

## DISCRETE RESULTS OF QI-TYPE INEQUALITY

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ABSTRACT. In the present paper, we give several discrete results for the open problem posed in the article [Feng Qi, *Several integral inequalities*, J. Inequal. Pure and Appl. Math. **1** (2000), no. 2, Art. 19].

### 1. Introduction

The following problem was posed by Qi in his article [10]: “*Under what condition does the inequality*

$$(1.1) \quad \int_a^b [f(x)]^t dx \geq \left( \int_a^b f(x) dx \right)^{t-1}$$

*hold for  $t > 1$ ?*”.

There are numerous answers and extension results to this open problem [1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 14]. These results were obtained by different approaches, such as, e.g. Jensen’s inequality, the convexity method [14]; functional inequalities in abstract spaces [1, 2]; probability measures view [4, 6]; Hölder inequality and its reversed variants [2, 9]; analytical methods [8, 13]; Cauchy’s mean value theorem [3, 11].

In this paper we introduce the discrete version of (1.1) as follows, “*Under what condition does the inequality*

$$(1.2) \quad \sum_{i=1}^n x_i^\alpha a_i \geq \left( \sum_{i=1}^n x_i a_i \right)^\beta$$

*hold for  $\alpha, \beta > 0$ ?*”. (For the infinite series, the same method in the above finite series can be discussed.)

Here and in what follows we write  $X$  for the discrete random variable (r.v.) with values  $x_1, x_2, \dots, x_n$ . Accordingly, let us denote  $E(X)$  the mathematical expectation of r.v.  $X$ , i.e.,  $E(X) = \sum_{i=1}^n x_i P(X = x_i)$ , where  $P(X = x_i)$  denotes the probability of the event  $\{X = x_i\}$ .

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We will consider some moment type inequalities for the discrete random variable  $X$  with values  $x_1, x_2, \dots, x_n$ . Precisely, in §2 we obtain results concerning the direct inequality (1.2) by taking the probability distribution function. In §3, we derive some inequalities reversed to (1.2). Finally, in §4 several further results will be treated by constructing suitable probability measures for arriving at answers to (1.2).

## 2. Direct inequality

In this section we consider two important cases of (1.2). At first let  $\alpha > \max\{1, \beta\}$ , then we take  $\alpha > 0, \beta > 1$ .

### 2.1. The case $\alpha > \max\{1, \beta\}$ .

**Theorem 1.** *Let  $\{x_i, i = 1, 2, \dots, n\}, \{a_i, i = 1, 2, \dots, n\}$  be two sequences of nonnegative real numbers such that*

$$(2.1) \quad \left( \sum_{i=1}^n x_i a_i \right)^{\beta-\alpha} \leq \left( \sum_{i=1}^n a_i \right)^{1-\alpha}.$$

*Then the inequality (1.2) holds.*

*Proof.* Let  $X$  denote the discrete random variable with

$$P(X = x_i) = \frac{a_i}{\sum_{i=1}^n a_i} \quad \text{and} \quad E(X) = \sum_{i=1}^n x_i \frac{a_i}{\sum_{i=1}^n a_i}.$$

Then it is obvious that

$$(2.2) \quad \sum_{i=1}^n x_i a_i = \left( \sum_{i=1}^n a_i \right) E(X) \quad \text{and} \quad \sum_{i=1}^n x_i^\alpha a_i = \left( \sum_{i=1}^n a_i \right) E(X^\alpha).$$

Thus it is sufficient to show

$$(2.3) \quad \left( \sum_{i=1}^n a_i \right) E(X^\alpha) \geq \left[ \left( \sum_{i=1}^n a_i \right) E(X) \right]^\beta.$$

Indeed, by Jensen's inequality and (2.1), we conclude

$$\begin{aligned} \left[ \left( \sum_{i=1}^n a_i \right) E(X) \right]^\beta &= \left( \sum_{i=1}^n a_i \right)^\beta [E(X)]^\alpha [E(X)]^{\beta-\alpha} \\ &\leq \left( \sum_{i=1}^n a_i \right)^\beta [E(X^\alpha)] [E(X)]^{\beta-\alpha} \leq \left( \sum_{i=1}^n a_i \right) E(X^\alpha), \end{aligned}$$

which implies our result.  $\square$

With the similar proof, we have the following

**Theorem 2.** Let  $\{x_i, i = 1, 2, \dots, n\}, \{a_i, i = 1, 2, \dots, n\}$  be two sequences of real numbers such that  $\beta \geq 0, \alpha = 2k/j > 1, j, k \in \mathbb{N}$ , and

$$(2.4) \quad \left( \sum_{i=1}^n x_i a_i \right)^{\beta-\alpha} \leq \left( \sum_{i=1}^n a_i \right)^{1-\alpha}.$$

Then the inequality (1.2) holds.

## 2.2. The case $\alpha > 0, \beta > 1$ .

In this case we will need the help of an auxiliary result, which we clearly deduce by Hölder's inequality.

**Lemma 1.** Let  $Z, Y$  be two random variables with  $Z \geq 0, Y \geq 0, Z/Y \geq 0$  a.e. In addition, let the constants  $K, r > 0, p > 1$  and  $\mathbb{E}(Z/Y)^{rp} \leq K$ . Then

$$(2.5) \quad \mathbb{E}Z^r \leq [\mathbb{E}(Z/Y)^{rp}]^{1/p} \times [\mathbb{E}Y^{rq}]^{1/q} \leq K^{1/p} [\mathbb{E}Y^{rq}]^{1/q}$$

where  $1/p + 1/q = 1$ .

**Theorem 3.** Let  $\{x_i, i = 1, 2, \dots, n\}, \{a_i, i = 1, 2, \dots, n\}$  be two sequences of nonnegative real numbers such that

$$(2.6) \quad \sum_{i=1}^n a_i x_i^{(\beta-\alpha)/(\beta-1)} \leq 1.$$

Specifically, for  $\alpha > \beta$ , letting  $x_i \geq m > 0, i = 1, 2, \dots, n$  and

$$\frac{1}{m^{(\alpha-\beta)/(\beta-1)}} \sum_{i=1}^n a_i \leq 1.$$

Then the inequality (1.2) holds true.

*Proof.* From the proof of Theorem 1, it is enough to prove that

$$(2.7) \quad \left( \sum_{i=1}^n a_i \right) E(X^\alpha) \geq \left[ \left( \sum_{i=1}^n a_i \right) E(X) \right]^\beta.$$

Let  $q = \beta > 1, p = \beta/(\beta-1), Z^r = X$  and  $Y^{r\beta} = X^\alpha$ , in the formula of Lemma 1. Then  $(Z/Y)^r = X^{1-\alpha/\beta}$  readily follows, and consequently

$$\begin{aligned} \left[ \left( \sum_{i=1}^n a_i \right) E(X) \right]^\beta &\leq \left[ \left( \sum_{i=1}^n a_i \right) \left( E(X^{p-p\alpha/\beta}) \right)^{1/p} (E(X^\alpha))^{1/\beta} \right]^\beta \\ &= \left( \sum_{i=1}^n a_i \right)^{\beta-1} \left( E(X^{p-p\alpha/\beta}) \right)^{\beta/p} \left( \sum_{i=1}^n a_i \right) E(X^\alpha) \\ &= \left( \sum_{i=1}^n a_i \right)^{\beta-1} \left( E(X^{(\beta-\alpha)/(\beta-1)}) \right)^{\beta-1} \left( \sum_{i=1}^n a_i \right) E(X^\alpha) \end{aligned}$$

$$= \left( \sum_{i=1}^n a_i x_i^{(\beta-\alpha)/(\beta-1)} \right)^{\beta-1} \left( \sum_{i=1}^n a_i \right) E(X^\alpha),$$

which, by (2.6), yields the desired result.  $\square$

*Remark 1.* In fact, we do not need the condition  $\alpha > \beta$ , since supposing the converse  $\alpha < \beta$  and  $(\beta-\alpha)/(\beta-1) < 1$ , then the condition (2.6) can be replaced by the following condition, from using  $x^\gamma$ ,  $0 < \gamma < 1$ , is concave function,

$$\sum_{i=1}^n a_i x_i \leq \left( \sum_{i=1}^n a_i \right)^{\frac{1-\alpha}{\beta-\alpha}},$$

which is easier to check.

### 3. Reverse inequality

In this section, we mainly discuss reverse inequality of the inequality (1.2). For this purpose we list the following auxiliary inequality by Saitoh et al. [12], which is a reverse Hölder's inequality.

**Lemma 2.** For two positive functions  $f$  and  $g$  satisfying  $0 < \hat{m} \leq f^p/g^q \leq \hat{M} < \infty$  on the set  $X$ , and for  $p > 1$  and  $q > 1$  with  $1/p + 1/q = 1$ , we have

$$(3.1) \quad \left( \int_X f^p d\mu \right)^{1/p} \left( \int_X g^q d\mu \right)^{1/q} \leq \left( \frac{\hat{M}}{\hat{m}} \right)^{\frac{1}{pq}} \int_X f g d\mu.$$

**Theorem 4.** Let  $\{x_i, i = 1, 2, \dots, n\}$ ,  $\{a_i, i = 1, 2, \dots, n\}$  be two sequences of positive real numbers and assume that  $\alpha > 1$ ,  $\beta > 0$  and

$$(3.2) \quad \left( \frac{m}{M} \sum_{i=1}^n a_i \right)^{1-\alpha} \left( \sum_{i=1}^n x_i a_i \right)^{\alpha-\beta} \leq 1,$$

where  $M = \max\{x_1, x_2, \dots, x_n\}$  and  $m = \min\{x_1, x_2, \dots, x_n\}$ . Then we have the following reverse inequality to (1.2), i.e.,

$$\sum_{i=1}^n x_i^\alpha a_i \leq \left( \sum_{i=1}^n x_i a_i \right)^\beta.$$

*Proof.* The inequality (3.1) can be written in an equivalent form as

$$(3.3) \quad [E_\mu(f^p)]^{1/p} [E_\mu(g^q)]^{1/q} \leq \left( \frac{\hat{M}}{\hat{m}} \right)^{\frac{1}{pq}} E_\mu(fg),$$

where  $E_\mu$  denotes the mathematical expectation under a probability measure  $\mu$ . Furthermore, as the proof of Theorem 1, it is sufficient to show

$$(3.4) \quad \left( \sum_{i=1}^n a_i \right) E(X^\alpha) \leq \left[ \left( \sum_{i=1}^n a_i \right) E(X) \right]^\beta.$$

Taking  $f = X$ ,  $g = 1$ ,  $p = \alpha$ ,  $q = \alpha/(\alpha - 1)$ ,  $\hat{M} = M^\alpha$ ,  $\hat{m} = m^\alpha$ , then (3.3) becomes

$$(3.5) \quad [E_\mu(X^\alpha)]^{1/\alpha} \leq \left(\frac{M}{m}\right)^{\frac{\alpha-1}{\alpha}} E_\mu(X).$$

Thus by (3.4) and (3.5), we deduce

$$\begin{aligned} \left(\sum_{i=1}^n a_i\right) E(X^\alpha) &\leq \left(\sum_{i=1}^n a_i\right) \left[\left(\frac{M}{m}\right)^{\frac{\alpha-1}{\alpha}} E(X)\right]^\alpha \\ &= \left[\left(\sum_{i=1}^n a_i\right) E(X)\right]^\beta \left(\sum_{i=1}^n a_i\right)^{1-\beta} \left(\frac{M}{m}\right)^{\alpha-1} [E(X)]^{\alpha-\beta} \\ &\leq \left[\left(\sum_{i=1}^n a_i\right) E(X)\right]^\beta. \end{aligned}$$

The proof of the theorem is completed.  $\square$

The following reverse Hölder's inequality was obtained by Nehari [7].

**Lemma 3.** Let  $(X, \Sigma, \mu)$  be a finite positive measure space and let  $f_1 \in L^p(X, \Sigma, \mu)$  and  $f_2 \in L^q(X, \Sigma, \mu)$ , where  $p, q > 0$ ,  $1/p + 1/q = 1$ . If, in addition,  $f_1$  and  $f_2$  satisfy  $0 \leq m_v \leq f_v \leq M_v < \infty$ ,  $v = 1, 2$ , and if the numbers  $\eta_1, \eta_2$ , ( $0 \leq \eta_1, \eta_2 \leq 1$ ) are defined by

$$\int_X f_v d\mu = [m_v + \eta(M_v - m_v)]\mu(X), \quad v = 1, 2,$$

then

$$(3.6) \quad \left[\int_X f_1^p d\mu\right]^{1/p} \left[\int_X f_2^q d\mu\right]^{1/q} \leq D \int_X f_1 f_2 d\mu,$$

where

$$D = \frac{[m_1^p + (M_1^p - m_1^p)\eta_1]^{1/p} [m_2^q + (M_2^q - m_2^q)\eta_2]^{1/q}}{m_1 m_2 + m_1(M_2 - m_2)\eta_2 + m_2(M_1 - m_1)\eta_1 + \gamma(M_1 - m_1)(M_2 - m_2)}$$

and

$$\gamma = \max\{0, \eta_1 + \eta_2 - 1\}.$$

**Theorem 5.** Let  $\{x_i, i = 1, 2, \dots, n\}, \{a_i, i = 1, 2, \dots, n\}$  be two sequences of nonnegative real numbers and assume that  $\alpha > 1$ ,  $\beta > 0$  and

$$(3.7) \quad D^\alpha \left(\sum_{i=1}^n x_i a_i\right)^{\alpha-\beta} \leq 1,$$

where

$$D = \frac{[m^\alpha + (M^\alpha - m^\alpha) \frac{\sum_{i=1}^n a_i x_i - m}{M - m}]^{1/\alpha}}{\sum_{i=1}^n a_i x_i},$$

$M = \max\{x_1, x_2, \dots, x_n\}$  and  $m = \min\{x_1, x_2, \dots, x_n\}$ . Then we have the following reverse inequality to (1.2), i.e.,

$$\sum_{i=1}^n x_i^\alpha a_i \leq \left( \sum_{i=1}^n x_i a_i \right)^\beta.$$

*Proof.* As the proof of Theorem 1, it is sufficient to show

$$(3.8) \quad \left( \sum_{i=1}^n a_i \right) E(X^\alpha) \leq \left[ \left( \sum_{i=1}^n a_i \right) E(X) \right]^\beta.$$

Taking  $f = X$ ,  $g = 1$ ,  $p = \alpha$ ,  $q = \alpha/(\alpha - 1)$ ,  $\hat{M}_1 = M$ ,  $m_1 = m$ ,  $M_2 = m_2 = 1$  in Lemma 3, then  $D$  in (3.6) becomes

$$(3.9) \quad D = \frac{[m^\alpha + (M^\alpha - m^\alpha) \frac{\sum_{i=1}^n a_i x_i - m}{M - m}]^{1/\alpha}}{\sum_{i=1}^n a_i x_i}.$$

Thus by (3.6), (3.8) and (3.9), we deduce

$$\begin{aligned} \left( \sum_{i=1}^n a_i \right) E(X^\alpha) &\leq \left( \sum_{i=1}^n a_i \right) [DE(X)]^\alpha \\ &= \left[ \left( \sum_{i=1}^n a_i \right) E(X) \right]^\beta \left( \sum_{i=1}^n a_i \right)^{1-\beta} D^\alpha [E(X)]^{\alpha-\beta} \\ &\leq \left[ \left( \sum_{i=1}^n a_i \right) E(X) \right]^\beta. \end{aligned}$$

The proof of the theorem is completed.  $\square$

*Remark 2.* In Theorem 5, if taking  $m = 0$ , then we have the following simple form of  $D$ ,

$$D = \left( \frac{M}{\sum_{i=1}^n a_i x_i} \right)^{1 - \frac{1}{\alpha}}.$$

#### 4. Solving (1.2) constructing suitable probability measures

In this section we construct convenient probability measures to derive some new discrete Qi-type inequalities. In what follows, we define  $M = \max\{x_1, x_2, \dots, x_n\}$  and  $m = \min\{x_1, x_2, \dots, x_n\}$ .

**Theorem 6.** Let  $\{x_i, i = 1, 2, \dots, n\}$ ,  $\{a_i, i = 1, 2, \dots, n\}$  be two sequences of nonnegative real numbers and assume that  $\alpha > \beta > 1$  and

$$(4.1) \quad \frac{m^{\alpha-1}}{M^{\beta-1} [\sum_{i=1}^n a_i]^{\beta-1}} \geq 1.$$

Then we have

$$\sum_{i=1}^n x_i^\alpha a_i \geq \left( \sum_{i=1}^n x_i a_i \right)^\beta.$$

Moreover, the reverse inequality is valid when

$$(4.2) \quad \frac{M^{\alpha-1}}{m^{\beta-1} [\sum_{i=1}^n a_i]^{\beta-1}} \leq 1.$$

*Proof.* Define

$$P(X = x_i) = \frac{x_i a_i}{\sum_{i=1}^n x_i a_i}, \quad \forall i = 1, 2, \dots, n.$$

It is easy to see that  $x_i a_i / \sum_{i=1}^n x_i a_i, \forall i = 1, 2, \dots, n$  defines a probability measure on  $x_1, x_2, \dots, x_n$  and the following implications follow

$$(4.3) \quad \frac{\sum_{i=1}^n x_i^\alpha a_i}{[\sum_{i=1}^n x_i a_i]^\beta} = \frac{\sum_{i=1}^n x_i^{\alpha-1} \frac{x_i a_i}{\sum_{i=1}^n x_i a_i}}{[\sum_{i=1}^n x_i a_i]^{\beta-1}} \geq \frac{m^{\alpha-1}}{M^{\beta-1} [\sum_{i=1}^n a_i]^{\beta-1}},$$

which implies our result. The remainder part of proof is straightforward.  $\square$

*Remark 3.* The direct use of the definitions of  $m$  and  $M$  results in

$$\frac{\sum_{i=1}^n x_i^\alpha a_i}{[\sum_{i=1}^n x_i a_i]^\beta} \geq \frac{m^\alpha}{M^\beta [\sum_{i=1}^n a_i]^{\beta-1}} =: \mathfrak{M}_1.$$

For our purposes we need the case  $\mathfrak{M}_1 \geq 1$ . However, it is easy to check that

$$\mathfrak{M}_1 \leq \frac{m^{\alpha-1}}{M^{\beta-1} [\sum_{i=1}^n a_i]^{\beta-1}},$$

hence, (4.1) generalizes the simplest possible  $\mathfrak{M}_1 \geq 1$ . By similar reasons,

$$\mathfrak{M}_2 := \frac{M^\alpha}{m^\beta [\sum_{i=1}^n a_i]^{\beta-1}} \leq 1$$

implies (4.2), so, the settings of Theorem 6 are optimal.

**Corollary 1.** Let  $\{x_i, i = 1, 2, \dots, n\}, \{a_i, i = 1, 2, \dots, n\}$  be two sequences of nonnegative real numbers and assume that  $0 < \beta < \alpha < 1$  and

$$(4.4) \quad \frac{M^{\alpha-1}}{m^{\beta-1} [\sum_{i=1}^n a_i]^{\beta-1}} \geq 1.$$

Then we have

$$\sum_{i=1}^n x_i^\alpha a_i \geq \left( \sum_{i=1}^n x_i a_i \right)^\beta.$$

Moreover, the reverse inequality is valid when

$$(4.5) \quad \frac{m^{\alpha-1}}{M^{\beta-1} [\sum_{i=1}^n a_i]^{\beta-1}} \leq 1.$$

**Corollary 2.** Let  $\{x_i, i = 1, 2, \dots, n\}, \{a_i, i = 1, 2, \dots, n\}$  be two sequences of nonnegative real numbers and assume that  $0 < \beta < 1 < \alpha$  and

$$(4.6) \quad \frac{m^{\alpha-\beta}}{[\sum_{i=1}^n a_i]^{\beta-1}} \geq 1.$$

Then it follows that

$$\sum_{i=1}^n x_i^\alpha a_i \geq \left( \sum_{i=1}^n x_i a_i \right)^\beta.$$

Otherwise, when  $0 < \beta < \alpha < 1$ , and

$$(4.7) \quad \frac{M^{\alpha-\beta}}{[\sum_{i=1}^n a_i]^{\beta-1}} \leq 1,$$

the reverse inequality

$$\sum_{i=1}^n x_i^\alpha a_i \leq \left( \sum_{i=1}^n x_i a_i \right)^\beta$$

is deduced.

Finally, let us construct another probability measure

$$(4.8) \quad P(X = x_i) = \frac{x_i^\beta a_i}{\sum_{i=1}^n x_i^\beta a_i}, \quad \forall i = 1, 2, \dots, n, \quad \beta \neq 1.$$

Taking into account the previous procedure for getting above inequalities and their reversed variants, we arrive at the following results.

**Theorem 7.** Let  $\{x_i, i = 1, 2, \dots, n\}, \{a_i, i = 1, 2, \dots, n\}$  be two sequences of nonnegative real numbers and assume that  $1 < \beta < \alpha$  and

$$(4.9) \quad \frac{m^{\alpha-\beta}}{[\sum_{i=1}^n a_i]^{\beta-1}} \geq 1.$$

Then it follows that

$$\sum_{i=1}^n x_i^\alpha a_i \geq \left( \sum_{i=1}^n x_i a_i \right)^\beta.$$

Otherwise, when  $0 < \beta < 1, \alpha > \beta$  and

$$(4.10) \quad \frac{M^{\alpha-\beta}}{[\sum_{i=1}^n a_i]^{\beta-1}} \leq 1,$$

the reverse inequality

$$\sum_{i=1}^n x_i^\alpha a_i \leq \left( \sum_{i=1}^n x_i a_i \right)^\beta$$

holds.



*Proof.* Let us consider the probability measure defined in (4.8),  $\beta > 1$ :

$$\begin{aligned} \frac{\sum_{i=1}^n x_i^\alpha a_i}{[\sum_{i=1}^n x_i a_i]^\beta} &= \frac{\sum_{i=1}^n x_i^\alpha a_i}{[\sum_{i=1}^n a_i]^\beta [E(X)]^\beta} \\ &\geq \frac{\sum_{i=1}^n x_i^\alpha a_i}{[\sum_{i=1}^n a_i]^\beta E(X^\beta)} \\ &= \frac{\sum_{i=1}^n x_i^{\alpha-\beta} x_i^\beta a_i}{[\sum_{i=1}^n a_i]^{\beta-1} \sum_{i=1}^n x_i^\beta a_i} \geq \frac{m^{\alpha-\beta}}{[\sum_{i=1}^n a_i]^{\beta-1}} \geq 1. \end{aligned}$$

This is the first desired result of Theorem 7.

Next we shall give the proof of the second case. As the same proof as the above discussion, by the assumptions  $0 < \beta < 1$  and (4.10), we have

$$\begin{aligned} \frac{\sum_{i=1}^n x_i^\alpha a_i}{[\sum_{i=1}^n x_i a_i]^\beta} &= \frac{\sum_{i=1}^n x_i^\alpha a_i}{[\sum_{i=1}^n a_i]^\beta [E(X)]^\beta} \\ &\leq \frac{\sum_{i=1}^n x_i^\alpha a_i}{[\sum_{i=1}^n a_i]^\beta E(X^\beta)} \\ &= \frac{\sum_{i=1}^n x_i^{\alpha-\beta} x_i^\beta a_i}{[\sum_{i=1}^n a_i]^{\beta-1} \sum_{i=1}^n x_i^\beta a_i} \leq \frac{M^{\alpha-\beta}}{[\sum_{i=1}^n a_i]^{\beta-1}} \leq 1. \quad \square \end{aligned}$$

Repeating the proving procedure of the previous theorem, we get the following interesting result.

**Theorem 8.** Let  $\{x_i, i = 1, 2, \dots, n\}, \{a_i, i = 1, 2, \dots, n\}$  be two sequences of nonnegative real numbers and assume that  $\beta > \max\{\alpha, 1\}$ ,  $\alpha > 0$  and

$$(4.11) \quad \frac{M^{\alpha-\beta}}{[\sum_{i=1}^n a_i]^{\beta-1}} \geq 1.$$

Then it follows that

$$\sum_{i=1}^n x_i^\alpha a_i \geq \left( \sum_{i=1}^n x_i a_i \right)^\beta.$$

In addition, for  $0 < \alpha < \beta < 1$ , and

$$(4.12) \quad \frac{m^{\alpha-\beta}}{[\sum_{i=1}^n a_i]^{\beta-1}} \leq 1$$

the reverse inequality

$$\sum_{i=1}^n x_i^\alpha a_i \leq \left( \sum_{i=1}^n x_i a_i \right)^\beta$$

holds.

Because of the similarity of the proofs of last two theorems the proof of the last one is omitted.

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