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STABILITY OF DERIVATIONS ON PROPER LIE CQ^*-ALGEBRAS

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ABSTRACT. In this paper, we obtain the general solution and the generalized Hyers–Ulam–Rassias stability for a following functional equation

$$\sum_{i=1}^{m} f(x_i + \frac{1}{m}\sum_{\substack{j=1\\ j\neq i}}^{m} x_j) + f(\frac{1}{m}\sum_{i=1}^{m} x_i) = 2f(\sum_{i=1}^{m} x_i)$$

for a fixed positive integer m with $m \geq 2$. This is applied to investigate derivations and their stability on proper Lie CQ^*-algebras. The concept of Hyers–Ulam–Rassias stability originated from the Th. M. Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

1. Introduction and preliminaries

Ulam [26] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(x*y), f(x)\diamond f(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $T : G_1 \to G_2$ with

$$d(f(x), T(x)) < \epsilon$$

for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism $f(xy) = f(x)f(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

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Hyers [10] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [24] for linear mappings by considering an *unbounded Cauchy difference.* The paper of Th. M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. In 1994, a generalization of Th. M. Rassias' theorem was obtained by Găvruta [9]. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [12], [18]-[23]). We also refer the readers to the books [1], [8], [11], [13] and [25].

We recall some basic facts concerning quasi $*$-algebras.

Definition 1.1. Let A be a linear space and A_0 be a $*$-algebra contained in A as a subspace. We say that A is a *quasi $*$-algebra* over A_0 if

(i) the right and left multiplications of an element of A and an element of A_0 are always defined and linear;
(ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;
(iii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$, whenever the multiplication is defined.

Quasi $*$-algebras [14, 15] arise in natural way as completions of locally convex $*$-algebras whose multiplication is not jointly continuous; in this case one has to deal with topological quasi $*$-algebras.

A quasi $*$-algebra (A, A_0) is called *topological* if a locally convex topology τ on A is given such that:

(i) the involution $a \mapsto a^*$ is continuous for each $a \in A$,
(ii) the mappings $a \mapsto ab$ and $a \mapsto ba$ are continuous for each $a \in A$ and $b \in A_0$,
(iii) A_0 is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra (A, A_0) is complete. For an overview on partial $*$-algebra and related topics we refer to [2].

In a series of papers [4], [5], [6], [7] many authors have considered a special class of quasi $*$-algebras, called proper CQ^*-algebras, which arise as completions of C^*-algebras. They can be introduced in the following way:

Definition 1.2. Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|.\|$;
(ii) $(ab)^* = b^*a^*$ whenever the multiplication is defined;
(iii) $\|y\|_0 = \max\{ \sup_{a\in A, \|a\|\le 1} \|ay\|, \sup_{a\in A,\|a\|\le 1} \|ya\| \}$ for all $y \in A_0$.

A proper CQ^*-algebra (A, A_0) is said to have a unit e if there exists an element $e \in A_0$ such that $ae = ea = a$ for all $a \in A$. In this paper we will always assume that the proper CQ^*-algebra under consideration have an identity.

Definition 1.3. A proper CQ^*-algebra (A, A_0), endowed with a bilinear multiplication $[,] : (A \times A_0) \cup (A_0 \times A) \to A$, called the bracket, which satisfies two simple properties:

(i) $[x_1, x_2] = -[x_2, x_1]$ for all $(x_1, x_2) \in (A \times A_0) \cup (A_0 \times A)$;
(ii) $[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_1, [x_2, x_3]]$ for all $x_1, x_2, x_3 \in A_0$

is called a *proper Lie CQ^*-algebra.*

Definition 1.4. Let (A, A_0) be a proper Lie CQ^*-algebras. A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *Lie derivation* if

$$\delta([z, x]) = [\delta(z), x] + [z, \delta(x)]$$

for all $x, z \in A_0$ (see [21]).

Throughout this paper, we assume that m is a fixed positive integer with $m \ge 2$.

In this paper, we obtain the general solution and the generalized Hyers–Ulam–Rassias stability for the following functional equation

$$\sum_{i=1}^{m} f\Big(x_i + \frac{1}{m}\sum_{\substack{j=1\\ j\neq i}}^{m} x_j\Big) + f\Big(\frac{1}{m}\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} x_i\Big), \tag{1.1}$$

where m is a fixed positive integer with $m \ge 2$. This is applied to investigate derivations and their stability on proper Lie CQ^*-algebras.

2. Solution of functional equation (1.1)

Throughout this section, let both X and Y be real vector spaces. We here present the general solution of (1.1).

Theorem 2.1. *A mapping $f : X \to Y$ satisfies* (1.1) *if and only if the mapping $f : X \to Y$ is additive.*

Proof. We first assume that the mapping $f : X \to Y$ satisfies (1.1). Setting $x_1 = mx$ and $x_2 = \cdots = x_m = 0$ in (1.1), we get

$$f(mx) = mf(x) \tag{2.1}$$

for all $x \in X$. Replacing $x_1 = x$ and $x_2 = \cdots = x_m = \frac{y}{m-1}$ in (1.1), and using (2.1), we get

$$f(mx + y) + (m-1)f(x + 2y) = (2m-1)f(x + y) \tag{2.2}$$

for all $x, y \in X$. Setting $x_1 = x$, $x_2 = y$ and $x_3 = \cdots = x_m = 0$ in (1.1) and using (2.1), we get

$$f(mx + y) + f(x + my) = (m+1)f(x + y) \tag{2.3}$$

for all $x, y \in X$. Therefore, it follows from (2.2) and (2.3) that

$$f(x + my) - (m-1)f(x + 2y) = (2-m)f(x + y) \tag{2.4}$$

for all $x, y \in X$. Replacing x and y by y and x in (2.2), respectively, we get

$$f(x + my) + (m-1)f(2x + y) = (2m-1)f(x + y) \tag{2.5}$$

for all $x, y \in X$. By using (2.4) and (2.5), we get

$$3f(x + y) = f(x + 2y) + f(2x + y) \tag{2.6}$$

for all $x, y \in X$. Setting $y = 0$ in (2.6), we get

$$f(2x) = 2f(x) \tag{2.7}$$

for all $x \in X$. Replacing y by x in (2.6) and using (2.7), we get

$$f(3x) = 3f(x) \tag{2.8}$$

for all $x \in X$. Replacing x and y by $\frac{2x-y}{3}$ and $\frac{2y-x}{3}$ in (2.6), respectively, and using (2.8), we get

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. So the mapping $f : X \to Y$ is additive.

Conversely, let the mapping $f : X \to Y$ be additive. By a simple computation, one can show that the mapping f satisfies the functional equation (1.1). □

3. Stability of derivation on proper Lie CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Lie CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$. For convenience, we use the following abbreviation for a given mapping $f : A_0 \to A$

$$D_\mu f(x_1, \ldots, x_m) := \sum_{i=1}^{m} f\Big(\mu x_i + \frac{1}{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \mu x_j\Big) + f\Big(\frac{1}{m}\sum_{i=1}^{m} \mu x_i\Big) - 2\mu f\Big(\sum_{i=1}^{m} x_i\Big)$$

for all $x_1, \ldots, x_m \in A_0$, where $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$.

We will use the following lemma:

Lemma 3.1 ([22]). *Let $f : A_0 \to A$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A_0$ and all $\mu \in \mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Theorem 3.2. *Let $\varphi : \underbrace{A_0 \times A_0 \times \cdots \times A_0}_{m-times} \to [0, \infty)$ and $\psi : A_0 \times A_0 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{1}{m^n}\varphi(m^n x_1, \ldots, m^n x_m) = 0, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{m^n}\psi(m^n x_1, x_2) = 0, \tag{3.2}$$

$$\widetilde{\varphi_j}(x) := \sum_{i=1}^{\infty} \frac{1}{m^i}\varphi(0, \ldots, \underbrace{m^i x}_{j\,th}, \ldots, 0) < \infty \tag{3.3}$$

for some $1 \le j \le m$ and all $x, x_1, \ldots, x_m \in A_0$. Suppose that $f : A_0 \to A$ is a mapping such that

$$\|D_\mu f(x_1, \ldots, x_m)\|_A \le \varphi(x_1, \ldots, x_m), \tag{3.4}$$

$$\big\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f_1(x_2)]\big\|_A \le \psi(x_1, x_2) \tag{3.5}$$

for all $x_1, \ldots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \widetilde{\varphi_j}(x) \tag{3.6}$$

for all $x \in A_0$.

Proof. Letting $\mu = 1$, $x_j = mx$ and $x_i = 0$ for all $1 \le i \le m$ with $i \ne j$ in (3.4), we get

$$\|f(mx) - mf(x)\|_A \le \varphi(0, \ldots, \underbrace{mx}_{j\,th}, \ldots, 0) \tag{3.7}$$

for all $x \in A_0$. Replacing x by $m^n x$ in (3.7) and dividing both sides of (3.7) by m^{n+1}, we get

$$\left\|\frac{1}{m^{n+1}} f(m^{n+1}x) - \frac{1}{m^n} f(m^n x)\right\|_A \le \frac{1}{m^{n+1}}\varphi(0, \ldots, \underbrace{m^{n+1}x}_{j\,th}, \ldots, 0) \tag{3.8}$$

for all $x \in A_0$ and all non-negative integers n. Hence

(3.9)

$$\begin{aligned}\left\|\frac{1}{m^{n+1}} f(m^{n+1}x) - \frac{1}{m^k} f(m^k x)\right\|_A &\le \sum_{i=k}^{n} \left\|\frac{1}{m^{i+1}} f(m^{i+1}x) - \frac{1}{m^i} f(2^i x)\right\|_A \\ &\le \sum_{i=k+1}^{n+1} \frac{1}{m^i}\varphi(0, \ldots, \underbrace{m^i x}_{j\,th}, \ldots, 0)\end{aligned}$$

for all $x \in A_0$ and all non-negative integers n and k with $n \geq k$. Therefore, we conclude from (3.3) and (3.9) that the sequence $\{\frac{1}{m^n}f(m^n x)\}$ is a Cauchy sequence in A for all $x \in A_0$. Since A is complete, the sequence $\{\frac{1}{m^n}f(m^n x)\}$ converges in A for all $x \in A_0$. So one can define the mapping $\delta : A_0 \to A$ by

$$\delta(x) := \lim_{n\to\infty} \frac{1}{m^n} f(m^n x) \tag{3.10}$$

for all $x \in A_0$. Letting $k = 0$ and passing the limit $n \to \infty$ in (3.9), we get (3.6). Now, we show that δ is a $\mathbb{C}$-linear mapping. It follows from (3.1), (3.4) and (3.10) that

$$\begin{aligned}\|D_1\delta(x_1, \ldots, x_m)\|_A &= \lim_{n\to\infty} \frac{1}{m^n}\big\|D_1 f(m^n x_1, \ldots, m^n x_m)\big\|_A \\ &\leq \lim_{n\to\infty} \frac{1}{m^n}\varphi(m^n x_1, \ldots, m^n x_m) = 0\end{aligned}$$

for all $x_1, \ldots, x_m \in A_0$. So the mapping δ satisfies (1.1). By Lemma 2.1, the mapping δ is additive.

Letting $x_j = mx$ and $x_i = 0$ for all $1 \leq i \leq m$ with $i \neq j$ in (3.4), we get

$$\|f(\mu m x) + mf(\mu x) - 2\mu f(mx)\|_A \leq \varphi(0, \ldots, \underbrace{mx}_{j\,th}, \ldots, 0) \tag{3.11}$$

for all $x \in A_0$. Replacing x by $m^n x$ in (3.11) and dividing both sides of (3.11) by m^{n+1}, we get

$$\begin{aligned}&\Big\|\frac{1}{m^{n+1}}f(\mu m^{n+1}x) + \frac{1}{m^n}f(\mu m^n x) - \frac{2\mu}{m^{n+1}}f(m^{n+1}x)\Big\|_A \\ &\leq \frac{1}{m^{n+1}}\varphi(0, \ldots, \underbrace{m^{n+1}x}_{j\,th}, \ldots, 0)\end{aligned} \tag{3.12}$$

for all $x \in A_0$ and all non-negative integers n. Passing the limit $n \to \infty$ in (3.12) and using (3.1) and (3.10), we get

$$\delta(\mu x) = \mu\delta(x)$$

for all $\mu \in \mathbb{T}^1$ and for all $x \in A_0$. So by Lemma 3.1, we infer that the mapping $\delta : A_0 \to A$ is $\mathbb{C}$-linear. To prove the uniqueness of δ, let $\delta' : A_0 \to A$ be another additive mapping satisfying (3.6). It follows from (3.6) and (3.10) that

$$\begin{aligned}\|\delta(x) - \delta'(x)\|_A &= \lim_{n\to\infty} \frac{1}{m^n}\big\|f(m^n x) - \delta'(m^n x)\big\|_A \\ &\leq \lim_{n\to\infty} \frac{1}{m^n}\widetilde{\varphi_j}(m^n x) = 0\end{aligned}$$

for all $x \in A_0$. So $\delta = \delta'$.

It follows from (3.2), (3.5) and (3.10) that

$$\begin{aligned}&\big\|\delta([x_1,x_2])-[\delta(x_1),x_2]-[x_1,\delta(x_2)]\big\|_A\\ &=\lim_{n\to\infty}\frac{1}{m^n}\big\|f(m^n[x_1,x_2])-[f(m^nx_1),x_2]-[m^nx_1,f(x_2)]\big\|_A\\ &\le\lim_{n\to\infty}\frac{1}{m^n}\psi(m^nx_1,x_2)=0\end{aligned}$$

for all $x_1,x_2\in A_0$. So

$$\delta([x_1,x_2])=[\delta(x_1),x_2]+[x_1,\delta(x_2)]$$

for all $x_1,x_2\in A_0$. Hence the mapping $\delta:A_0\to A$ is a unique Lie derivation satisfying (3.6). □

Remark 3.3. Theorem 3.2 holds if we replace the condition (3.2) by one of the following conditions

(i) $\lim_{n\to\infty}\frac{1}{m^n}\psi(x_1,m^nx_2)=0$,
(ii) $\lim_{n\to\infty}\frac{1}{m^{2n}}\psi(m^nx_1,m^nx_2)=0$

for all $x_1,x_2\in A_0$.

Corollary 3.4. *Let $\delta,\alpha_1,\alpha_2,s_1,s_2,\{\theta_i\}_{i=1}^m$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $0<s_j<1$ for some $j=1,2$ (or $0<s_1,s_2<2$), and $0<r_i<1$ for all $1\le i\le m$. Suppose that $f:A_0\to A$ is a mapping such that*

$$\|D_\mu f(x_1,\ldots,x_m)\|_A\le\delta+\sum_{i=1}^m\theta_i\|x_i\|_{A_0}^{r_i},$$

$$\|f([x_1,x_2])-[f(x_1),x_2]-[x_1,f(x_2)]\|_A\le\delta+\alpha_1\|x_1\|_{A_0}^{s_1}+\alpha_2\|x_2\|_{A_0}^{s_2}$$

for all $x_1,\ldots,x_m\in A_0$ and all $\mu\in\mathbb{T}^1$. Then there exists a unique Lie derivation $\delta:A_0\to A$ such that

$$\|f(x)-\delta(x)\|_A\le\frac{\delta}{m-1}+\gamma(x)$$

for all $x\in A_0$, where

$$\gamma(x):=\min_{1\le i\le m}\Big\{\frac{\theta_i m^{r_i}}{m-m^{r_i}}\|x\|_{A_0}^{r_i}\Big\}.$$

Remark 3.5. If $\delta=\theta_i=0$ in Corollary 3.4 for some $1\le i\le m$, the mapping $\delta:A_0\to A$ is a Lie derivation.

Theorem 3.6. *Let $\Phi:\underbrace{A_0\times A_0\times\cdots\times A_0}_{m-times}\to[0,\infty)$ and $\Psi:A_0\times A_0\to[0,\infty)$ be mappings such that*

$$\lim_{n\to\infty}m^n\Phi\Big(\frac{x_1}{m^n},\ldots,\frac{x_m}{m^n}\Big)=0,$$

$$\lim_{n\to\infty}m^n\Psi\Big(\frac{x_1}{m^n},x_2\Big)=0, \tag{3.13}$$

$$\widetilde{\Phi_j}(x) := \sum_{i=0}^{\infty} m^i \Phi(0,\ldots,\underbrace{\frac{x}{m^i}}_{j\,th},\ldots,0) < \infty$$

for some $1 \leq j \leq m$ and all $x, x_1, \ldots, x_m \in A_0$. Suppose that $f : A_0 \to A$ is a mapping such that

$$\|D_\mu f(x_1,\ldots,x_m)\|_A \leq \Phi(x_1,\ldots,x_m),$$
$$\big\|f([x_1,x_2]) - [f(x_1),x_2] - [x_1,f_1(x_2)]\big\|_A \leq \Psi(x_1,x_2)$$

for all $x_1,\ldots,x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \widetilde{\Phi_j}(x) \tag{3.14}$$

for all $x \in A_0$.

Proof. Similarly to the proof of Theorem 3.2, we have

$$\|f(mx) - mf(x)\|_A \leq \Phi(0,\ldots,\underbrace{mx}_{j\,th},\ldots,0) \tag{3.15}$$

for all $x \in A_0$. Replacing x by $\frac{x}{m^{n+1}}$ in (3.15) and multiplying both sides of (3.15) to m^n, we get

$$\Big\|m^{n+1}f(\frac{x}{m^{n+1}}) - m^n f(\frac{x}{m^n})\Big\|_A \leq m^n\Phi(0,\ldots,\underbrace{\frac{x}{m^n}}_{j\,th},\ldots,0)$$

for all $x \in A_0$ and all non-negative integers n. Hence

$$\begin{aligned}\Big\|m^{n+1}f(\frac{x}{m^{n+1}}) - m^k f(\frac{x}{m^k})\Big\|_A &\leq \sum_{i=k}^{n}\Big\|m^{i+1}f(\frac{x}{m^{i+1}}) - m^i f(\frac{x}{m^i})\Big\|_A \\ &\leq \sum_{i=k}^{n} m^i\Phi(0,\ldots,\underbrace{\frac{x}{m^i}}_{j\,th},\ldots,0)\end{aligned} \tag{3.16}$$

for all $x \in A_0$ and all non-negative integers n and k with $n \geq k$. Therefore the sequence $\{m^n f(x/m^n)\}$ is a Cauchy sequence in A for all $x \in A_0$. Since A is complete, the sequence $\{m^n f(x/m^n)\}$ converges in A for all $x \in A_0$. So one can define the mapping $\delta : A_0 \to A$ by

$$\delta(x) := \lim_{n\to\infty} m^n f(\frac{x}{m^n})$$

for all $x \in A_0$. Letting $k = 0$ and passing the limit $n \to \infty$ in (3.16), we get (3.14).

The rest of the proof is similar to the proof of Theorem 3.2. □

Remark 3.7. Theorem 3.6 holds if we replace the condition (3.13) by one of the following conditions

(i) $\lim_{n\to\infty} m^n\Psi(x_1, \frac{x_2}{m^n}) = 0$,

(ii) $\lim_{n\to\infty} m^{2n}\Psi(\frac{x_1}{m^n}, \frac{x_2}{m^n}) = 0$
for all $x_1, x_2 \in A_0$.

Corollary 3.8. *Let $\alpha_1, \alpha_2, s_1, s_2, \{\theta_i\}_{i=1}^m$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $s_1, s_2 > 2$ and $r_i > 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*

$$\|D_\mu f(x_1, \ldots, x_m)\|_A \le \sum_{i=1}^m \theta_i \|x_i\|_{A_0}^{r_i},$$

$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2}$$

for all $x_1, \ldots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \Gamma(x)$$

for all $x \in A_0$, where

$$\Gamma(x) := \min_{1\le i\le m} \Big\{ \frac{\theta_i m^{r_i}}{m^{r_i} - 1} \|x\|_{A_0}^{r_i} \Big\}.$$

Remark 3.9. If $\theta_i = 0$ in Corollary 3.8 for some $1 \le i \le m$, the mapping $\delta : A_0 \to A$ is a Lie derivation.

Lemma 3.10. *Let X, Y be real vector spaces and $k \in \mathbb{R} \setminus \{\pm 1\}$. Suppose that $f : X \to Y$ is a mapping such that*

$$f(kx + y) + f(x + ky) = (k + 1)f(x + y) \tag{3.17}$$

for all $x, y \in X$, then the mapping $f : X \to Y$ is additive.

Proof. Replacing x and y by $\frac{kx-y}{k^2-1}$ and $\frac{ky-x}{k^2-1}$ in (3.17), respectively, we get

$$f(x) + f(y) = (k + 1)f(\frac{x + y}{k + 1}) \tag{3.18}$$

for all $x, y \in X$. Replacing x by $(k + 1)x$ and letting $y = 0$ in (3.18), we get

$$f((k + 1)x) = (k + 1)f(x) \tag{3.19}$$

for all $x \in X$. Using (3.18) and (3.19), we get

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. So the mapping $f : X \to Y$ is additive. □

Theorem 3.11. *Let δ, θ, s_1, s_2 and $\{r_i\}_{i\in J}$ be non-negative real numbers such that $s_i \ne 1$ for some $i = 1, 2$ and $r_j > 0$ for all $j \in J$, where J is a non-empty subset of $\{1, \ldots, m\}$ and $m \ge 3$. Suppose that $f : A_0 \to A$ is a mapping such that*

$$\|D_\mu f(x_1, \ldots, x_m)\|_A \le \theta \Pi_{j\in J} \|x_j\|_{A_0}^{r_j} \tag{3.20}$$

$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \delta \|x_1\|_{A_0}^{s_1} \|x_2\|_{A_0}^{s_2} \tag{3.21}$$

for all $x_1, \dots, x_m \in A_0$ *and all* $\mu \in \mathbb{T}^1$ *(by letting* $\|\cdot\|_{A_0}^0 = 1$*). Then the mapping* $f: A_0 \to A$ *is a Lie derivation.*

Proof. We first show that f is $\mathbb{C}$-linear. We have two cases:

(i) $|J| = m$.

Letting $x_1 = \cdots = x_m = 0$ in (3.20), we get that $f(0) = 0$. Letting $\mu = 1$, $x_1 = mx$ and $x_2 = \cdots = x_m = 0$ in (3.20), we get

$$f(mx) = mf(x) \tag{3.22}$$

for all $x \in A_0$. Letting $\mu = 1$, $x_1 = x$, $x_2 = y$ and $x_3 = \cdots = x_m = 0$ in (3.20) and using (3.22), we get

$$f(mx + y) + f(x + my) = (m + 1)f(x + y) \tag{3.23}$$

for all $x, y \in A_0$. So by Lemma 3.10, the mapping f is additive. Letting $x_1 = mx$ and $x_2 = \cdots = x_m = 0$ in (3.20) and using (3.22), we get

$$f(\mu x) = \mu f(x)$$

for all $x \in A_0$ and all $\mu \in \mathbb{T}^1$. Hence it follows from Lemma 3.1 that the mapping $f: A_0 \to A$ is $\mathbb{C}$-linear.

(ii) $0 < |J| < m$.

Let $j_0 \in J$, $i_0 \notin J$ and $k_0 \neq i_0, j_0$ for some $1 \leq i_0, j_0, k_0 \leq m$. Letting $\mu = 1$, $x_{i_0} = mx$ and $x_j = 0$ in (3.20) for all $j \in J$, we get (3.22). Letting $x_{i_0} = x$, $x_{k_0} = y$ and $x_j = 0$ in (3.20) for all $j \in J \setminus \{k_0\}$ and using (3.22), we get (3.23). So the mapping f is additive. Letting $x_{i_0} = mx$ and $x_j = 0$ in (3.20) for all $j \in J$ and using (3.22), we get $f(\mu x) = \mu f(x)$ for all $x \in A_0$ and all $\mu \in \mathbb{T}^1$. Hence it follows from Lemma 3.1 that the mapping $f: A_0 \to A$ is $\mathbb{C}$-linear.

Let $s_1 < 1$ (we have a similar proof when $s_1 > 1, s_2 < 1$ or $s_2 > 1$). Since f is $\mathbb{C}$-linear, it follows from (3.21) that

$$\begin{aligned}
&\big\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\big\|_A \\
&= \lim_{n\to\infty} \frac{1}{n}\big\|f([nx_1, x_2]) - [f(nx_1), x_2] - [nx_1, f(x_2)]\big\|_A \\
&\leq \delta \lim_{n\to\infty} \frac{n^{s_1}}{n}\|x_1\|_{A_0}^{s_1}\|x_2\|_{A_0}^{s_2} = 0
\end{aligned}$$

for all $x_1, x_2 \in A_0$. So

$$f([x_1, x_2]) = [f(x_1), x_2] + [x_1, f(x_2)]$$

for all $x_1, x_2 \in A_0$. Hence the mapping $f: A_0 \to A$ is a Lie derivation. □

For $m = 2$, we have the following theorem.

Theorem 3.12. *Let* δ, θ, s_1, s_2 *and* $\{r_i\}_{i\in J}$ *be non-negative real numbers such that* $s_i \neq 1$ *for some* $i = 1, 2$ *and* $\lambda := \sum_{j\in J} r_j \neq 1$*, where* J *is a non-empty subset of* $\{1, 2\}$*. Suppose that* $f: A_0 \to A$ *is a mapping such that*

$$\begin{aligned}
&\|D_\mu f(x_1, x_2)\|_A \leq \theta \Pi_{j\in J}\|x_j\|_{A_0}^{r_j}, \\
&\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \leq \theta\|x_1\|_{A_0}^{s_1}\|x_2\|_{A_0}^{s_2}
\end{aligned} \tag{3.24}$$

for all $x_1, x_2 \in A_0$ and all $\mu \in \mathbb{T}^1$ (by letting $\|\cdot\|_{A_0}^0 = 1$). Then the mapping $f : A_0 \to A$ is a Lie derivation.

Proof. Without loss of generality, we may assume that $2 \in J$. Letting $x_1 = x_2 = 0$ in (3.24), we get that $f(0) = 0$. Letting $\mu = 1$, $x_1 = 2x$ and $x_2 = 0$ in (3.24), we get that $f(2x) = 2f(x)$ for all $x \in A_0$. Hence

$$2^n f\left(\frac{x}{2^n}\right) = f(x), \qquad \frac{1}{2^n} f(2^n x) = f(x) \tag{3.25}$$

for all $x \in A_0$ and all $n \in \mathbb{N}$. Let $\lambda < 1$ (we have a similar proof when $\lambda > 1$). By (3.24) and (3.25) we have

$$\begin{aligned}\|D_\mu f(x_1, x_2)\|_A &= \lim_{n\to\infty} \frac{1}{2^n} \|D_\mu f(2^n x_1, 2^n x_2)\|_A \\ &\leq \theta \lim_{n\to\infty} \frac{2^{\lambda n}}{2^n} \Pi_{j\in J} \|x_j\|_{A_0}^{r_j} = 0\end{aligned}$$

for all $x_1, x_2 \in A_0$ and all $\mu \in \mathbb{T}^1$. Hence it follows from (3.25) that

$$f(2\mu x_1 + \mu x_2) + f(\mu x_1 + 2\mu x_2) + f(\mu x_1 + \mu x_2) = 4\mu f(x_1 + x_2) \tag{3.26}$$

for all $x_1, x_2 \in A_0$ and all $\mu \in \mathbb{T}^1$. Letting $\mu = 1$ in (3.26) and applying Lemma 3.10, we get that the mapping f is additive. Letting $x_2 = 0$ in (3.26) and using (3.25), we get that $f(\mu x_1) = \mu f(x_1)$ for all $x_1 \in A_0$ and all $\mu \in \mathbb{T}^1$. So by Lemma 3.1, the mapping f is $\mathbb{C}$-linear. The rest of the proof is similar to the proof of Theorem 3.11. □

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