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A NOTE ON THE FIRST LAYERS OF $\mathbb{Z}_p$-EXTENSIONS

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ABSTRACT. In this paper we explicitly compute a Minkowski unit of a real abelian field and give a criterion when the first layer of anti-cyclotomic $\mathbb{Z}_3$-extension of an imaginary quadratic field is unramified everywhere.

1. Introduction

For each prime number p, a $\mathbb{Z}_p$-extension of a number field k is an extension $k = k_0 \subset k_1 \subset \cdots \subset k_n \subset \cdots \subset k_\infty$ with $Gal(k_\infty/k) \simeq \mathbb{Z}_p$, the additive group of p-adic integers. Let k be an imaginary quadratic field, and K an abelian extension of k. The number field K is called an anti-cyclotomic extension of k if it is Galois over $\mathbb{Q}$, and $Gal(k/\mathbb{Q})$ acts on $Gal(K/k)$ by -1. The explicit construction of the first layer k_1^a of the anti-cyclotomic $\mathbb{Z}_p$-extension of k is given in [2, 3, 4].

In this paper, we prove two theorems (Theorem 1 and Theorem 2 in this paper) on questions raised from our previous paper [2, 3, 4]. First it is about the explicit construction of a Minkowski unit which plays a very important role in [4]. Let L be a finite real Galois extension of $\mathbb{Q}$. It is well-known that there exists a unit in L such that the set of units $\{\epsilon^\sigma | \sigma \neq 1, \sigma \in Gal(L/\mathbb{Q})\}$ is multiplicative independent and generates a subgroup of finite index in the full group of units. Such a unit is called a Minkowski unit. Theorem 1 gives an explicit construction of a Minkowski unit.

The first layer of anti-cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field k may be or may not be contained in the Hilbert class field of k. Hence it is a natural question when the compositum K of the $\mathbb{Z}_p$-extensions of a number field k and Hilbert class field of k are linearly disjoint over k. Theorem 2 gives an answer for this question when k is an imaginary quadratic field and $p = 3$.

Let $n \not\equiv 2 \bmod 4$, and let $n = \prod_{i=1}^{s} p_i^{e_i}$ be its prime factorization. Let I run through all subsets of $\{1, \ldots, s\}$, except $\{1, \ldots, s\}$, and let $n_I = \prod_{i \in I} p_i^{e_i}$. For

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integer $a, (a,n) = 1$, define (see [5, Theorem 8.3])

$$\xi_a = \zeta_n^{d_a} \prod_I \frac{1-\zeta_n^{an_I}}{1-\zeta_n^{n_I}}, \quad d_a = \frac{1}{2}(1-a)\sum_I n_I,$$

where ζ_n is a primitive n-th root of unity.

Theorem 1. *Let $n \not\equiv 2 \mod 4$, and*

$$(Z/n)^\times/\{\pm 1\} = \langle t_1\rangle \times \langle t_2\rangle \times \cdots \times \langle t_m\rangle.$$

Then

$$\xi_{(n)} := \xi_{t_1}\xi_{t_2}\cdots\xi_{t_m}$$

is a Minkowski unit for $\mathbb{Q}(\zeta_n)^+$

Let K be the compositum of all $\mathbb{Z}_3$-extensions of k, H_k the 3-part of Hilbert class field of a number field k and A_k the 3-part of the ideal class group of k. Then we have the following theorem.

Theorem 2. *Let $d \not\equiv 3 \mod 9$ be a positive integer and $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. Then*

$$H_k \cap K = k \Longleftrightarrow \mathrm{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} = \mathrm{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})}.$$

Remark 1. It is well-known that

$$\mathrm{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} \le \mathrm{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})} \le \mathrm{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} + 1.$$

2. Proof of theorems

First we prove Theorem 1. To prove Theorem 1 we need lemmas.

Lemma 3. *For integers a, b relatively prime to $n \not\equiv 2 \mod 4$, we have*

$$\xi_a^{\sigma_b}\xi_b = \xi_{ab}, \quad \xi_a = \pm\xi_{-a},$$

where σ_b is the Frobenius map.

Proof. The proof comes directly from simple computations. □

Lemma 4. *Let notations be as above and χ be a nontrivial even character mod n. Then*

$$S_\chi := \sum_{(b,n)=1}^{n} \chi^{-1}(b)(\sum_I \log|1-\zeta_n^{bn_I}|) \neq 0.$$

Proof. In fact $S_\chi = \frac{-1}{2}\tau(\chi^{-1})L(1,\chi)\prod_{p_i \nmid f_\chi}(\phi(p_i^{e_i}) + 1 - \chi^{-1}(p_i))$. See [5, Theorem 8.3] for details. □

Now we are ready to prove Theorem 1. It suffices to prove that

$$e_\chi \log|\xi_{(n)}| \neq 0,$$

for any nontrivial even character mod n. We may assume that the real number ξ_a is positive for any integer a relatively prime to n. First let us compute

$$\begin{aligned} e_\chi \log \xi_a &= \frac{2}{\phi(n)} \sum_{(b,n)=1, b\leq \frac{n}{2}} \chi^{-1}(b)\sigma_b \log \xi_a \\ &= \frac{2}{\phi(n)} \sum_{(b,n)=1, b\leq \frac{n}{2}} \chi^{-1}(b) \log \xi_a^{\sigma_b} \\ &= \frac{2}{\phi(n)} \sum_{(b,n)=1, b\leq \frac{n}{2}} \chi^{-1}(b) \log \xi_{ab}\xi_b^{-1} \\ &= \frac{2}{\phi(n)}(\chi(a)-1) \sum_{(b,n)=1, b\leq \frac{n}{2}} \chi^{-1}(b) \log \xi_b \\ &= \frac{1}{\phi(n)}(\chi(a)-1)S_\chi. \end{aligned}$$

Therefore

$$\begin{aligned} e_\chi \log |\xi_{(n)}| &= \sum_{i=1}^{m} e_\chi \log \xi_{t_i} \\ &= \sum_{i=1}^{m} \frac{1}{\phi(n)}(\chi(t_i)-1)S_\chi = \frac{2}{\phi(n)}(\sum_{i=1}^{m}\chi(t_i)-m)S_\chi. \end{aligned}$$

Note that $(\sum_{i=1}^m \chi(t_i) - m) \neq 0$ since χ is a nontrivial even character mod n. By Lemma 4, $S_\chi \neq 0$. This completes the proof of Theorem 1.

Corollary 5. *Let L be a real abelian field contained in $\mathbb{Q}(\zeta_n + \zeta_n{}^{-1})$. Then $N_{\mathbb{Q}(\zeta_n+\zeta_n{}^{-1})/L}(\xi_{t_1}\cdots\xi_{t_m})$ is a Minkowski unit of L.*

Proof. This directly comes from the finiteness of the index

$$[E_L : N_{\mathbb{Q}(\zeta_n+\zeta_n{}^{-1})/L}(E_{\mathbb{Q}(\zeta_n+\zeta_n{}^{-1})})]$$

and Theorem 1. □

Now we will prove Theorem 2. We assume that $p = 3$. For $d \not\equiv 3 \mod 9$, let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, $F = \mathbb{Q}(\sqrt{-d}, \sqrt{-3})$ a biquadratic field, M_F and M_k the maximal abelian p-extension of F and k unramified outside above p, respectively.

Let $X_F := Gal(M_F/F)/pGal(M_F/F)$ and $X_{F,\chi}$ be the χ-component of X_F for the nontrivial character χ of $Gal(k/\mathbb{Q})$. Let S be a subset of $F^\times/(F^\times)^3$ corresponding to the X_F. Then, by Kummer theory, we have a perfect pairing $S_{\chi\omega} \times X_{F,\chi} \longrightarrow \mu_p$, where ω is the nontrivial character of $Gal(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$ and $S_{\chi\omega}$ is the $\chi\omega$-component of S. Note that

$$X_{F,\chi} \simeq X_{k,\chi}.$$

By [1, Proposition 6.B], $S \simeq E_F/E_F{}^p \times A_F/A_F{}^p \times \langle p\rangle/\langle p\rangle^p$, where E_F is the group of units of F and A_F is the p-part of the ideal class group of F. Since the

$\chi\omega$-component $E_{F,\chi\omega}$ of the group of units E_F is the group of the units of the real quadratic subfield $F^+(=\mathbb{Q}(\sqrt{3d}))$ of F, the $\text{rank}_{\mathbb{Z}/p}(E_F/{E_F}^p)_{\chi\omega}$ is equal to 1. Therefore

$$\begin{aligned} H_k \cap K = k &\Longleftrightarrow \text{rank}_{\mathbb{Z}/p} X_{k,\chi} = 1 + \text{rank}_{\mathbb{Z}/p} A_k \\ &\Longleftrightarrow \text{rank}_{\mathbb{Z}/p} X_{F,\chi} = 1 + \text{rank}_{\mathbb{Z}/p} A_k \\ &\Longleftrightarrow \text{rank}_{\mathbb{Z}/p} S_{\chi\omega} = 1 + \text{rank}_{\mathbb{Z}/p} A_k \\ &\Longleftrightarrow \text{rank}_{\mathbb{Z}/p} (A_F/A_F^p)_{\chi\omega} = \text{rank}_{\mathbb{Z}/p} A_k \\ &\Longleftrightarrow \text{rank}_{\mathbb{Z}/p} A_{F^+} = \text{rank}_{\mathbb{Z}/p} A_k \\ &\Longleftrightarrow \text{rank}_{\mathbb{Z}/p} A_{\mathbb{Q}(\sqrt{3d})} = \text{rank}_{\mathbb{Z}/p} A_{\mathbb{Q}(\sqrt{-d})}. \end{aligned}$$

This completes the proof of Theorem 2.

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