w(n+1)] \\
&\quad - \sum_{n=0}^{T-1} p(n+1)\phi(\Delta w(n+1))\Delta w(n+1) \\
&= p(T)\phi(\Delta w(T))w(T+1) - p(0)\phi(\Delta w(0))w(1) \\
&\quad - \sum_{n=0}^{T-1} p(n+1)\phi(\Delta w(n+1))\Delta w(n+1) \\
&= p(0)\Big(\phi(\Delta w(T))w(T+1) - \phi(\Delta w(0))w(1)\Big) \\
&\quad - \sum_{n=0}^{T-1} p(n+1)\phi(\Delta w(n+1))\Delta w(n+1) \\
&= -\sum_{n=0}^{T-1} p(n+1)\phi(\Delta w(n+1))\Delta w(n+1).
\end{aligned}$$

Since $\phi(x)x \geq 0$ for all $x \in R$ and $p(n) \geq 0$ for all $n \in Z$, we get

$$\sum_{n=0}^{T-1} \Delta[p(n)\phi(\Delta w(n))]w(n+1) \leq 0.$$

So, we get

$$\sum_{n=0}^{T-1} f(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))w(n+1) \leq 0.$$

It follows that

$$\begin{aligned}
&\beta \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \\
&\leq \sum_{n=0}^{T-1} g(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))w(n+1)
\end{aligned}$$

$$\leq -\sum_{n=0}^{T-1} h(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))w(n+1)$$
$$\leq \sum_{n=0}^{T-1} |h(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))|\,|w(n+1)|$$
$$\leq \sum_{n=0}^{T-1} p_0(n)|w(n+1)|^{\theta+1}$$
$$+\sum_{i=1}^{m}\sum_{n=0}^{T-1} p_i(n)|w(n-\tau_i(n))|^{\theta}|w(n+1)| + \sum_{n=0}^{T-1} r(n)|w(n+1)|$$
$$\leq ||p_0|| \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}$$
$$+\sum_{i=1}^{m} ||p_i|| \sum_{n=0}^{T-1} |w(n-\tau_i(n))|^{\theta}|w(n+1)| + ||r|| \sum_{n=0}^{T-1} |w(n+1)|.$$

For $x_i \geq 0, y_i \geq 0$, Holder's inequality implies

$$\sum_{i=1}^{s} x_i y_i \leq \left(\sum_{i=1}^{s} x_i^p\right)^{1/p} \left(\sum_{i=1}^{s} y_i^q\right)^{1/q}, \quad 1/p + 1/q = 1, \quad q > 0, \ p > 0.$$

It follows that

$$\sum_{n=0}^{T-1} |w(n+1)| \leq T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$

and

$$\sum_{n=0}^{T-1} |w(n-\tau_i(n))|^{\theta}|w(n+1)|$$
$$\leq \left(\sum_{n=0}^{T-1} |w(n-\tau_i(n))|^{\theta+1}\right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$

Then

$$\beta \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}$$
$$\leq ||p_0|| \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$
$$+\sum_{i=1}^{m} ||p_i|| \left(\sum_{n=0}^{T-1} |w(n-\tau_i(n))|^{\theta+1}\right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$

$$= ||p_0|| \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \right)^{\frac{1}{\theta+1}}$$

$$+ \sum_{i=1}^{m} ||p_i|| \left(\sum_{u \in \{n-\tau_i(n)-1:\, n=0,\ldots,T-1\}} |w(u+1)|^{\theta+1} \right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \right)^{\frac{1}{\theta+1}}$$

$$\leq ||p_0|| \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \right)^{\frac{1}{\theta+1}}$$

$$+ T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} + \sum_{n=T}^{T+\delta} |\psi(n+1)|^{\theta+1} + \sum_{n=-\tau}^{-1} |\gamma(n+1)|^{\theta+1} \right)^{\frac{\theta}{\theta+1}}$$

$$\times \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \right)^{\frac{1}{\theta+1}}$$

$$= ||p_0|| \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \right)^{\frac{1}{\theta+1}}$$

$$+ T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \left(1 + \frac{\sum_{n=T}^{T+\delta} |\psi(n+1)|^{\theta+1} + \sum_{n=-\tau}^{-1} |\gamma(n+1)|^{\theta+1}}{\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}} \right)^{\frac{\theta}{\theta+1}} \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}.$$

Then we get

$$\left(\beta - ||p_0|| - T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \right) \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}$$

$$\leq ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \right)^{\frac{1}{\theta+1}} + T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i||$$

$$\times \left[\left(1 + \frac{\sum_{n=T}^{T+\delta} |\psi(n+1)|^{\theta+1} + \sum_{n=-\tau}^{-1} |\gamma(n+1)|^{\theta+1}}{\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}} \right) - 1 \right]^{\frac{\theta}{\theta+1}} \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}.$$

It follows from (6) that $\beta - ||p_0|| - T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| > 0$. Then

$$0 < \beta - ||p_0|| - T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i||$$

$$\leq ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \right)^{-\frac{\theta}{\theta+1}} + T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i||$$

$$\times \left[\left(1 + \frac{\sum_{n=T}^{T+\delta} |\psi(n+1)|^{\theta+1} + \sum_{n=-\tau}^{-1} |\gamma(n+1)|^{\theta+1}}{\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}} \right) - 1 \right]^{\frac{\theta}{\theta+1}}.$$

We now prove that there exists $M_1 > 0$ such that $\sum_{u=0}^{T-1} |w(u+1)|^{\theta+1} \leq M_1$ for each $w = x+x_0$ with $(x,y) \in \Omega_1$. In fact, if there is a sequence $\{w_l = x_l+(x_0)_l\}$ with $(x_l, y_l) \in \Omega_1$ such that $\sum_{u=0}^{T-1} |w_l(u+1)|^{\theta+1} \to +\infty$ as $l \to +\infty$. Then

$$\begin{aligned} 0 &< \beta - ||p_0|| - T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \\ &\leq ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w_l(n+1)|^{\theta+1} \right)^{-\frac{\theta}{\theta+1}} + T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \\ &\quad \times \left[\left(1 + \frac{\sum_{n=T}^{T+\delta} |\psi(n+1)|^{\theta+1} + \sum_{n=-\tau}^{-1} |\gamma(n+1)|^{\theta+1}}{\sum_{n=0}^{T-1} |w_l(n+1)|^{\theta+1}} \right) - 1 \right]^{\frac{\theta}{\theta+1}} \\ &\to 0 \text{ as } l + \infty, \end{aligned}$$

a contradiction. Hence there is $M_1 > 0$ such that $\sum_{u=0}^{T-1} |w(u+1)|^{\theta+1} \leq M_1$. Then $|w(n+1)| \leq (M_1/T)^{1/(\theta+1)}$ for all $n \in [0, T-1]$. Thus we get

$$|x(n+1)| \leq |w(n+1)| + |x_0(n+1)| \leq (M_1/T)^{1/(\theta+1)} + ||x_0||, \quad n \in [0, T-1].$$

Hence

$$||x|| \leq (M_1/T)^{1/(\theta+1)} + ||x_0|| =: H_1.$$

It follows from (7) that

$$\Delta x(n) = \lambda \phi^{-1} \left(\frac{y(n)}{p(n)} \right), \quad n = 0, \ldots, T,$$

and

$$\Delta y(n) = \lambda f(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))), \quad n = 0, \ldots, T-1.$$

Since $x(0) = x(T)$, it is easy to see that there exists $k_0 \in [0, T]$ such that

$$\Delta x(k_0) \geq 0, \quad \Delta x(k_0+1) < 0$$

or

$$\Delta x(k_0) \leq 0, \quad \Delta x(k_0+1) > 0.$$

Case 1. $\Delta x(k_0) \geq 0$, $\Delta x(k_0+1) < 0$. At this case, we get $y(k_0) \geq 0$ and $y(k_0+1) < 0$. Then one sees that there exists a real number $\xi \in [k_0, k_0+1]$ such that

$$\frac{y(k_0+1) - y(k_0)}{k_0+1-k_0} = \frac{0 - y(k_0)}{\xi - k_0}.$$

It follows that

$$\begin{aligned} |y(k_0)| &= |\xi - k_0||\Delta y(k_0)| \leq |\Delta y(k_0)| \\ &\leq |f(k_0, w(k_0+1), w(k_0-\tau_1(k_0)), \ldots, w(k_0-\tau_m(k_0)))| \\ &\leq \max_{n \in [0,T-1], |x_i| \leq H_1, i=0,\ldots,m} |f(n, x_0, \ldots, x_m)| = H_0. \end{aligned}$$

Then, for $n > k_0$, one sees that

$$\begin{aligned} |y(n)| &= \left| y(k_0) + \sum_{s=k_0}^{n-1} \Delta y(s) \right| \le |y(k_0)| + \sum_{s=0}^{n-1} |\Delta y(s)| \\ &\le H_0 + T \max_{n\in[0,T-1],|x_i|\le H_1, i=0,\dots,m} |f(n,x_0,\dots,x_m)| = (1+T)H_0. \end{aligned}$$

For $n < k_0$, we have

$$\begin{aligned} |y(n)| &= \left| y(k_0) - \sum_{s=n}^{k_0-1} \Delta y(s) \right| \le |y(k_0)| + \sum_{s=n}^{k_0-1} |\Delta y(s)| \\ &\le H_0 + T \max_{n\in[0,T-1],|x_i|\le H_1, i=0,\dots,m} |f(n,x_0,\dots,x_m)| = (1+T)H_0. \end{aligned}$$

Thus

$$||y|| \le (1+T)H_0 = H_2.$$

Case 2. $\Delta x(k_0) \le 0, \ \ \Delta x(k_0+1) > 0$. Similar to Case 1, one gets that

$$||y|| \le (1+T)H_0 = H_2.$$

So Ω_1 is bounded and $\overline{\Omega_1} \neq X$ since $||(x,y)|| \le \max\{H_1, H_2\}$ for all $(x,y) \in \Omega_1$. Let

$$A = \max\{H_1, \ H_2\} + 1.$$

Then $A < +\infty$. Choose $\Omega = \{(x,y) \in X : ||(x,y)|| < A\}$. It is easy to see that $\Omega \supset \overline{\Omega_1}$ is an open bounded nonempty subset of X with $\overline{\Omega} \neq X$. It is easy to see from the definition of Ω that $L(x,y) \neq \lambda N_1(x,y) + (1-\lambda)N_2(x,y)$ for all $(x,y) \in D(L) \cap \partial\Omega$ and $\lambda \in (0,1)$, and $Lx = N_2 x$ has a solution in $\Omega \subset X$. It follows from Lemma 2.1 that equation $L(x,y) = N_1(x,y)$ has at least one solution (x,y), then $x + x_0$ is a solution of BVP (1) and (2). The proof is complete. □

3. An example

In this section, we present an example to illustrate the main results obtained in Section 2.

Example 3.1. Consider the following boundary value problem

$$(8) \qquad \begin{cases} \Delta(\phi(\Delta x(n))) = \beta[x(n+1)]^{2k+1} + \sum_{i=0}^{m} p_i(n)[x(n-i)]^{2k+1} \\ +\sum_{j=0}^{l} q_j(n)[x(n+j)]^{2k+1} + r(n), \\ x(0) = x(T), \quad \Delta x(0) = \Delta x(T), \\ x(i) = \gamma(i), \quad i \in [-m, -1], \\ x(i) = \psi(i), \quad i \in [T+2, T+l], \end{cases}$$

where $\beta > 0$, T, l, m are positive integers, $k \ge 0$ an integer, ψ, ϕ, r, p_i, q_j are sequences. Corresponding to BVP (1) and (2), let

$$g(n, x_0, \dots, x_{m+l}) = \beta x_0^{2k+1},$$

and

$$h(n, x_0, \ldots, x_{m+l}) = \sum_{i=0}^{m} p_i(n) x_i^{2k+1} + \sum_{j=0}^{l} q_j(n) x_{m+j}^{2k+1} + r(n).$$

It is easy to show from Theorem L that BVP (8) has at least one solution if

$$||p_0|| + ||q_0|| + T^{\frac{2k+1}{2k+2}} \left(\sum_{i=0}^{m} ||p_i|| + \sum_{j=0}^{l} ||q_j|| \right) < \beta.$$

Remark. If $k = 0$ and $\phi(x) = x$, (8) becomes

(9)
$$\begin{cases} \Delta^2 x(n) = \beta x(n+1) + \sum_{i=0}^{m} p_i(n) x(n-i) + \sum_{j=0}^{l} q_j(n) x(n+j) + r(n), \\ x(0) = x(T), \quad \Delta x(0) = \Delta x(T), \\ x(i) = \gamma(i), \quad i \in [-m, -1], \\ x(i) = \psi(i), \quad i \in [T+2, T+l], \end{cases}$$

BVP (9) is a linear periodic boundary value problem. It follows from Example 3.1 that BVP (9) has at least one solution if

$$||p_0|| + ||q_0|| + T^{\frac{1}{2}} \left(\sum_{i=0}^{m} ||p_i|| + \sum_{j=0}^{l} ||q_j|| \right) < \beta.$$

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